Advanced Algorithms

南京大学

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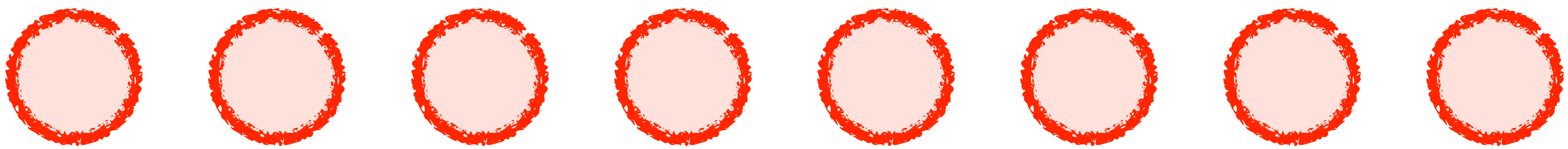
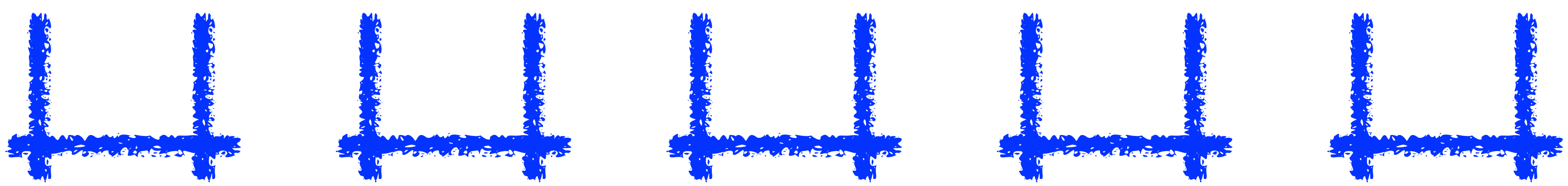
Load Balancing

m balls



uniformly & independently

n bins



loads: X_1 X_2 ⋯ ⋯ X_n

(# of balls
in each bin)

maximum load?

Balls-into-Bins Model

"m balls are thrown into n bins uniformly & independently at random"

uniform random $h: [m] \to [n]$

- birthday problem: probability of h being 1-1

$$\prod_{i=1}^{m-1}\left(1-\frac{i}{n}\right) \approx \exp\left(-\frac{m^2}{2n}\right) = \delta \quad \text{for some} \quad m = \Theta\left(\sqrt{n \ln \frac{1}{\delta}}\right)$$

- coupon collector: probability of h being onto

$$\Pr[\text{not on-to}] \le n\left(1-\frac{1}{n}\right)^m \le e^{-c} \quad \text{if} \quad m \ge n\ln n + c\ln n$$

throwing balls until no empty bin:

$\mathbf{E}[\text{ \# of balls used}]$
$= n\,H(n) = n \ln n + O(n)$

- load balancing: # of balls in bin i is $X_i = |h^{-1}(i)|$

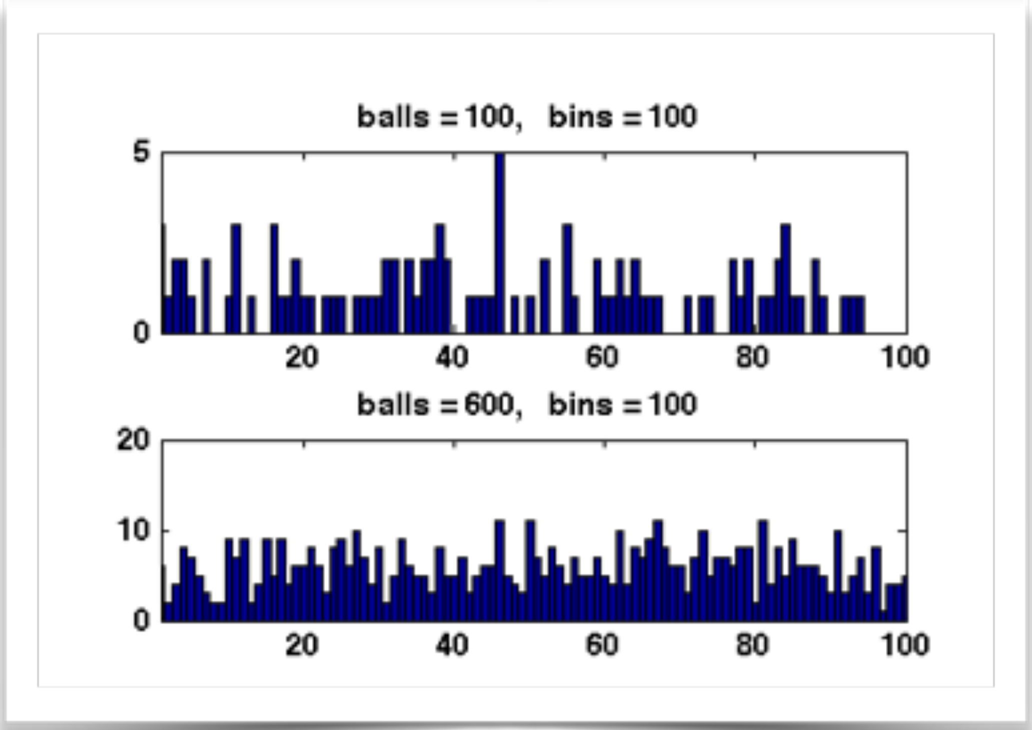
max loads?

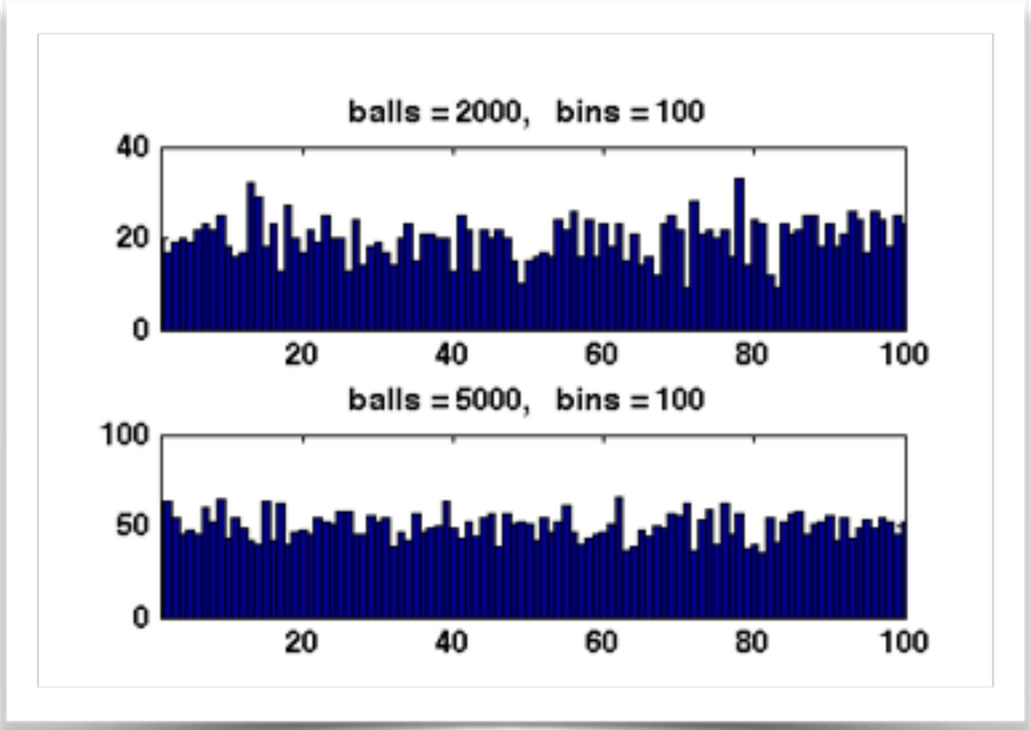
Balls-into-Bins Model

"m balls are thrown into n bins uniformly & independently at random"

uniform random $h: [m] \to [n]$

- load balancing: # of balls in bin i is $X_i = |h^{-1}(i)|$





Balls-into-Bins Model

"m balls are thrown into n bins
uniformly & independently at random"

uniform random $h: [m] \to [n]$

- load balancing: # of balls in bin i is $X_i = |h^{-1}(i)|$

$$\max_i \mathbf{E}[X_i] = \frac{m}{n}$$

symmetry: every X_i is identically distributed

all $\mathbf{E}[X_i]$ are the same

linearity of expectation:

$$\sum_i \mathbf{E}[X_i] = \mathbf{E}[\sum_i X_i] = \mathbf{E}[m] = m$$

Balls-into-Bins Model

"m balls are thrown into n bins uniformly & independently at random"

uniform random $h: [m] \to [n]$

- load balancing: # of balls in bin i is $X_i = |h^{-1}(i)|$

When $m = \theta(n)$:

max load $\max_i X_i = O\left(\frac{\log n}{\log \log n}\right)$ with high probability.

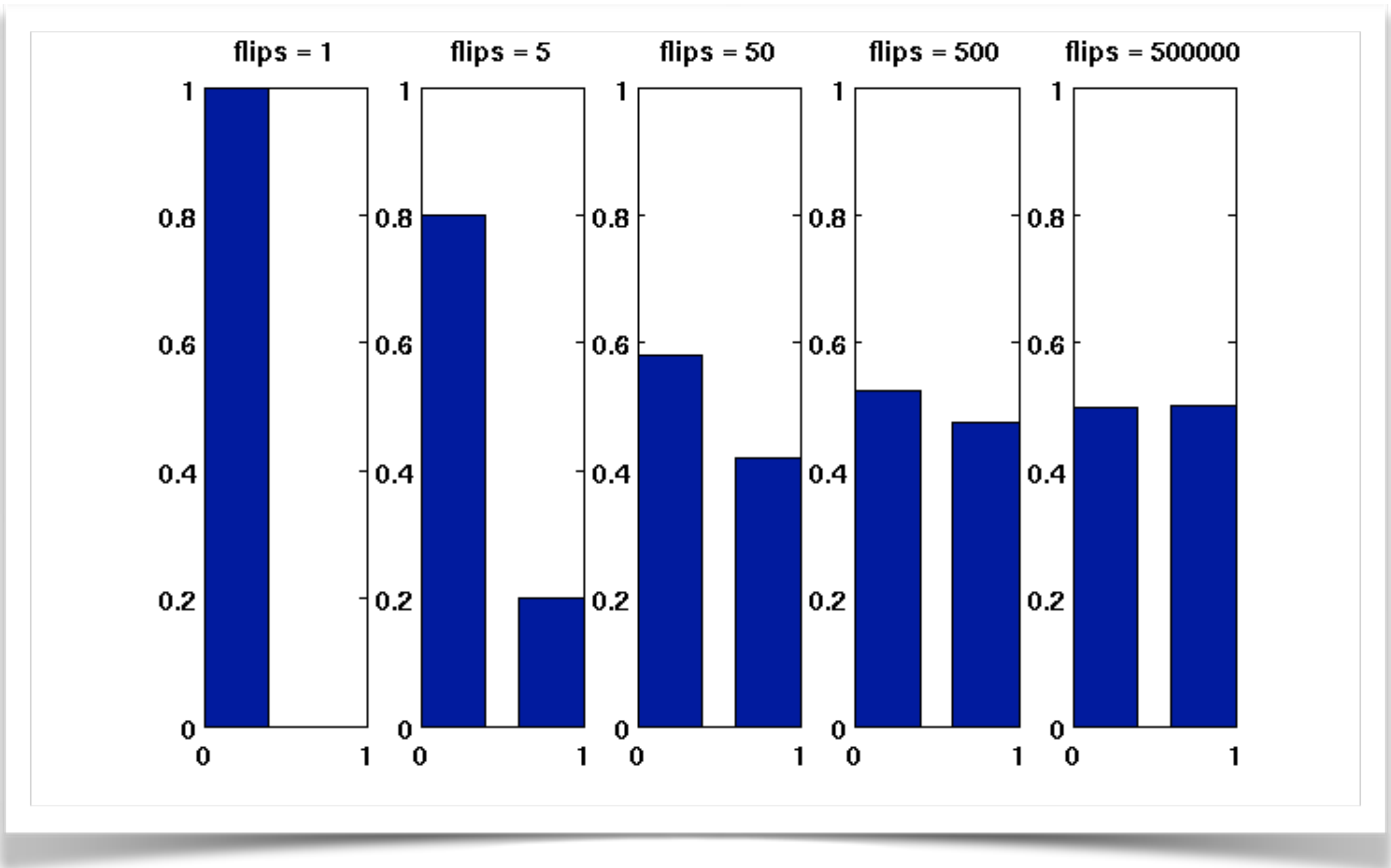
When $m = \Omega(n \log n)$:

max load $\max_i X_i = O\left(\frac{m}{n}\right)$ with high probability.

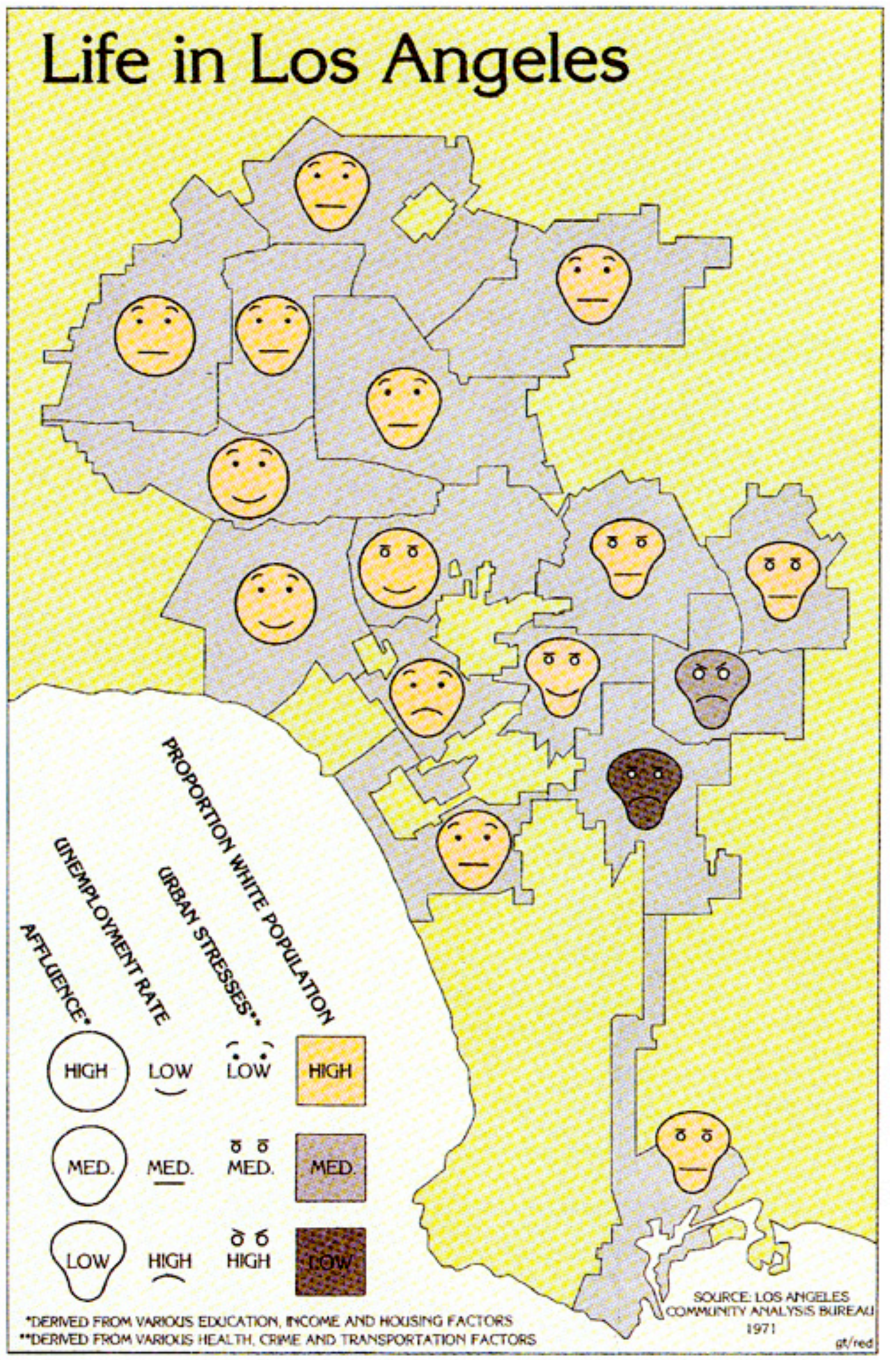
with high probability (*w.h.p.*): with probability $1\text{-}O(1/n)$

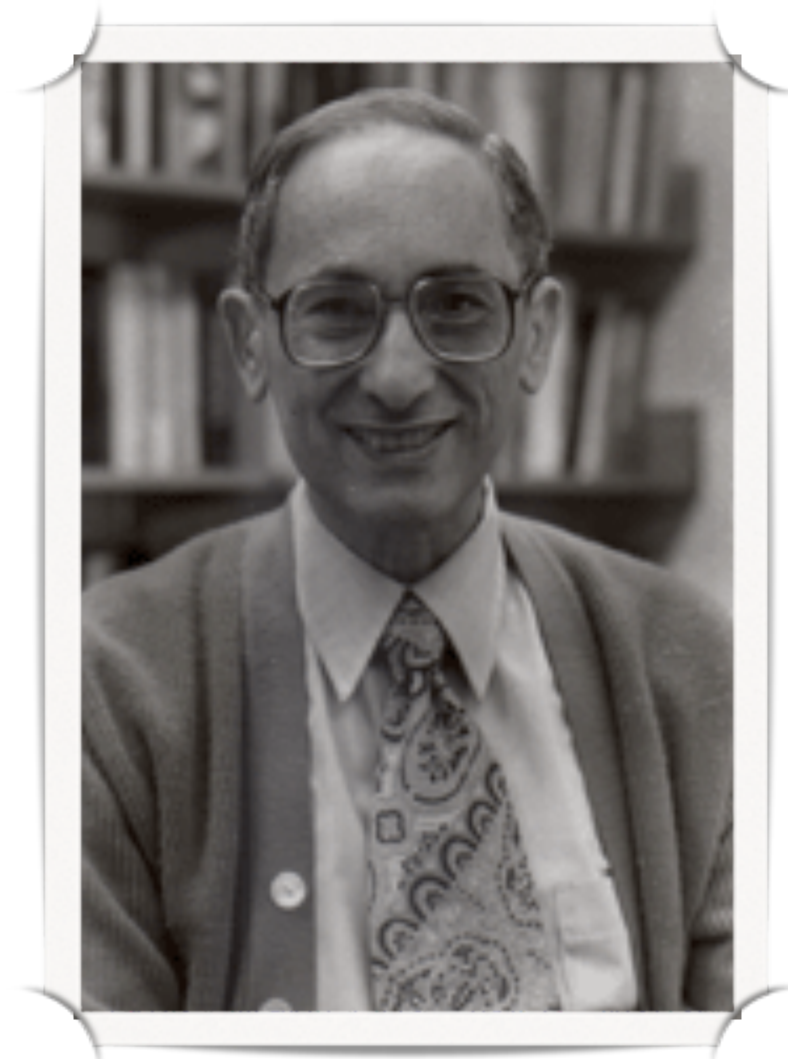
Concentration

Flip a coin for many times:



Chernoff Bounds





Herman Chernoff

Chernoff bound:

For independent trials $X_1, X_2, \ldots, X_n \in \{0, 1\}$.
Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathbf{E}[X]$.

For any $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \right)^{\mu}$$

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mu}$$

Chernoff Bound

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$0 < \delta \leq 1$:

$$\Pr[X \geq (1+\delta)\mu] < \exp\left(-\frac{\mu\delta^2}{3}\right)$$

$$\Pr[X \leq (1-\delta)\mu] < \exp\left(-\frac{\mu\delta^2}{2}\right)$$

$t \geq 2\mathrm{e}\mu$:

$$\Pr[X \geq t] \leq 2^{-t}$$

Balls-into-Bins

- *m*-balls-into-*n*-bins: max load <? w.h.p.

X_1: load of the first bin

$$X_1 = \sum_{j=1}^{m} X_{1j} \qquad X_{ij} = \begin{cases} 1 & \text{ball } j \text{ goes to bin } i \\ 0 & \text{otherwise} \end{cases}$$

$$X_1 = \text{Binomial}\left(m, \frac{1}{n}\right) \qquad \mu = \mathbf{E}[X_1] = \frac{m}{n}$$

Chernoff bound:

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

X_1: load of the first bin

$$X_1 = \text{Binomial}\left(m, \frac{1}{n}\right) \qquad \mu = \mathbf{E}[X_1] = \frac{m}{n}$$

Chernoff bound:

$$\Pr[X_1 \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu$$

For $m = n$, $\mu = 1$.

$$\Pr[X_1 \geq L] \leq \frac{e^L}{eL^L} \leq \frac{1}{n^2} \qquad \text{when } L = \frac{e \ln n}{\ln \ln n}$$

union bound:

$$\Pr\left[\max_{1\leq i\leq n} X_i \geq L\right] \leq n \cdot \Pr[X_1 \geq L] \leq \frac{1}{n}$$

max load is $O(\log n / \log\log n)$ w.h.p.

X_1: load of the first bin

$$X_1 = \text{Binomial}\left(m, \frac{1}{n}\right) \qquad \mu = \mathbf{E}[X_1] = \frac{m}{n}$$

Chernoff bound:

$$\Pr[X_1 \geq t] \leq 2^{-t} \text{ for } t \geq 2e\mu$$

For $m \geq n \ln n$, $\mu \geq \ln n$.

$$\Pr\left[X_1 \geq \frac{2em}{n}\right] = \Pr[X_1 \geq 2e\mu] \leq 2^{-2e\mu} \leq 2^{-2e\ln n} < \frac{1}{n^2}$$

union bound:

$$\Pr\left[\max_{1\leq i\leq n} X_i \geq \frac{2em}{n}\right] \leq n \cdot \Pr\left[X_1 \geq \frac{2em}{n}\right] \leq \frac{1}{n}$$

max load is $O(m/n)$ w.h.p.

Balls-into-Bins Model

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uniformly & independently at random"

uniform random $h: [m] \to [n]$

- load balancing: # of balls in bin i is $X_i = |h^{-1}(i)|$

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For independent trials $X_1, X_2, \ldots, X_n \in \{0, 1\}$.
Let $X = \sum_{i=1}^{n} X_i$, and $\mu = \mathbf{E}[X]$.

For any $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

A Generalization of Markov's Inequality

Theorem:

For any X, for $h : X \mapsto \mathbb{R}^+$, for any $t > 0$,

$$\Pr[h(X) \geq t] \leq \frac{\mathbf{E}[h(X)]}{t}.$$

Moment Generating Functions

Definition (moment generating functions):

The moment generating function of X is

$$M(\lambda) = \mathbf{E}\left[e^{\lambda X}\right].$$

Taylor's expansion:

$$\mathbf{E}\left[e^{\lambda X}\right] = \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} X^k\right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbf{E}\left[X^k\right]$$

independent $X_1, X_2, \ldots, X_n \in \{0,1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] \quad \leq ? \qquad \text{for } \lambda > 0$$

$$\leq \Pr\left[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\right] \leq \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(1+\delta)\mu}}$$

Markov

$$= \mathbf{E}\left[e^{\lambda \sum_{i=1}^{n} X_i}\right] = \mathbf{E}\left[\prod_{i=1}^{n} e^{\lambda X_i}\right] = \prod_{i=1}^{n} \mathbf{E}\left[e^{\lambda X_i}\right]$$

Independence!

independent $X_1, X_2, \ldots, X_n \in \{0,1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] \qquad \text{for } \lambda > 0$$

$$\leq \Pr\left[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\right] \leq \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(1+\delta)\mu}}$$

$$= \prod_{i=1}^{n} \mathbf{E}\left[e^{\lambda X_i}\right] \leq \prod_{i=1}^{n} e^{p_i(e^{\lambda}-1)} = e^{(e^{\lambda}-1)\mu}$$

$$p_i = \Pr[X_i = 1] \qquad \mu = \sum_{i=1}^{n} p_i$$

$$= p_i \cdot e^{\lambda \cdot 1} + (1-p_i) \cdot e^{\lambda \cdot 0} \leq e^{p_i(e^{\lambda}-1)}$$

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

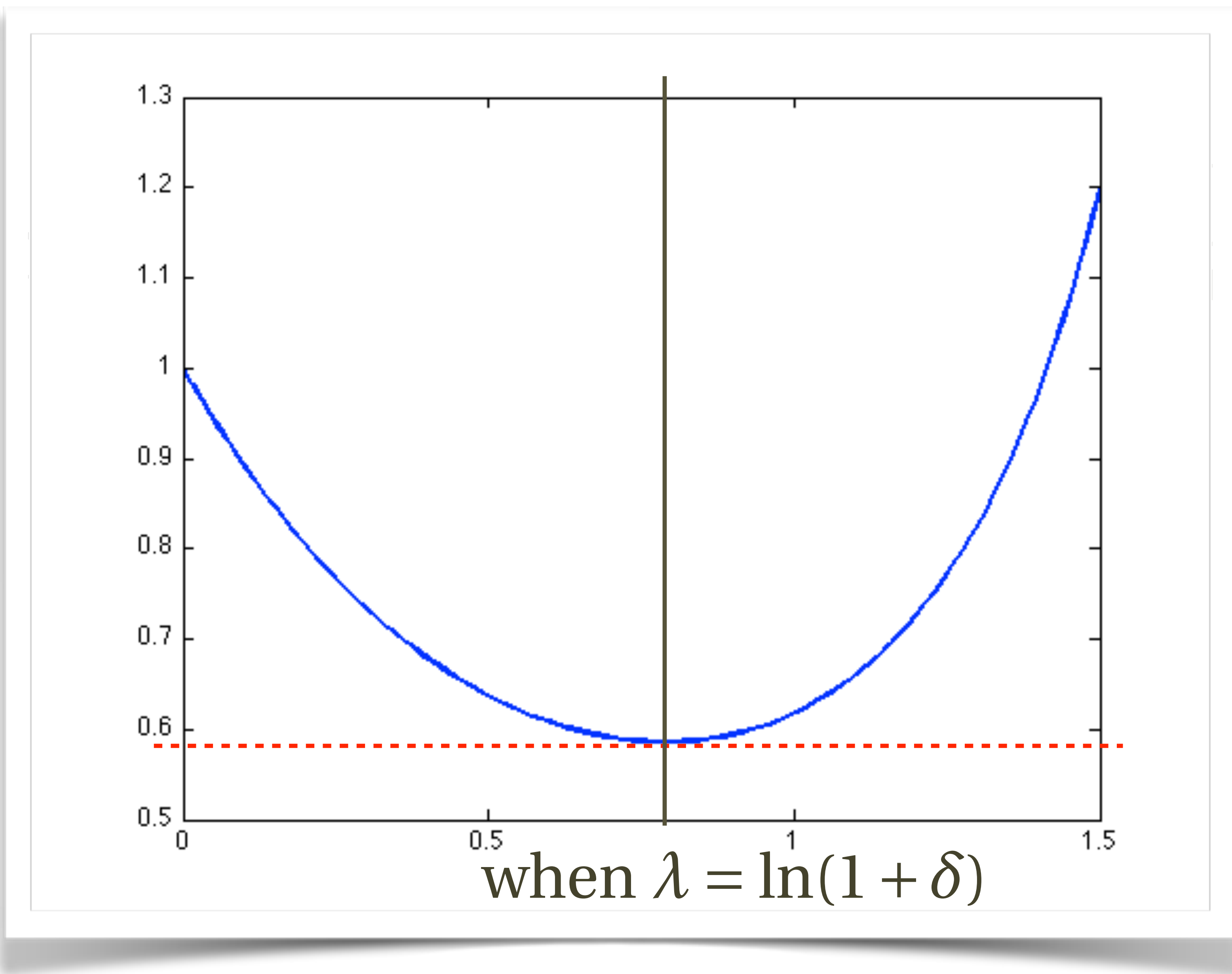
$$\Pr[X \geq (1+\delta)\mu] \qquad \text{for } \lambda > 0$$

$$\leq \Pr\left[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\right] \leq \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(1+\delta)\mu}} \leq \left(\frac{e^{(e^{\lambda}-1)}}{e^{\lambda(1+\delta)}}\right)^{\mu}$$

$$\leq e^{(e^{\lambda}-1)\mu}$$

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$



$\lambda > 0$

$1) \quad \delta) \quad)^{\mu}$

independent $X_1, X_2, \ldots, X_n \in \{0,1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] \qquad \text{for } \lambda > 0$$

$$\leq \Pr\left[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}\right] \leq \left(\frac{e^{(e^{\lambda}-1)}}{e^{\lambda(1+\delta)}}\right)^{\mu}$$

when $\lambda = \ln(1+\delta)$ $$= \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] \qquad \text{for } \lambda > 0$$

$$\leq \Pr\left[\mathrm{e}^{\lambda X} \geq \mathrm{e}^{\lambda(1+\delta)\mu}\right] \leq \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(1+\delta)\mu}} \leq \left(\frac{e^{(e^{\lambda}-1)}}{e^{\lambda(1+\delta)}}\right)^{\mu}$$

$$= \mathbf{E}\left[\prod_{i=1}^{n} e^{\lambda X_i}\right] = \prod_{i=1}^{n} \mathbf{E}\left[e^{\lambda X_i}\right] \leq \prod_{i=1}^{n} e^{p_i(e^{\lambda}-1)} = e^{(e^{\lambda}-1)\mu}$$

$$p_i = \Pr[X_i = 1] \qquad \mu = \sum_{i=1}^{n} p_i$$

$$\leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu} \qquad \text{when } \lambda = \ln(1+\delta)$$

Chernoff Bound

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] = \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

Chernoff Bound

independent $X_1, X_2, \ldots, X_n \in \{0,1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mu}$$

For any $\lambda < 0$,

$$\Pr[X \le (1-\delta)\mu] = \Pr\left[e^{\lambda X} \ge e^{\lambda(1-\delta)\mu}\right] \le \frac{\mathbf{E}\left[e^{\lambda X}\right]}{e^{\lambda(1-\delta)\mu}}$$

Chernoff Bound

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

$$X = \sum_{i=1}^{n} X_i \qquad \mathbf{E}[X] = \mu$$

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \right)^{\mu}$$

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mu}$$

Chernoff Bound

independent $X_1, X_2, \ldots, X_n \in \{0, 1\}$

let $X = \sum_{i=1}^{n} X_i$

$t > 0:$

$$\Pr[X \geq \mathbf{E}[X] + t] \leq \exp\left(-\frac{2t^2}{n}\right)$$

$$\Pr[X \leq \mathbf{E}[X] - t] \leq \exp\left(-\frac{2t^2}{n}\right)$$

Hoeffding's Bound

independent $X_1, X_2, \ldots, X_n$ with $X_i \in [a_i, b_i]$

let $X = \sum_{i=1}^{n} X_i$

$t > 0:$

$$\Pr[X \geq \mathbf{E}[X] + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

$$\Pr[X \leq \mathbf{E}[X] - t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Hoeffding's Lemma

random variable $Y \in [a, b]$ with $\mathbb{E}[Y] = 0$

$$\Rightarrow \quad \mathbb{E}[e^{\lambda Y}] \le e^{\lambda^2 (b-a)^2/8}$$

independent $X_1, X_2, \ldots, X_n$ with $X_i \in [a_i, b_i]$

$X = \sum_{i=1}^{n} X_i$ let $Y = X - \mathbb{E}[X]$, $Y_i = X_i - \mathbb{E}[X_i]$ $\Rightarrow$ $\begin{cases} \mathbb{E}[Y] = \mathbb{E}[Y_i] = 0 \\ Y = \sum_{i=1}^{n} Y_i \end{cases}$

$\Pr[X - \mathbb{E}[X] \ge t] = \Pr[Y \ge t] \le e^{-\lambda t}\mathbb{E}[e^{\lambda Y}]$ for $\lambda \ge 0$

$$= e^{-\lambda t} \prod_{i=1}^{n} \mathbb{E}[e^{\lambda Y_i}] \le \exp\left(-\lambda t + \frac{\lambda^2}{8}\sum_{i=1}^{n}(b_i - a_i)^2\right) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

when $\lambda = \frac{4t}{\sum_{i=1}^{n}(b_i - a_i)^2}$

Hoeffding's Lemma

random variable $Y \in [a, b]$ with $\mathbb{E}[Y] = 0$

$$\Rightarrow \mathbb{E}[e^{\lambda Y}] \le e^{\lambda^2 (b-a)^2/8}$$

independent $X_1, X_2, \ldots, X_n$ with $X_i \in [a_i, b_i]$

$$X = \sum_{i=1}^{n} X_i$$

let $Y = X - \mathbb{E}[X]$, $Y_i = X_i - \mathbb{E}[X_i]$ $\Rightarrow$ $\begin{cases} \mathbb{E}[Y] = \mathbb{E}[Y_i] = 0 \\ Y = \sum_{i=1}^{n} Y_i \end{cases}$

$$\Pr[X - \mathbb{E}[X] \le -t] = \Pr[Y \le -t] \le e^{\lambda t}\mathbb{E}[e^{\lambda Y}]$$ for $\lambda \le 0$

$$= e^{\lambda t} \prod_{i=1}^{n} \mathbb{E}[e^{\lambda Y_i}] \le \exp\left(\lambda t + \frac{\lambda^2}{8} \sum_{i=1}^{n} (b_i - a_i)^2\right) \le \exp\left(-\frac{2t^2}{\sum_{i=1}^{n} (b_i - a_i)^2}\right)$$

when $\lambda = \frac{-4t}{\sum_{i=1}^{n} (b_i - a_i)^2}$

Hoeffding's Bound

independent $X_1, X_2, \ldots, X_n$ with $X_i \in [a_i, b_i]$

let $X = \sum_{i=1}^{n} X_i$

$t > 0:$

$$\Pr[X \geq \mathbf{E}[X] + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

$$\Pr[X \leq \mathbf{E}[X] - t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

The method of bounded differences

Independent random variables: $X=(X_1, X_2, \ldots, X_n)$.

$f(x_1, x_2, \ldots, x_n)$ satisfies the Lipschitz condition:

$$| f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) | \leq c_i$$

for arbitrary possible values $x_1, \ldots, x_n, y_i$.

$t > 0$:

$$\Pr[f(\boldsymbol{X}) \geq \mathbf{E}[f(\boldsymbol{X})] + t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right)$$

$$\Pr[f(\boldsymbol{X}) \leq \mathbf{E}[f(\boldsymbol{X})] - t] \leq \exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right)$$

Occupancy Problem

- m-balls-into-n-bins:
- number of empty bins?

$$X_i = \begin{cases} 1 & \text{bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$$

empty bins: $X = \sum_{i=1}^{n} X_i$

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] \quad = n\left(1 - \frac{1}{n}\right)^m$$

Xi are dependent

deviation: $\Pr[|X - \mathbf{E}[X]| \geq t] \leq ?$

Occupancy Problem

- m-balls-into-n-bins:
- number of empty bins?

empty bins: X

deviation:

$$\Pr[|X - \mathbf{E}[X]| \geq t] \leq ?$$

Y_j : the bin of ball *j* (Independent!)

$$X = f(Y_1, \ldots, Y_m) \quad = |[n] - \{Y_1, \ldots, Y_m\}|$$

Lipschitz:

changing any Y_j can change X for at most 1

$$\Rightarrow \quad \Pr[|X - \mathbf{E}[X]| \geq t\sqrt{m}] \leq 2e^{-t^2/2}$$

Pattern Matching

- a random string of length n,
- a pattern of length k,
- # of matched substrings?

alphabet Σ

$$|\Sigma| = m$$

a fixed pattern: $\pi \in \Sigma^k$

uniform & independent: $X_1, \ldots, X_n \in \Sigma$

$$Y : \text{\#substrings } \pi \text{ in } (X_1, \ldots, X_n)$$

$$\mathbf{E}[Y] = (n-k+1)\left(\frac{1}{m}\right)^k$$

Deviation?

Pattern Matching

- a random string of length n,
- a pattern of length k,
- # of matched substrings?

alphabet Σ

$|\Sigma| = m$

a fixed **pattern**: $\pi \in \Sigma^k$

uniform & independent: $X_1, \ldots, X_n \in \Sigma$

$Y = f(X_1, \ldots, X_n)$

changing any X_i changes f for at most k

$$\Pr\left[|Y - \mathbf{E}[Y]| \geq tk\sqrt{n}\right] \leq 2e^{-t^2/2}$$

Martingale

Definition:

A sequence of random variables $X_0, X_1, \ldots$ is a **martingale** if for all $i > 0$,

$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

Azuma's Inequality:

Let $X_0, X_1, \ldots$ be a martingale such that, for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c_k,$$

Then

$$\Pr\left[|X_n - X_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

Generalization

Definition:

$Y_0, Y_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$ if, for all $i \geq 0$,

- Y_i is a function of $X_0, X_1, \ldots, X_i$;
- $\mathbf{E}[Y_{i+1} \mid X_0, \ldots, X_i] = Y_i$.

Definition:

$Y_0, Y_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$ if, for all $i \geq 0$,

- Y_i is a function of $X_0, X_1, \ldots, X_i$;
- $\mathbf{E}[Y_{i+1} \mid X_0, \ldots, X_i] = Y_i$.

- Betting on a fair game;
- X_i : win/loss of the *i*-th bet;
- Y_i : wealth after the *i*-th bet -- Martingale (fair game)

Azuma's Inequality (general version):

Let $Y_0, Y_1, \ldots$ be a martingale with respect to $X_0, X_1, \ldots$ such that, for all $k \geq 1$,

$$|Y_k - Y_{k-1}| \leq c_k,$$

Then

$$\Pr[|Y_n - Y_0| \geq t] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

Doob Sequence

Definition (Doob sequence):

The Doob sequence of a function f with respect to a sequence $X_1, \ldots, X_n$ is

$$Y_i = \mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i]$$

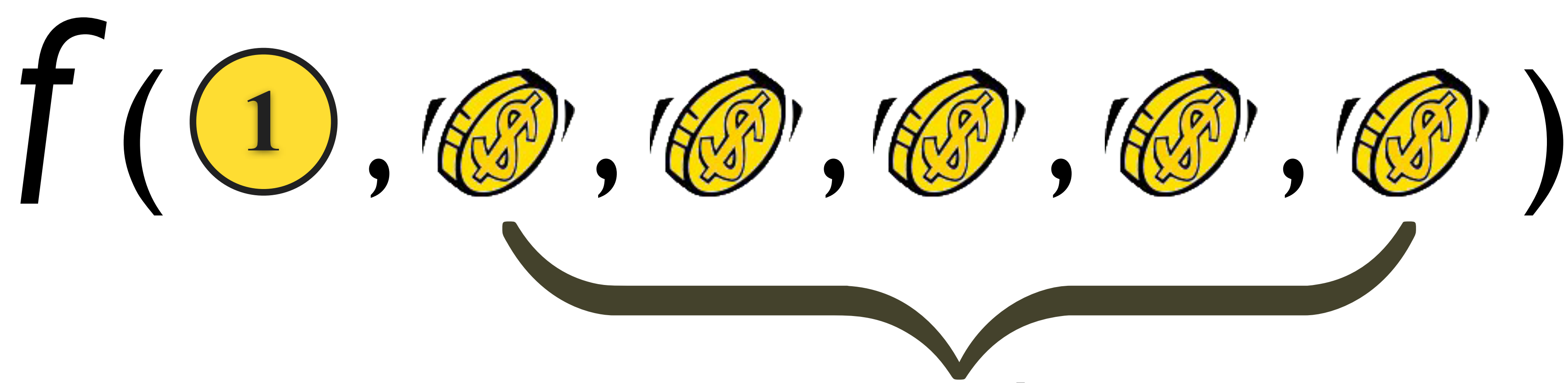
$$Y_0 = \mathbf{E}[f(X_1, \ldots, X_n)] \dashrightarrow Y_n = f(X_1, \ldots, X_n)$$

Doob Sequence

$$\mathbf{E}[f] = Y_0,$$

Doob Sequence

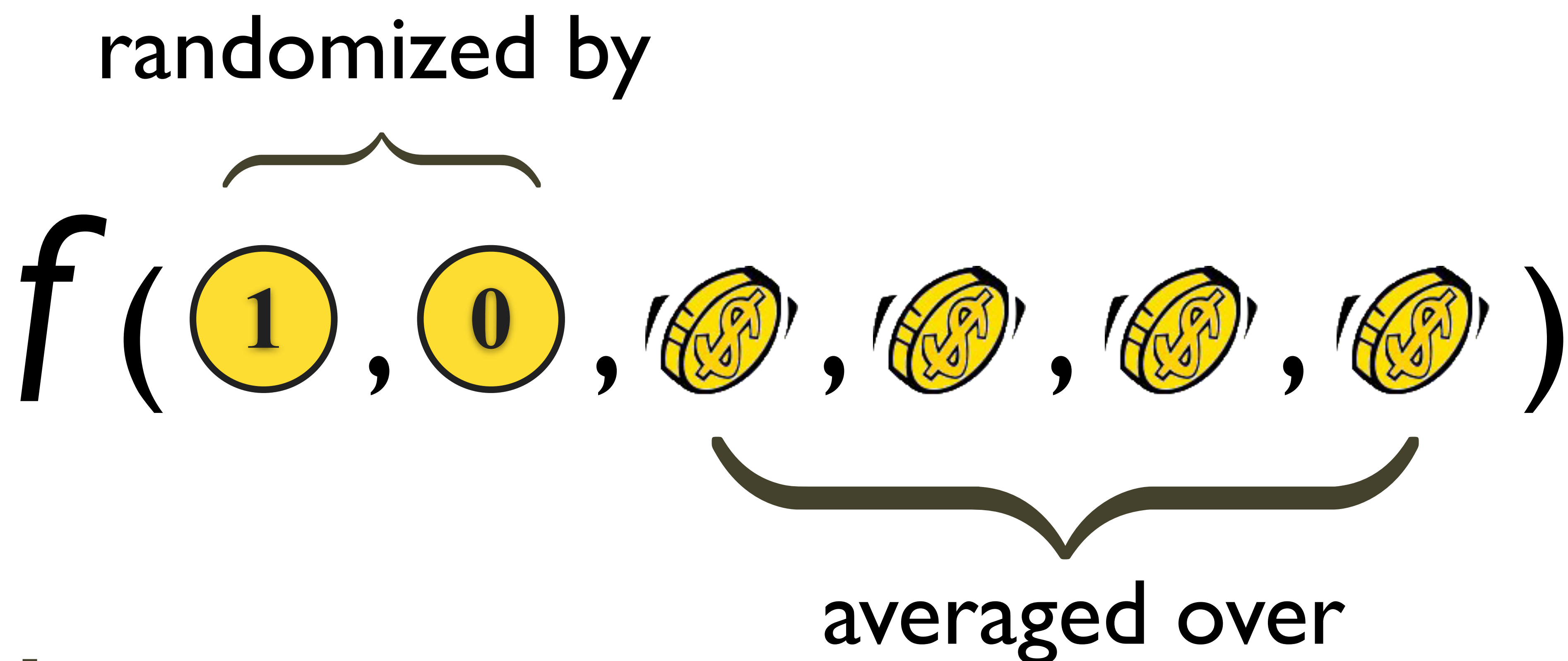
randomized by



averaged over

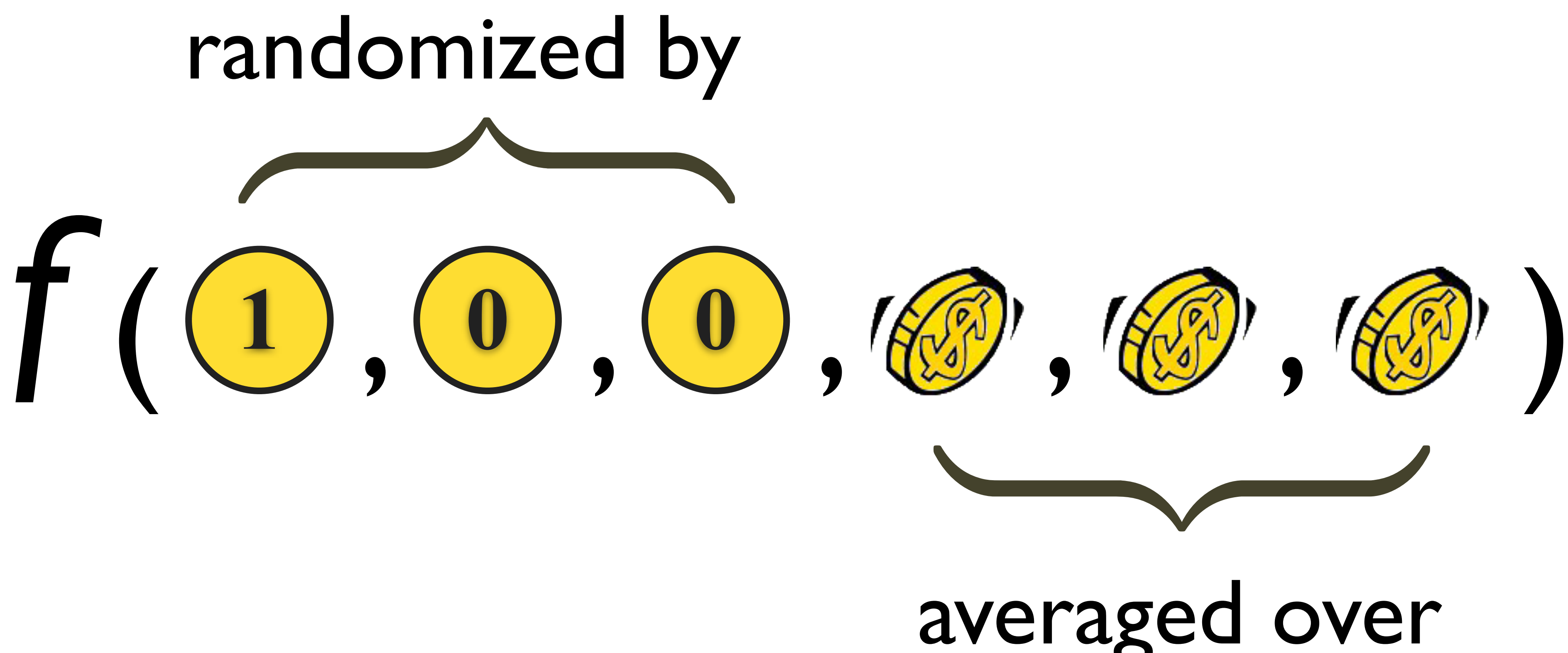
$$\mathbf{E}[f] = Y_0, \quad Y_1,$$

Doob Sequence



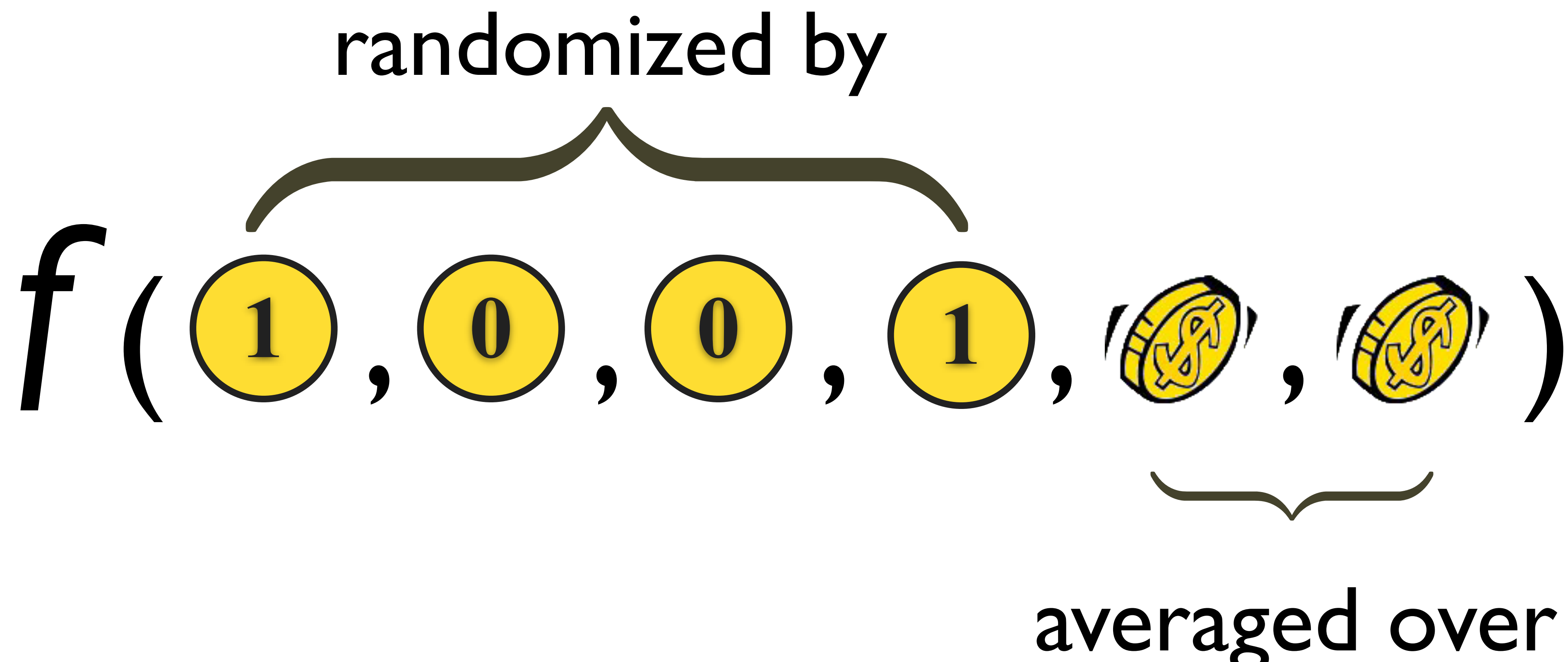
$\mathbf{E}[f] = Y_0, \quad Y_1, \quad Y_2,$

Doob Sequence



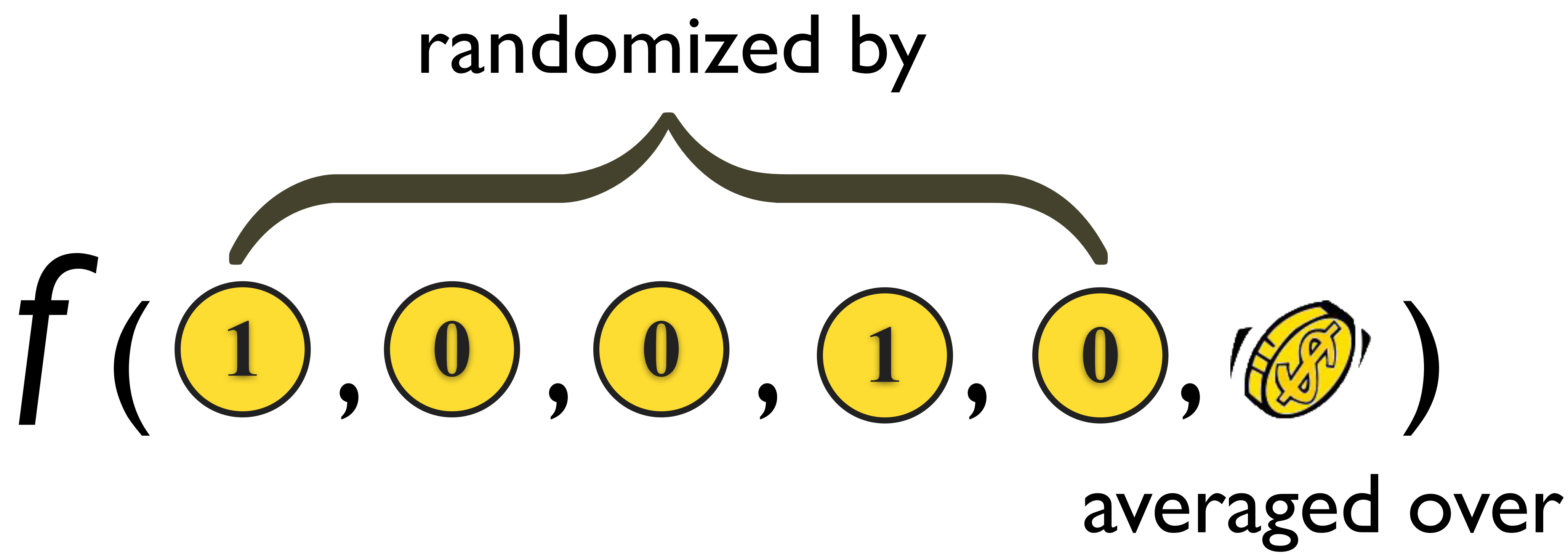
$$\mathbf{E}[f] = Y_0, \quad Y_1, \quad Y_2, \quad Y_3,$$

Doob Sequence



$$\mathbf{E}[f] = Y_0, \quad Y_1, \quad Y_2, \quad Y_3, \quad Y_4,$$

Doob Sequence



$$\mathbf{E}[f] = Y_0, \quad Y_1, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5,$$

Doob Sequence

randomized by

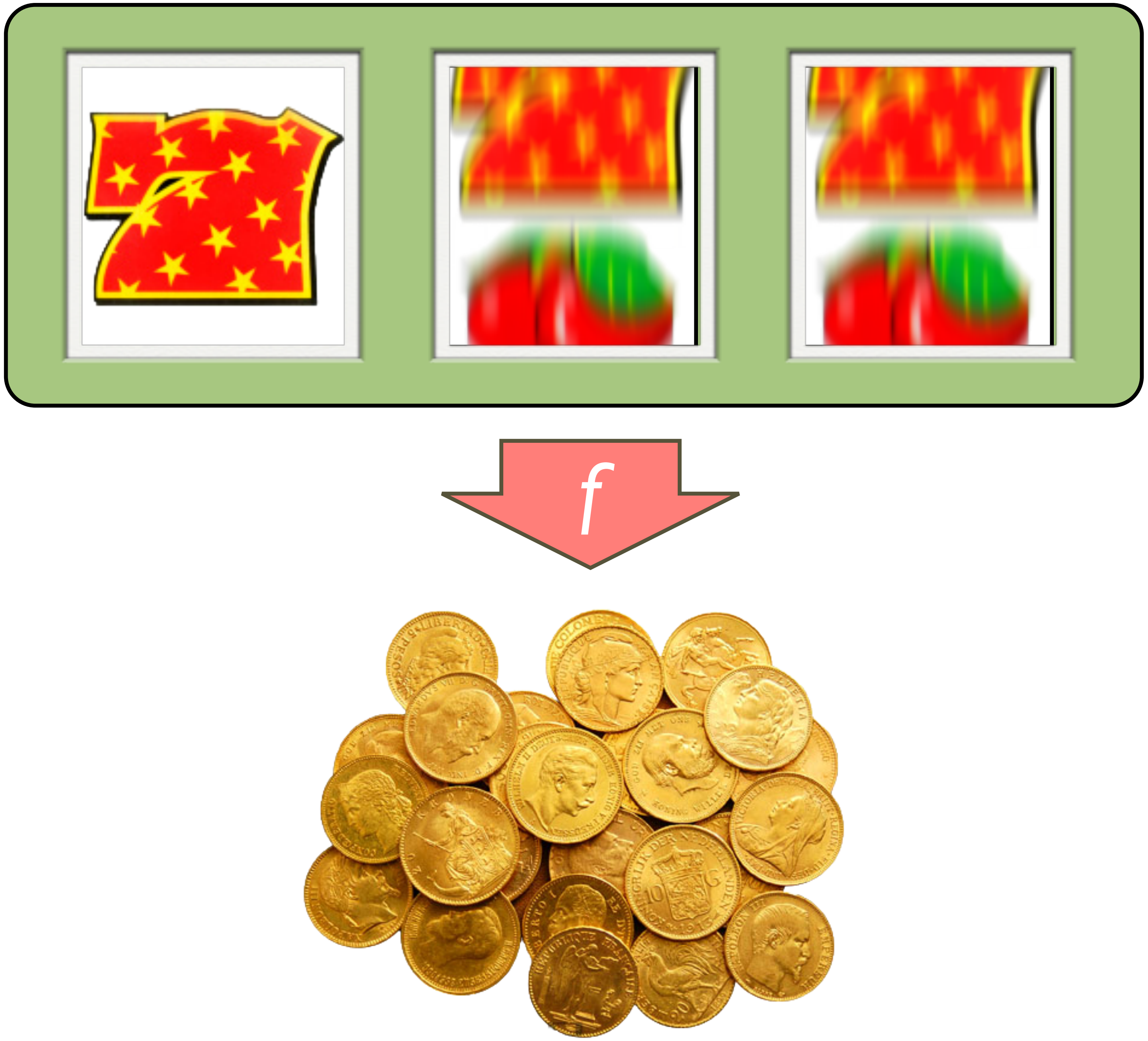
$$f(1, 0, 0, 1, 0, 1)$$

no information

full information

$$\mathbf{E}[f] = Y_0, \quad Y_1, \quad Y_2, \quad Y_3, \quad Y_4, \quad Y_5, \quad Y_6 \quad = f$$

Doob Martingale



Doob sequence:

$$Y_i = \mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_i]$$

Doob sequence is a martingale:

$$\mathbf{E}[Y_i \mid X_1,\ldots,X_{i-1}] = Y_{i-1}$$

Proof:

$$\begin{aligned}
&\mathbf{E}[Y_i \mid X_1,\ldots,X_{i-1}] \\
&= \mathbf{E}[\mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_i] \mid X_1,\ldots,X_{i-1}] \\
&= \mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_{i-1}] \\
&= Y_{i-1}
\end{aligned}$$

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and let f be a function of $X_0, X_1, \ldots, X_n$ satisfying that, for all $1 \leq i \leq n$,

$$|\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]| \leq c_i,$$

Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$

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$$\left|\underbrace{\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i]}_{Y_i} - \underbrace{\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]}_{Y_{i-1}}\right| \le c_i,$$

Then

(Azuma)

$$\Pr\left[\left|\underbrace{f(\boldsymbol{X})}_{Y_n} - \underbrace{\mathbf{E}[f(\boldsymbol{X})]}_{Y_0}\right| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

Doob martingale: $Y_i = \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i]$

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and let f be a function of $X_0, X_1, \ldots, X_n$ satisfying that, for all $1 \le i \le n$,

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hard to check!

Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$

$$|\mathbf{E}[f(\mathbf{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\mathbf{X}) \mid X_1, \ldots, X_{i-1}]| \le c_i,$$

Lipschitz Condition:

$f(x_1, \ldots, x_n)$ satisfies the **Lipschitz condition** with constants c_i, $1 \le i \le n$, if

$$\Big| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \Big| \le c_i.$$

Average-case:

$$|\mathbf{E}[f(\boldsymbol{X}) \mid X_1,\ldots,X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1,\ldots,X_{i-1}]| \le c_i,$$

Worst-case:

Lipschitz Condition:

$f(x_1,\ldots,x_n)$ satisfies the **Lipschitz condition** with constants c_i, $1 \le i \le n$, if

$$\Big| f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n) \Big| \le c_i.$$

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n independent random variables and let f be a function satisfying the Lipschitz condition with constants c_i, $1 \le i \le n$. Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$

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Proof:

Lipschitz condition

\+

independence

bounded averaged differences