# Advanced Algorithms

## Introduction: Min-Cut and Max-Cut

尹一通 Nanjing University, 2022 Fall

# Course Info

- Instructor: 尹一通
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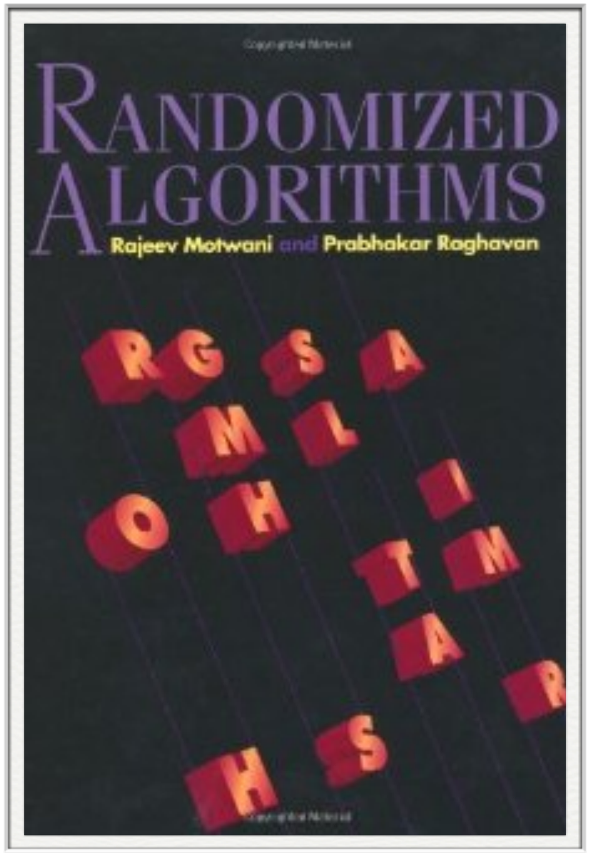
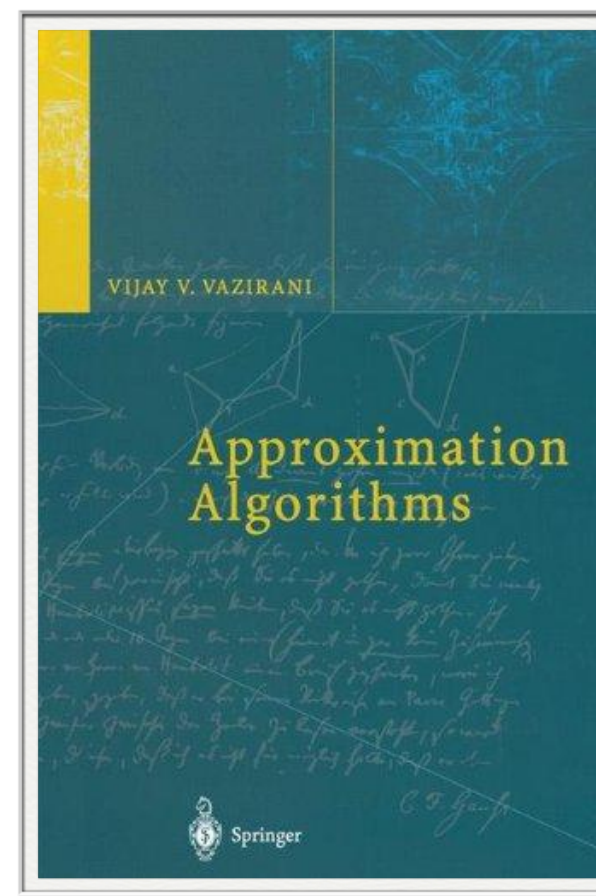
- Office hour: 804 Thursday 2-4pm
  - QQ group: 945849735
- Course webpage:
  - (校内暂时) http://114.212.81.7/wiki/
  - (稍后) http://tcs.nju.edu.cn/wiki/

# Textbooks




Rajeev Motwani and Prabhakar Raghavan.
***Randomized Algorithms.***
Cambridge University Press, 1995.

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***Approximation Algorithms.***
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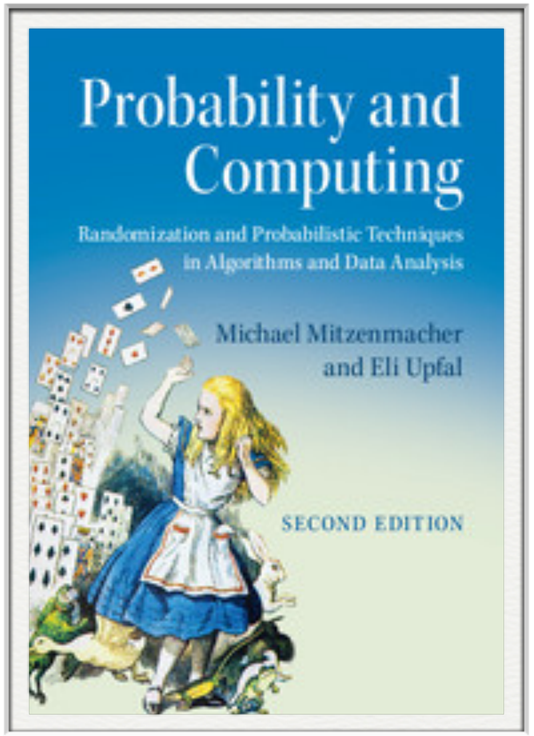
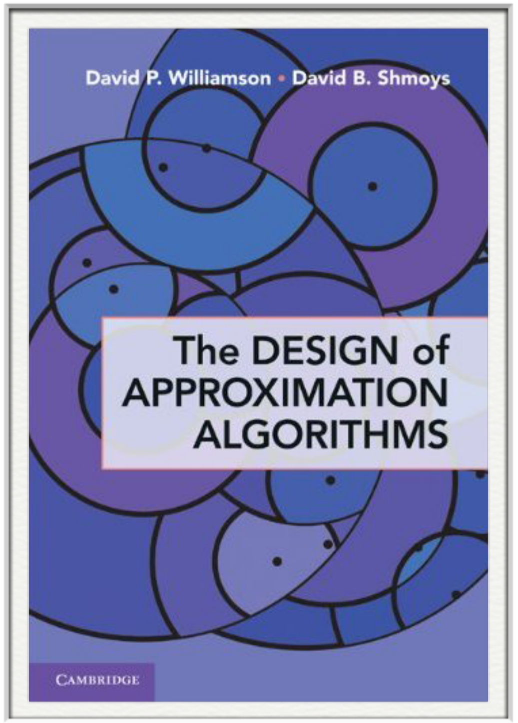
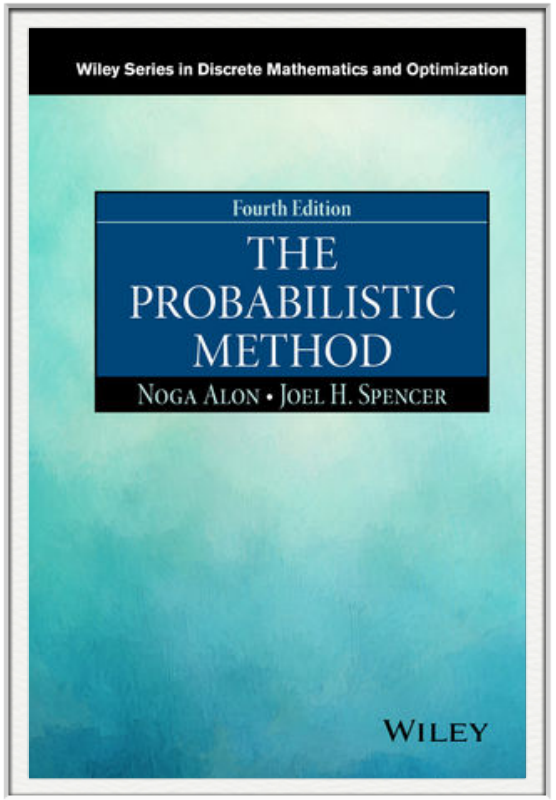
# References




Mitzenmacher and Upfal.
***Probability and Computing,***
*2nd Ed.*

Williamson and Shmoys
***The Design of Approximation Algorithms***




Alon and Spencer
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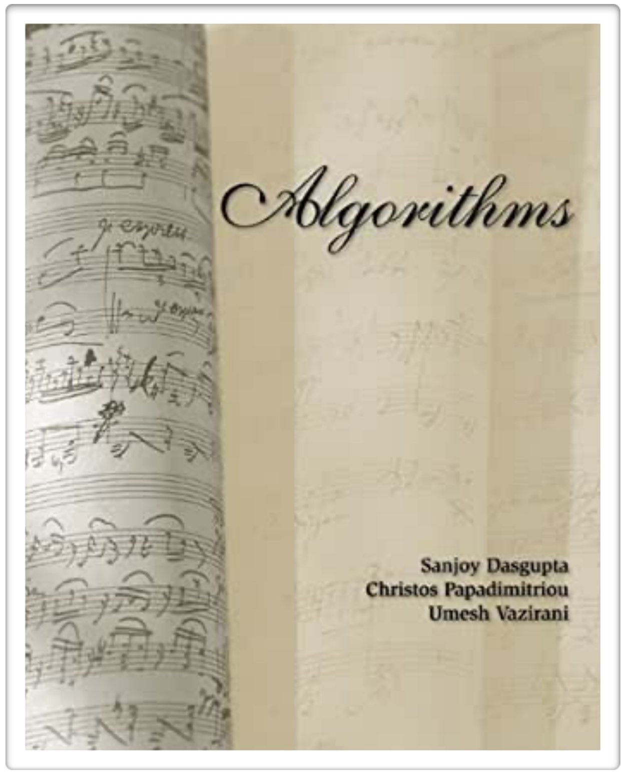
DPV
***Algorithms***



Muḥammad ibn Mūsā
al-Khwārizmī
(780-850)

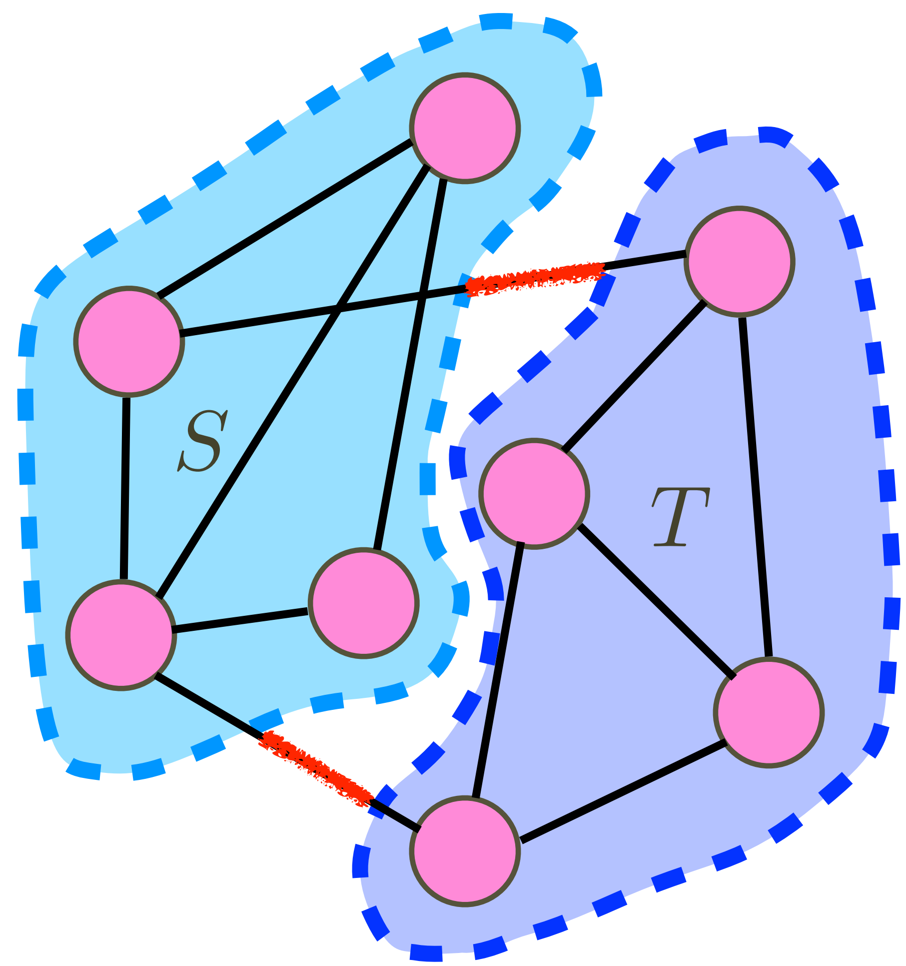
# *Advanced* Algorithms

# Minimum Cut

# Min-Cut

$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$

- Undirected graph $G(V,E)$
- Bi-partition of $V$ into nonempty $S$ and $T$
- Find a cut $E(S,T)$ of smallest size (***global*** min-cut)
- **Deterministic** algorithms:
  - max-flow min-cut (**duality**)
  - best upper bound (2021): $m^{1+o(1)}$

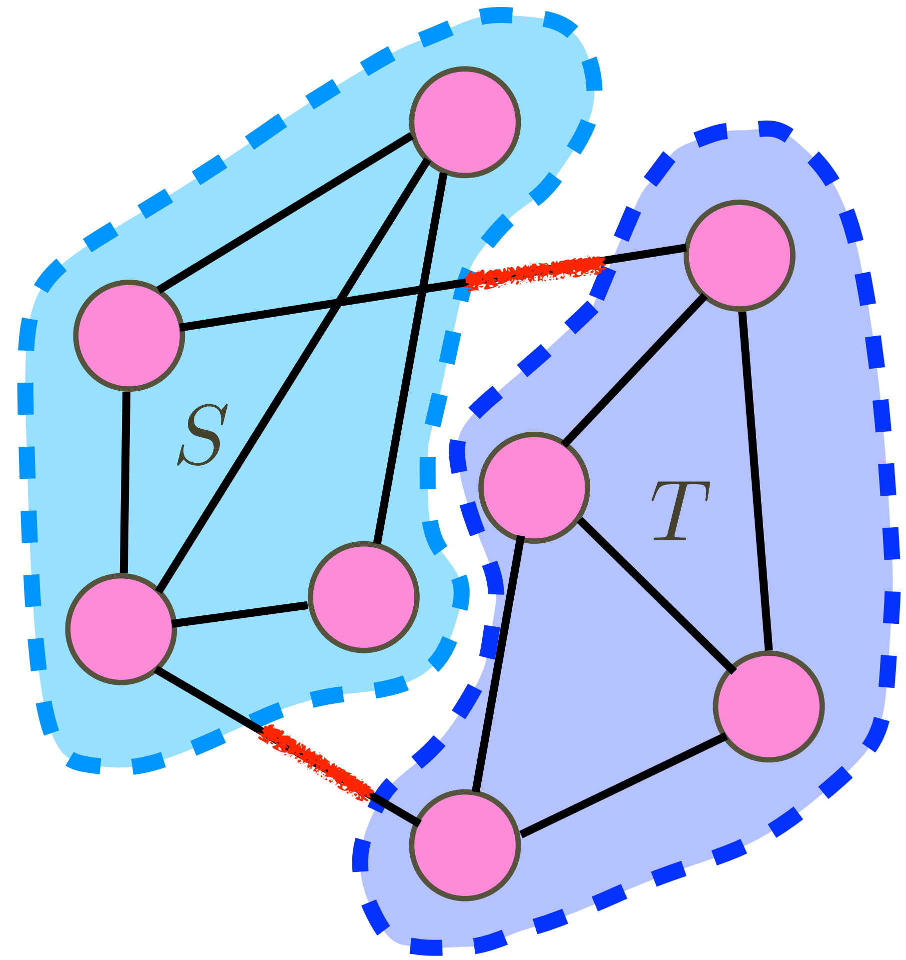



$\tilde{O}(\,\cdot\,)$ : ignore poly-logarithmic factors

# Min-Cut

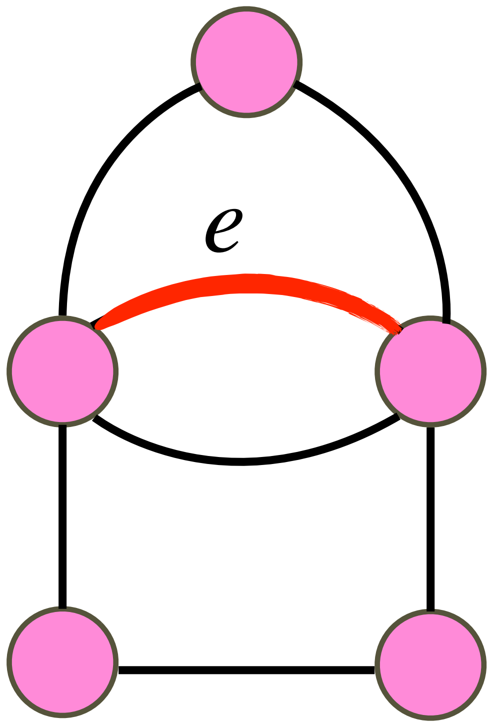
- Undirected graph $G(V, E)$
- Too many cuts:
  - there are $2^{\Omega(n)}$ bi-partitions
  - (will see later) only at most $O(n^2)$ min-cuts
- **Generate a random cut that is a min-cut with large enough probability?**

$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$



# Contraction

- Undirected multigraph $G(V, E)$
- multigraph:
  - allow parallel edges
  - but no self-loop
- contract($e$): $e = uv$ is an edge
  - merges two endpoints $u$ and $v$



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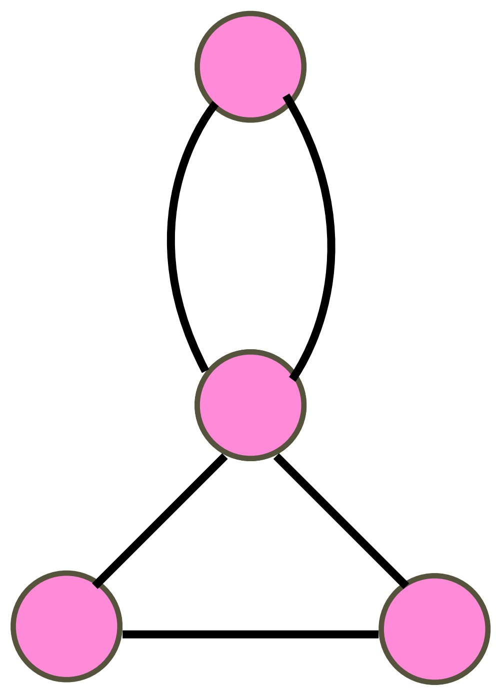


# Karger's Min-Cut Algorithm

- multigraph $G(V,E)$

**Karger's Algorithm:**

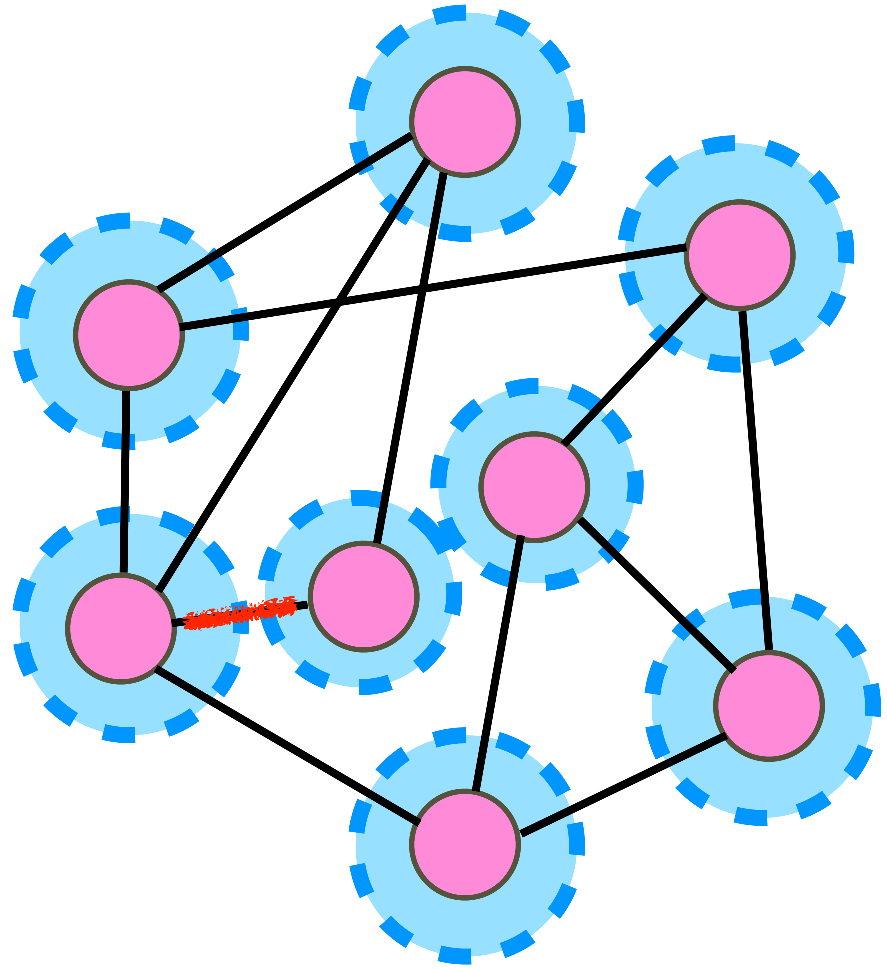
while $|V| > 2$ do:

  pick random $e \in E$;

  contract$(e)$;

return remaining edges;

random: uniformly and independently at random



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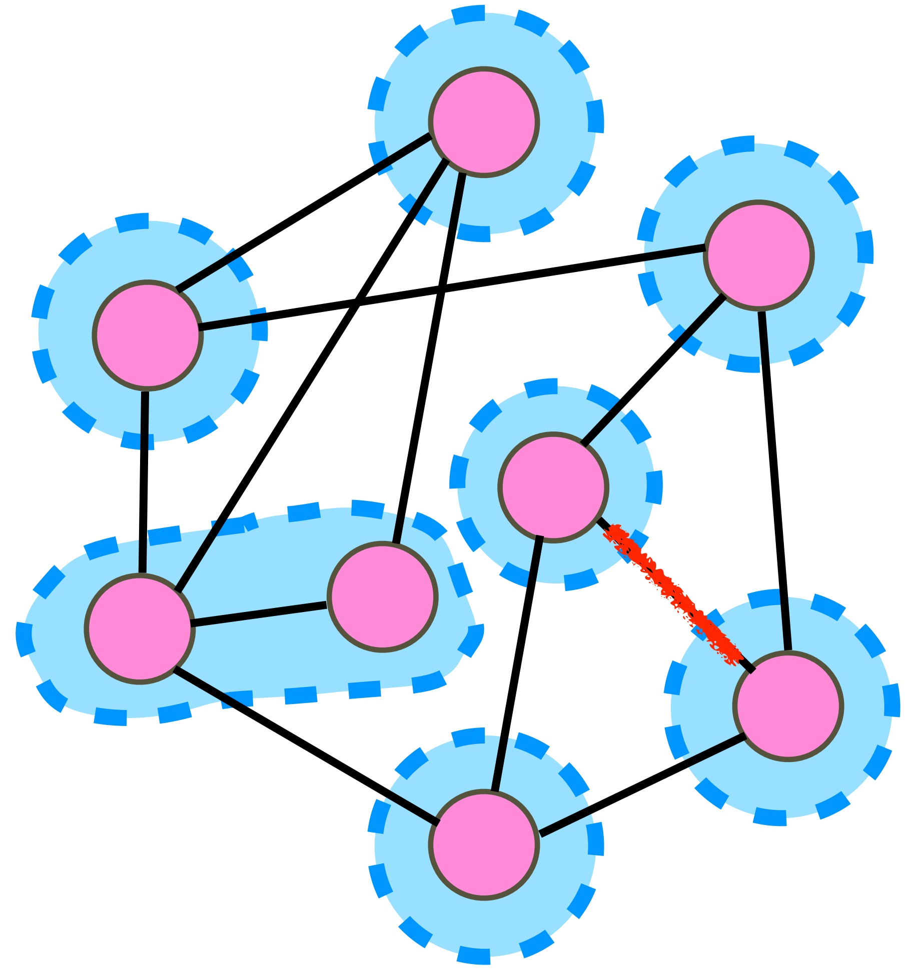

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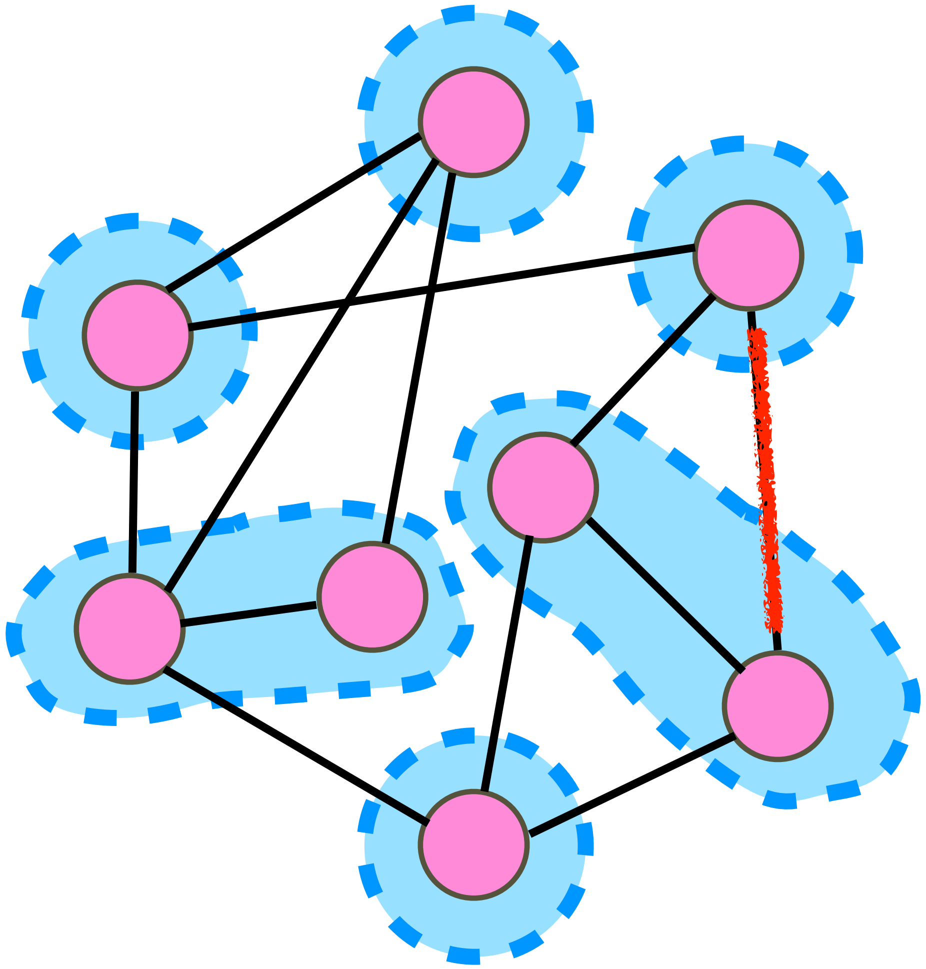

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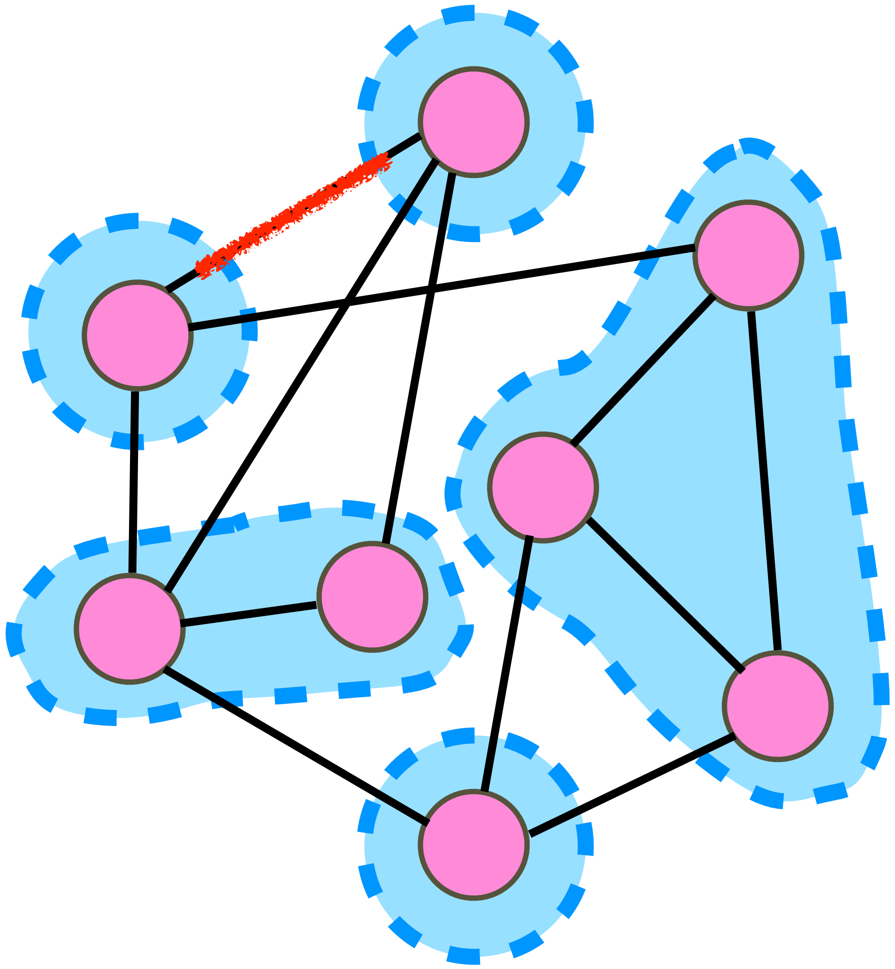

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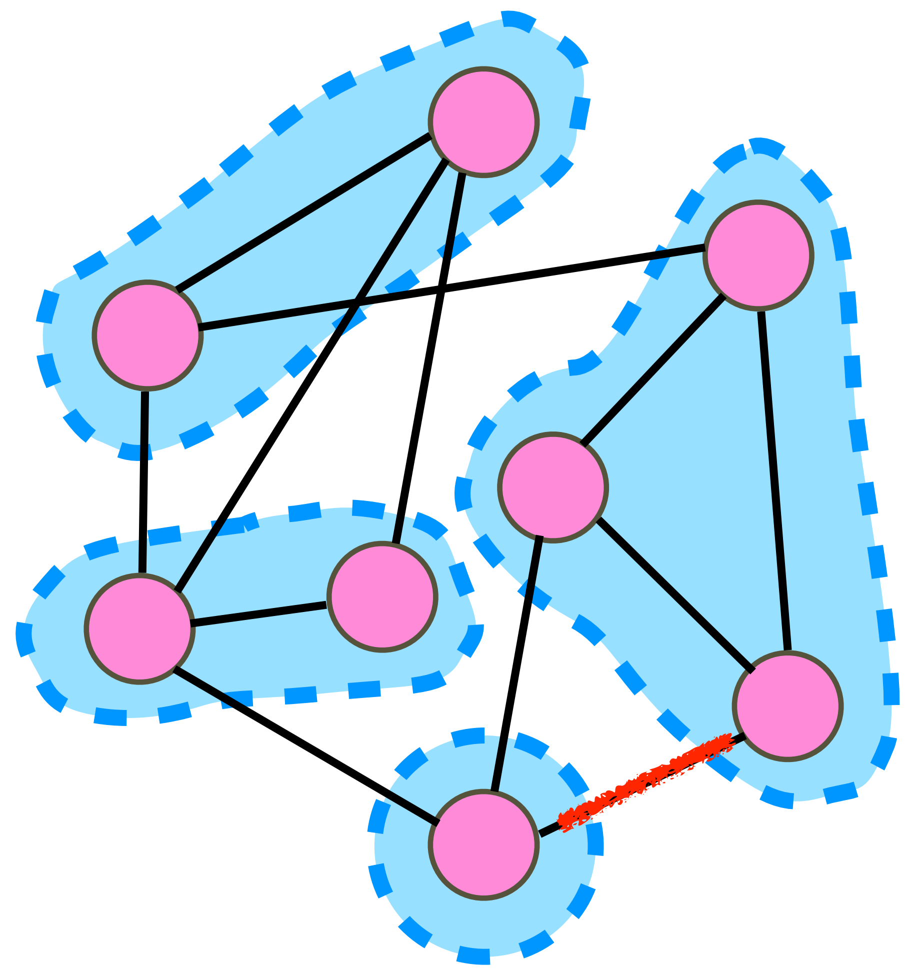

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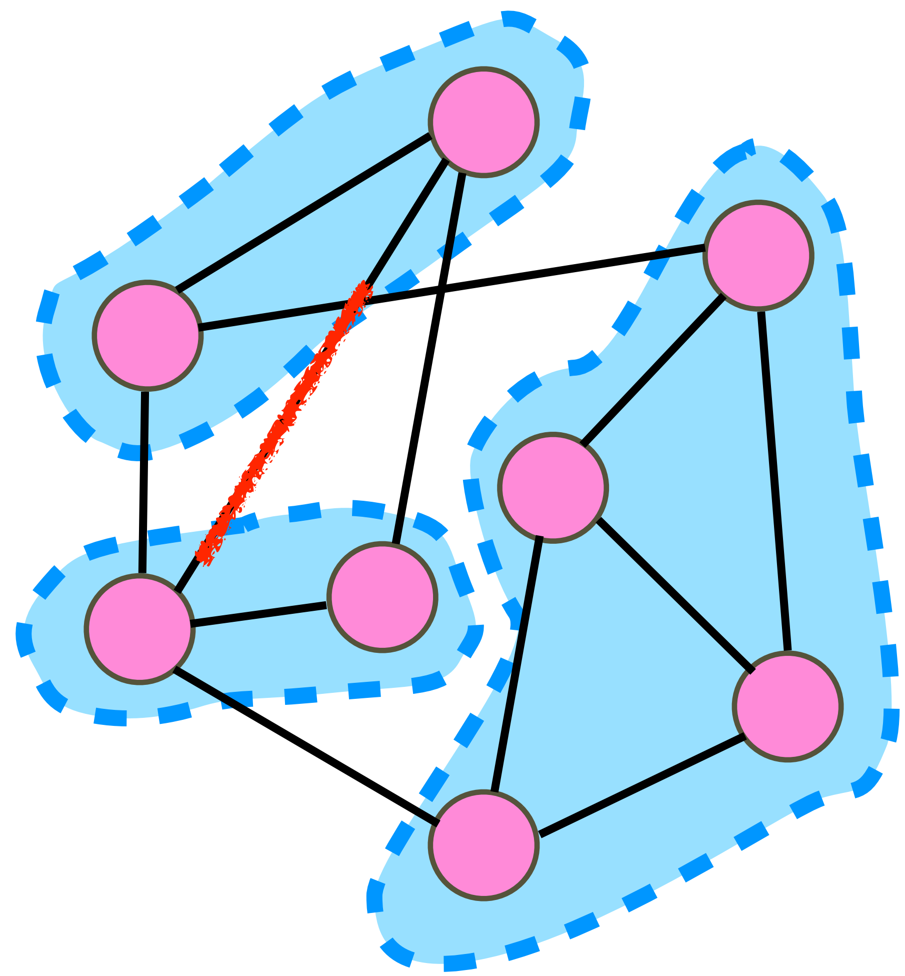

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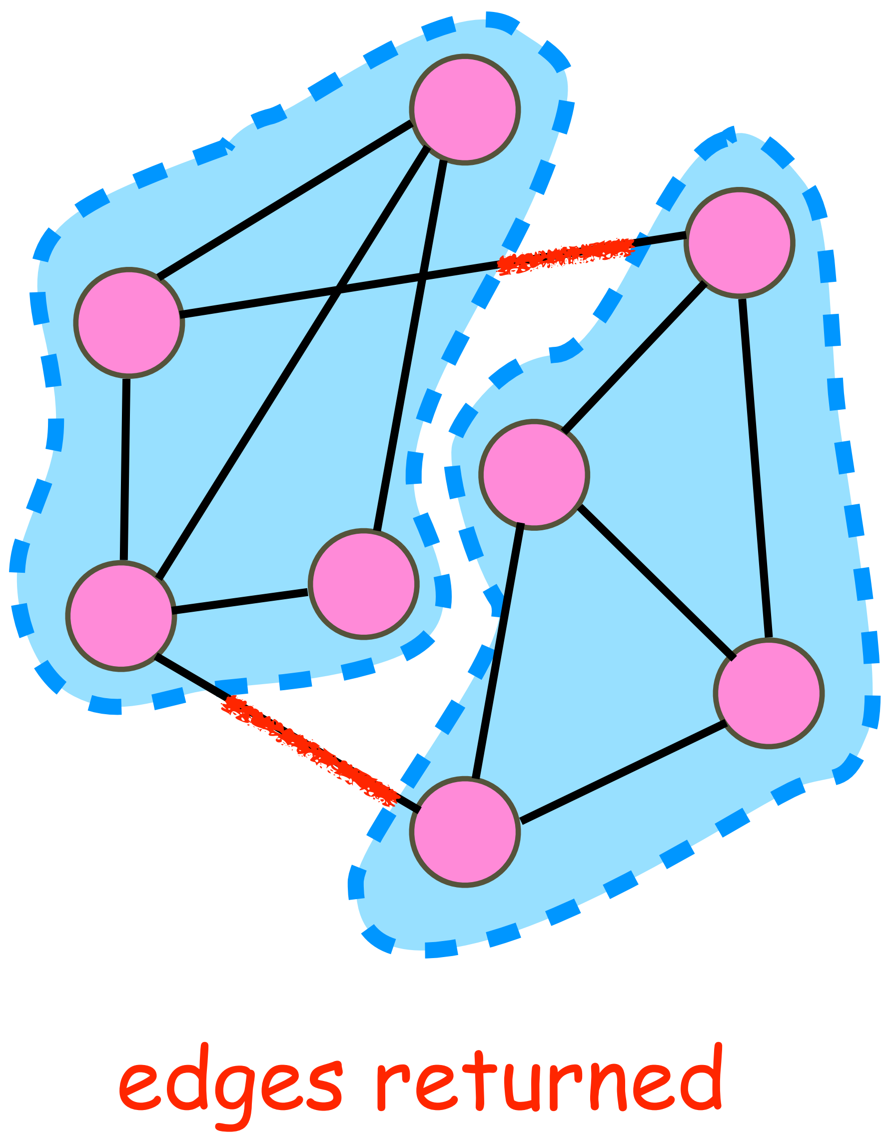

**Karger's Algorithm:**

while $|V| > 2$ do:

  pick random $e \in E$;

  contract($e$);

return remaining edges;

**Theorem** (Karger 1993).

$$\Pr[\text{ a min-cut is returned }] \geq \frac{2}{n(n-1)}$$

**repeat independently for $\frac{1}{2}n(n-1)\ln n$ times and return the smallest cut**

$$\Pr[\text{ fail to output a min-cut at last }]$$

$$= \Pr[\text{ fail to output a min-cut in one trial }]^{\frac{n(n-1)}{2}\ln n}$$

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^{\frac{n(n-1)}{2}\ln n} < \left(\frac{1}{\mathrm{e}}\right)^{\ln n} = \frac{1}{n}$$

**Succeed *with high probability* (*w.h.p.*)!**

**Karger's Algorithm:**

while $|V| > 2$ do:

pick random $e \in E$;

contract($e$);

return remaining edges;

Sequence of contracted edges:

$$e_1, e_2, \ldots, e_{n-2}$$

Initially: $G_0 = G$ input multigraph

$i$-th iteration:

$$G_i = \text{contract}(G_{i-1}, e_i)$$

**Observation**: Suppose $C \subseteq E_{i-1}$ is a min-cut in $G_{i-1}$.

$e_i \notin C \implies C$ is a min-cut in $G_i$

Fix an arbitrary min-cut $C$ in $G$.

$$\Pr[\, C \text{ is returned}\,] \geq \Pr[e_1, e_2, \ldots, e_{n-2} \notin C]$$

chain rule: $$= \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

Sequence of contracted edges: $e_1, e_2, \ldots, e_{n-2}$

Initially: $G_0 = G$ $\quad$ $i$-th iteration: $G_i = \text{contract}(G_{i-1}, e_i)$

**Observation:** Suppose $C \subseteq E_{i-1}$ is a min-cut in $G_{i-1}$.

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Fix an arbitrary min-cut $C$ in $G$.

$$\Pr[\, C \text{ is returned} \,] \geq \prod_{i=1}^{n-2} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

$C$ is a min-cut in $G_{i-1}$

**Observation:**

$C$ is a min-cut in $G(V, E)$

$\implies |E| \geq \frac{1}{2} |C| |V|$

$$\geq \prod_{i=1}^{n-2} \left(1 - \frac{2}{n-i+1}\right)$$

**Proof:**

min-degree of $G \geq |C|$

$$= \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} = \frac{2}{n(n-1)}$$

**Karger's Algorithm:**

while $|V| > 2$ do:

pick random $e \in E$;

contract($e$);

return remaining edges;

**Theorem** (Karger 1993)**.**

For any min-cut $C$ in an input graph $G(V, E)$,

$$\Pr\left[\, C \text{ is returned} \,\right] \geq \frac{2}{n(n-1)}$$

Time cost of 1 iteration: $O(n)$ Total time cost: $O(n^2)$

**repeat independently for** $\frac{1}{2}n(n-1)\ln n$ **times**

**and return the smallest cut**

Find a min-cut *with high probability* (w.h.p.) in $\tilde{O}(n^4)$ time.

# Number of Min-Cuts

**Theorem** (Karger 1993)**.**

For any min-cut $C$ in an input graph $G(V, E)$,

$$\Pr\left[\, C \text{ is returned} \,\right] \geq \frac{2}{n(n-1)}$$

**Corollary** (Karger 1993)**.**

The number of distinct min-cuts in any graph of $n$ vertices is at most $n(n-1)/2$.

Suppose there are $M$ distinct min-cuts: $C_1, C_2, \ldots, C_M$

$$1 \geq \Pr[\text{ a min-cut is returned }] = \sum_{i=1}^{M} \Pr[\, C_i \text{ is returned }]$$

$$\geq M \cdot \frac{2}{n(n-1)}$$

# An Observation

**Contract($G$, $t$):**

while $|V| > t$ do:

pick random $e \in E$;

contract($e$);

return remaining edges;

- multigraph $G(V, E)$
- sequence of contracted edges: $e_1, e_2, \ldots, e_{n-t}$

$C$: a min-cut in $G$.

$\mathscr{E}$: no edge in $C$ is contracted in Contract($G, t$).

$$\Pr[\,\mathscr{E}\,] \geq \prod_{i=1}^{n-t} \Pr[e_i \notin C \mid e_1, e_2, \ldots, e_{i-1} \notin C]$$

$$\geq \prod_{i=1}^{n-t} \frac{n-i-1}{n-i+1} = \frac{t(t-1)}{n(n-1)}$$

**only getting bad when $t$ is small**

# Faster Min-Cut

**Contract($G$, $t$):**

while $|V| > t$ do:

pick random $e \in E$;

contract($e$);

return remaining edges;

**FastCut($G$):**

if $|V| \le 6$ then return a min-cut by brute force;

else: ($t$ to be fixed later)

$$\left.\begin{aligned} G_1 &= \text{Contract}(G, t); \\ G_2 &= \text{Contract}(G, t); \end{aligned}\right\} \text{(independently)}$$

return $\min\left\{\text{FastCut}(G_1), \text{FastCut}(G_2)\right\}$;

$C$: a min-cut in $G$.

$\mathscr{E}$: no edge in $C$ is contracted in Contract($G$, $t$).

$$\Pr[\,\mathscr{E}\,] \ge \frac{t(t-1)}{n(n-1)}$$

**Contract($G$, $t$):**

while $|V| > t$ do:

pick random $e \in E$;

contract($e$);

return remaining edges;

**FastCut($G$):**

if $|V| \leq 6$ then return a min-cut by brute force;

else: set $t = n/\sqrt{2} + 1$

$G_1$ = Contract($G$, $t$);

$G_2$ = Contract($G$, $t$);

(independently)

return $\min\left\{\text{FastCut}(G_1), \text{FastCut}(G_2)\right\}$;

$C$: a min-cut in $G$.

$\mathscr{E}$: no edge in $C$ is contracted in Contract($G$, $t$).

$$\Pr[\mathscr{E}] \geq \frac{t(t-1)}{n(n-1)} \geq \frac{1}{2}$$

$$p(n) = \min_{G:|V|=n} \Pr\left[\text{FastCut}(G) \text{ succeeds}\right]$$

succeeds: returns a min-cut in $G$

$$\geq 1 - \left(1 - \Pr[\mathscr{E}] \Pr\left[\text{FastCut}(G_1) \text{ succeeds} \mid \mathscr{E}\right]\right)^2$$

$$\geq 1 - \left(1 - \frac{t(t-1)}{n(n-1)} p(t)\right)^2 \geq p\left(\frac{n}{\sqrt{2}} + 1\right) - \frac{1}{4} p\left(\frac{n}{\sqrt{2}} + 1\right)^2$$

**Contract($G$, $t$):**

while $|V| > t$ do:

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return remaining edges;

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$G_1 = \text{Contract}(G, t)$;
$G_2 = \text{Contract}(G, t)$; } (independently)

return $\min\left\{\text{FastCut}(G_1), \text{FastCut}(G_2)\right\}$;

$$p(n) = \min_{G:|V|=n} \Pr\left[\text{FastCut}(G) \text{ succeeds}\right]$$

$$\geq p\left(\frac{n}{\sqrt{2}} + 1\right) - \frac{1}{4}p\left(\frac{n}{\sqrt{2}} + 1\right)^2$$

- **Verified by induction:** $p(n) = \Omega\left(\frac{1}{\log n}\right)$
- **Recursion for time cost:** $T(n) \leq 2T\left(n/\sqrt{2} + 1\right) + O(n^2)$
- **Verified by induction:** $T(n) = O(n^2 \log n)$

**Contract($G$, $t$):**

while $|V| > t$ do:

    pick random $e \in E$;

    contract($e$);

return remaining edges;

**FastCut($G$):**

if $|V| \leq 6$ then return a min-cut by brute force;

else: set $t = n/\sqrt{2} + 1$

    $G_1 =$ Contract($G, t$);
    $G_2 =$ Contract($G, t$); } (independently)

return $\min \{$FastCut($G_1$), FastCut($G_2$)$\}$;

**Theorem** (Karger and Stein 1996)**.**

On any graph $G$ of $n$ vertices, FastCut($G$) runs in $O(n^2 \log n)$ time and returns a min-cut in $G$ with probability $\Omega(1/\log n)$.

**repeat independently for $O(\log^2 n)$ times and return the smallest cut**

Find a min-cut *with high probability* (w.h.p.) in $\tilde{O}(n^2)$ time.

# Min-Cut

- **Randomized** (*Monte Carlo*) algorithms: (correct w.h.p.)
  - Karger's *Contraction* algorithm (1993): $\tilde{O}(n^4)$
  - Karger-Stein Algorithm (1996): $\tilde{O}(n^2)$
  - Karger's *Tree-packing* Algorithm (2000): $\tilde{O}(m)$
- **Deterministic** algorithms:
  - max-flow min-cut (**duality**): $n \times$ max-flow computation
  - Stoer-Wagner Algorithm (1997): $\tilde{O}(mn)$
  - (only for single graphs) Kawarabayashi-Thorup (2015): $\tilde{O}(m)$
  - Jason Li (2021): $m^{1+o(1)}$

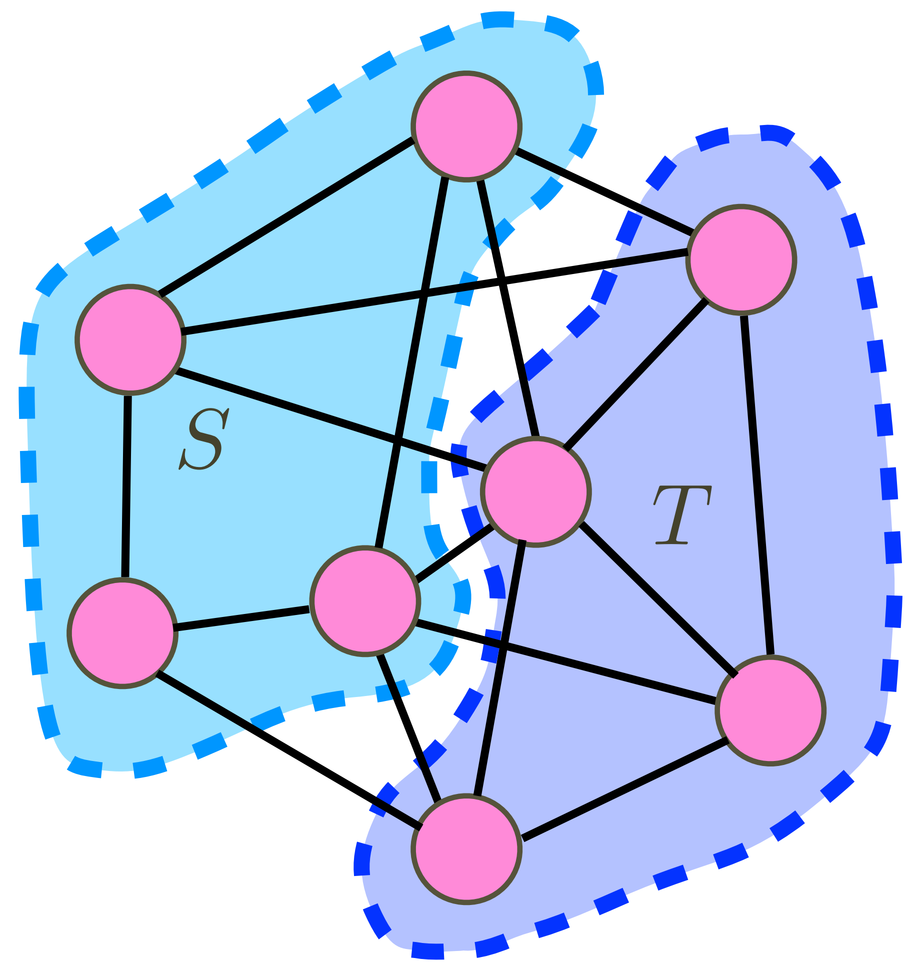
conductance-based

# Maximum Cut

# Max-Cut

- Undirected graph $G(V, E)$
- Bi-partition of $V$ into nonempty $S$ and $T$
- Find a cut $E(S, T)$ of largest size
- **NP-hard:**
  - one of Karp's $21$ NP-complete problems
- Approximation algorithms?

$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$



# Greedy *Heuristics*

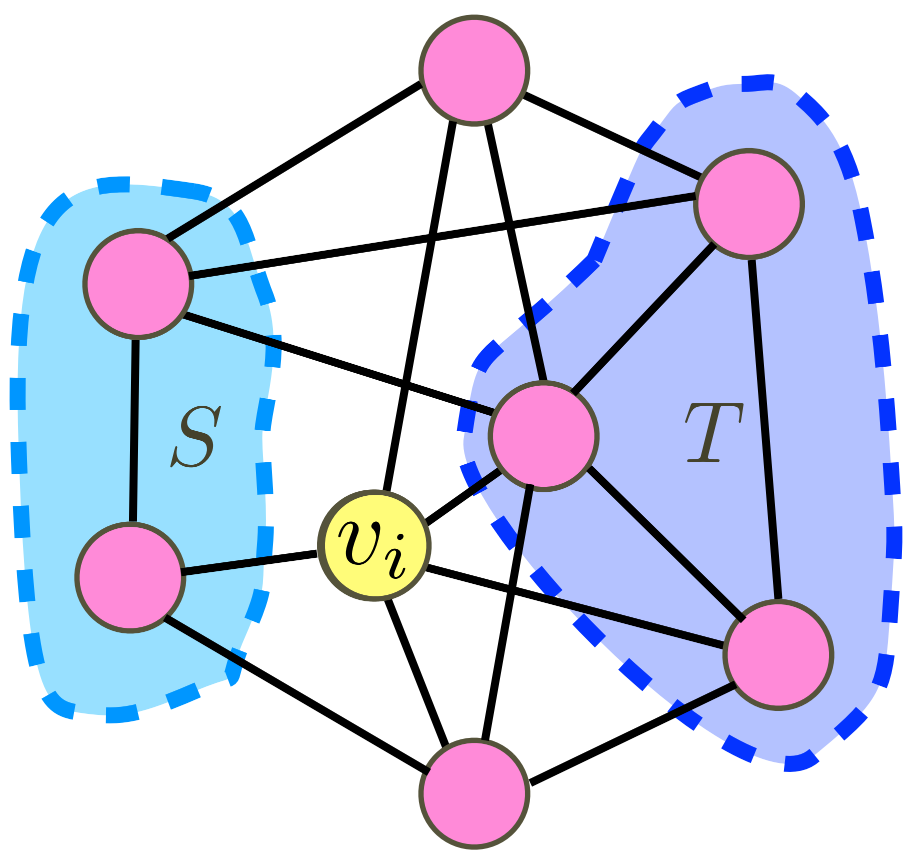
**Greedy Cut:**

initially, $S = T = \emptyset$;

for $i = 1, 2, \ldots, n$:

$v_i$ joins one of $S, T$

to maximize current $E(S, T)$;




$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$

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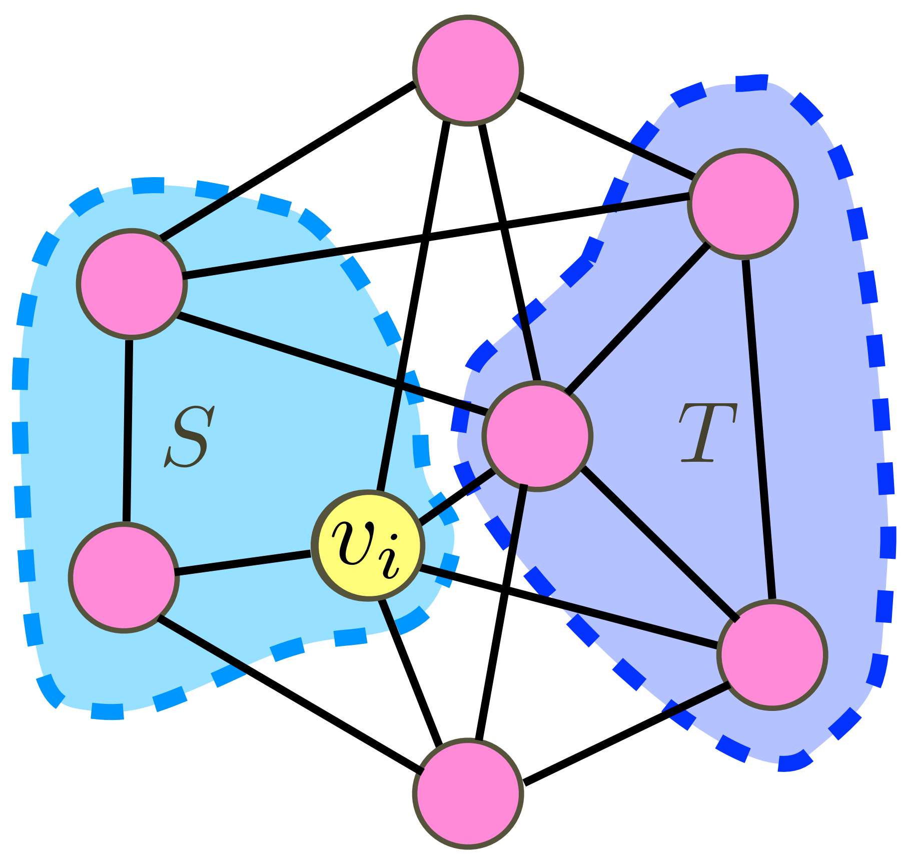


$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$

# Approximation Ratio

Algorithm $\mathscr{A}$:

**Greedy Cut:**

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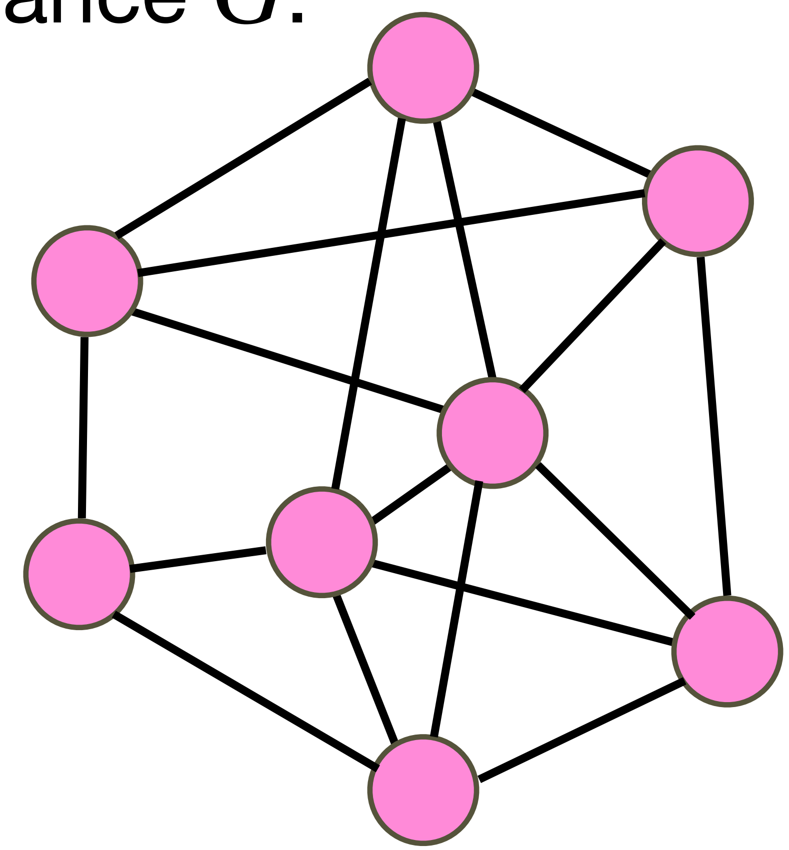
for $i = 1,2,\ldots,n$:

$v_i$ joins one of $S, T$

to maximize current $E(S, T)$;

Instance $G$:



$OPT_G$: value of max-cut in $G$

$SOL_G$: value of the cut returned by $\mathscr{A}$ on $G$

Algorithm $\mathscr{A}$ has approximation ratio $\alpha$ if

$$\forall \text{ instance } G, \quad \frac{SOL_G}{OPT_G} \geq \alpha$$

# Approximation Algorithm

**Greedy Cut:**

initially, $S = T = \varnothing$;

for $i = 1,2,\ldots,n$:

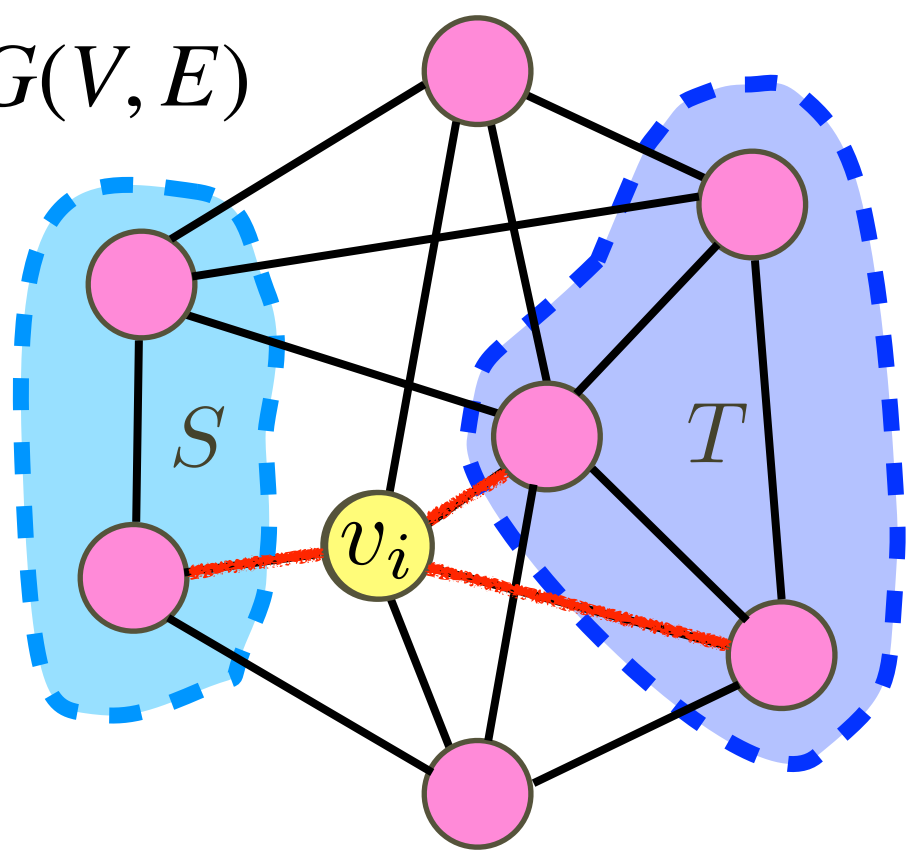
$v_i$ joins one of $S, T$

to maximize current $E(S,T)$;

$(S_i, T_i)$:

current $(S,T)$ in the beginning of $i$-th iteration

$G(V,E)$




$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$

$$\frac{SOL_G}{OPT_G} \ge \frac{SOL_G}{|E|} \ge \frac{1}{2}$$

$\forall v_i$, $\ge 1/2$ of $|E(S_i, v_i)| + |E(T_i, v_i)|$ contributes to $SOL_G$

$$|E| = \sum_{i=1}^{n} \left( |E(S_i, v_i)| + |E(T_i, v_i)| \right)$$

# Approximation Algorithm

**Greedy Cut:**

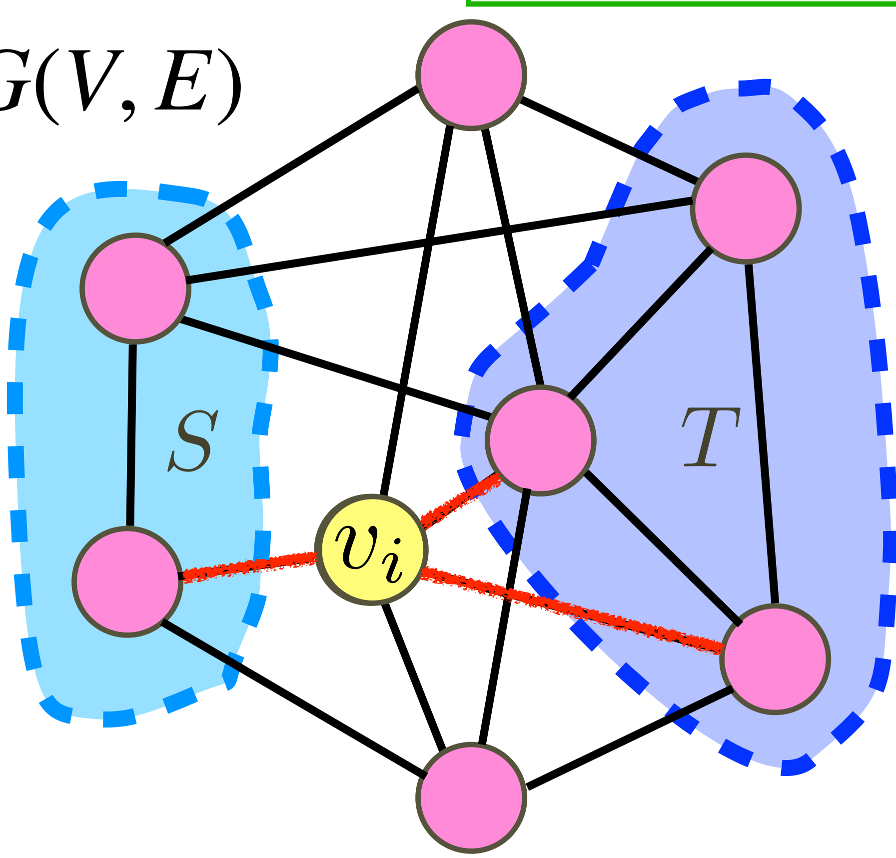
initially, $S = T = \varnothing$;

for $i = 1,2,\ldots,n$:

    $v_i$ joins one of $S, T$

    to maximize current $E(S, T)$;

$G(V, E)$




$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$

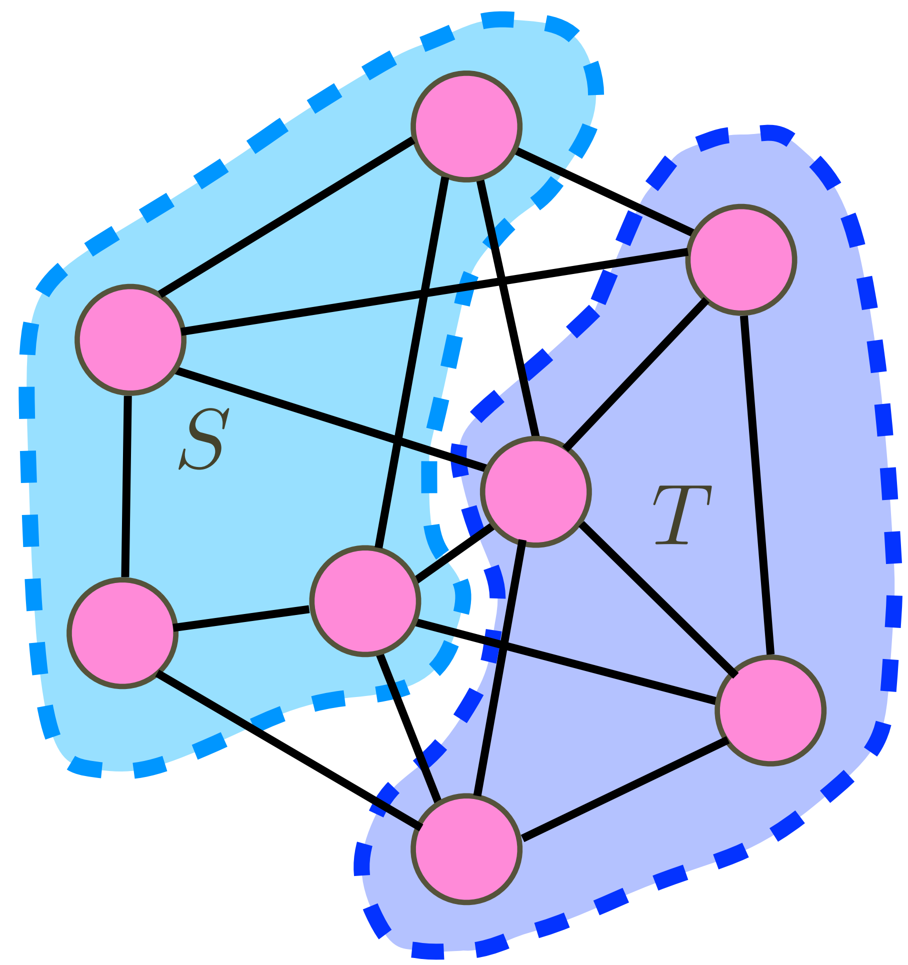
$$\frac{SOL_G}{OPT_G} \geq \frac{SOL_G}{|E|} \geq \frac{1}{2}$$

- **Approximation ratio:** $1/2$
- **Time cost:** $O(m)$

# Max-Cut

- Find a cut $E(S, T)$ of largest size
- **NP-hard:**
  - one of Karp's $21$ NP-complete problems
- Greedy algorithm: $0.5$-approximation

$$E(S,T) = \{uv \in E \mid u \in S, v \in T\}$$



# Random Cut

**Theorem:** For uniform random cut $E(S,T)$ in graph $G(V,E)$,

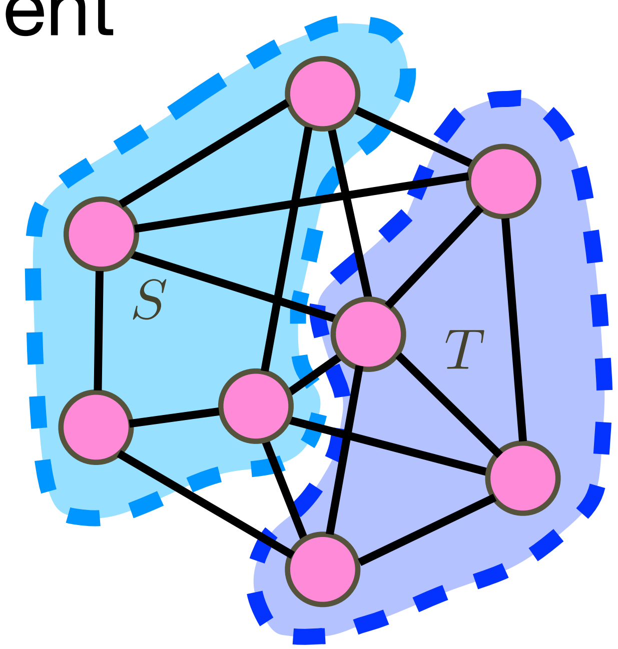
$$\mathbf{E}[\,|E(S,T)|\,] = \frac{|E|}{2}.$$

- $\forall v \in V$: let $X_v \in \{0,1\}$ be uniform & independent

$$X_v = 0 \Longrightarrow v \text{ joins } S$$
$$X_v = 1 \Longrightarrow v \text{ joins } T$$

$$|E(S,T)| = \sum_{uv \in E} I[X_u \neq X_v]$$

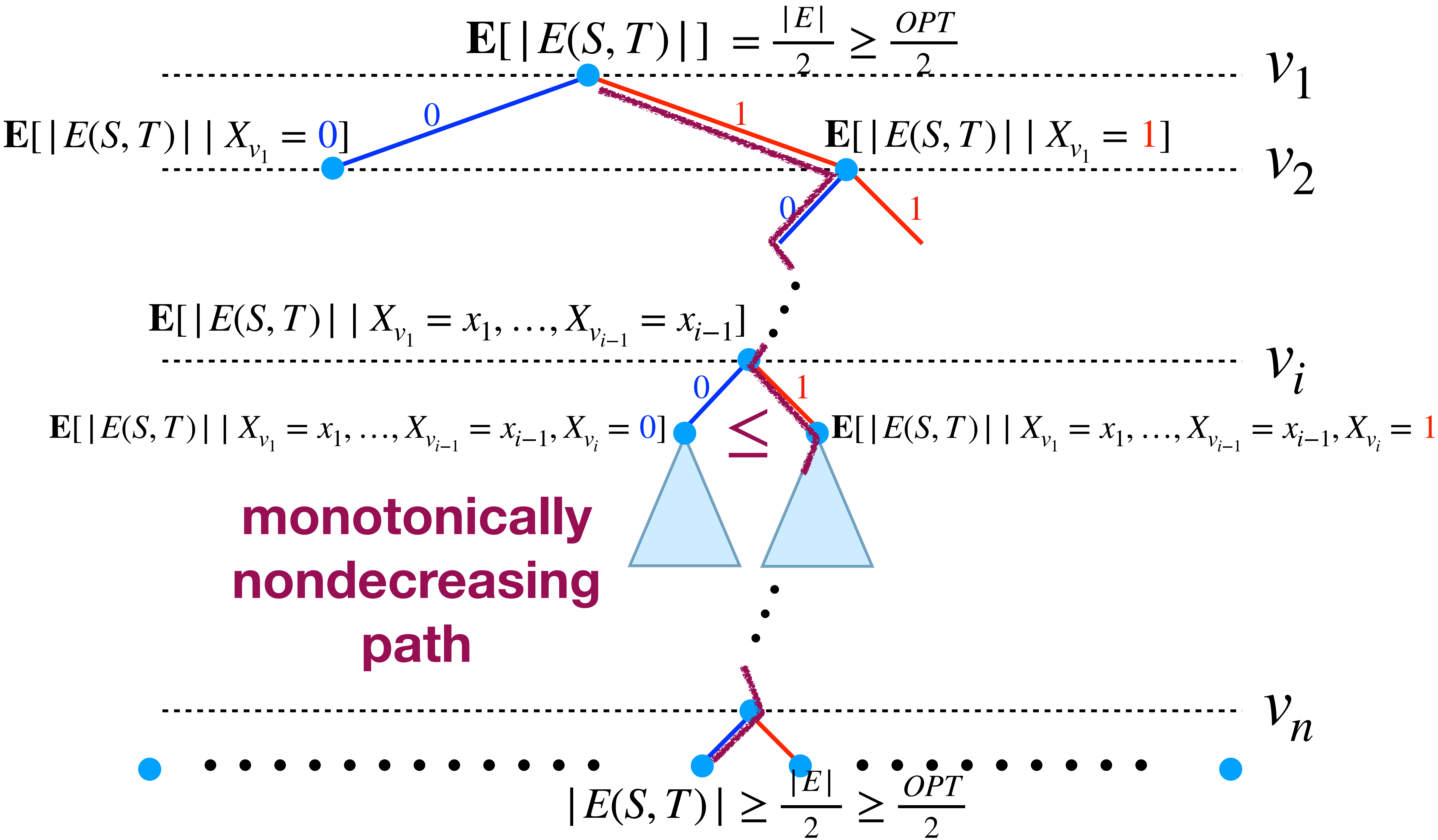
indicator of $X_u \neq X_v$




- Linearity of expectation:

$$\mathbf{E}[\,|E(S,T)|\,] = \sum_{uv \in E} \Pr[X_u \neq X_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

# De-randomization (By Conditional Expectation)




All $2^n$ possible bi-partitions $(S,T)$ of $V$.

# De-randomization (By Conditional Expectation)

**Greedy Cut:**

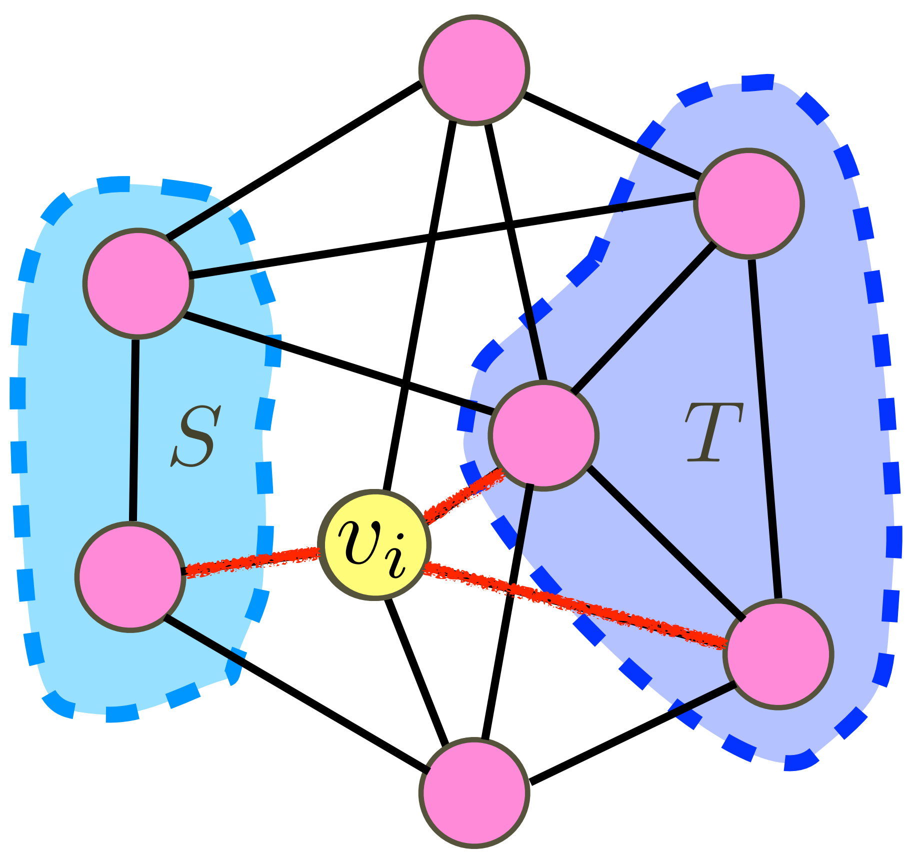
initially, $S = T = \emptyset$;

for $i = 1,2,\ldots,n$:

$v_i$ joins one of $S, T$

to maximize current $E(S, T)$;

with bigger expected cut conditional on current $(S, T)$



# Random Cut

**Theorem:** For uniform random cut $E(S,T)$ in graph $G(V,E)$,

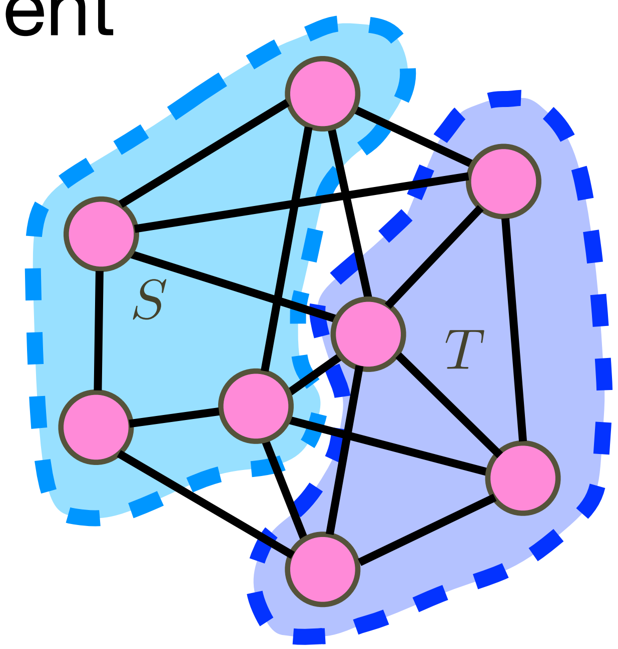
$$\mathbf{E}[\,|E(S,T)|\,] = \frac{|E|}{2}.$$

- $\forall v \in V$: let $X_v \in \{0,1\}$ be uniform & independent

$$X_v = 0 \implies v \text{ joins } S$$
$$X_v = 1 \implies v \text{ joins } T$$

$$|E(S,T)| = \sum_{uv \in E} I[X_u \neq X_v]$$

indicator of $X_u \neq X_v$




- **Linearity of expectation**:

$$\mathbf{E}[\,|E(S,T)|\,] = \sum_{uv \in E} \Pr[X_u \neq X_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

Holds for pairwise independent $X_v$'s!

# Mutual Independence

**Definition** (mutual independence):

Events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ are mutually independent if for any subset $I \subseteq \{1,\ldots,n\}$,

$$\Pr\left[\bigwedge_{i\in I} \mathcal{E}_i\right] = \prod_{i\in I} \Pr\left[\mathcal{E}_i\right]$$

**Definition** (mutual independence of random variables):

Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if for any subset $I \subseteq \{1,\ldots,n\}$ and any values $x_i$, where $i \in I$,

$$\Pr\left[\bigwedge_{i\in I} (X_i = x_i)\right] = \prod_{i\in I} \Pr\left[X_i = x_i\right]$$

# *k*-wise Independence

**Definition** ($k$-wise independence):

Events $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ are mutually independent if for any subset $I \subseteq \{1, \ldots, n\}$ of size at most $k$,

$$\Pr\left[\bigwedge_{i \in I} \mathscr{E}_i\right] = \prod_{i \in I} \Pr\left[\mathscr{E}_i\right]$$

**Definition** ($k$-wise independence of random variables):

Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if for any subset $I \subseteq \{1, \ldots, n\}$ of size at most $k$, and any values $x_i$, where $i \in I$,

$$\Pr\left[\bigwedge_{i \in I} (X_i = x_i)\right] = \prod_{i \in I} \Pr\left[X_i = x_i\right]$$

**Pairwise: 2-wise**

# Pairwise Independent Bits

- Mutually independent uniform random bits: (random source)

$$b_1, \ldots, b_l \in \{0,1\}$$

| a | b | a⊕b |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

Enumerate all nonempty subsets:

$$S_1, \ldots, S_{2^l-1} \subseteq \{1, \ldots, l\}$$

**Parity Construction:**

$$X_j = \bigoplus_{i \in S_j} b_i$$

**Theorem:**

$X_1, \ldots, X_{2^l-1}$ are pairwise independent uniform random bits.

- Pairwise independent uniform random bits: (for $n \gg l$)

$$X_1, \ldots, X_n \in \{0,1\}$$

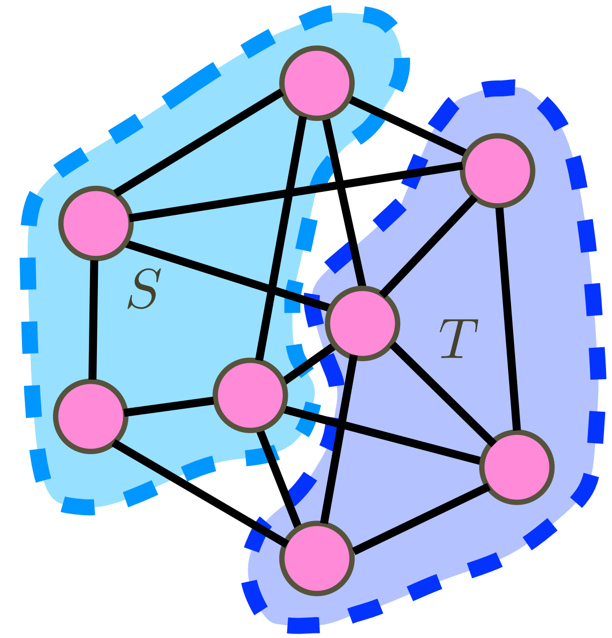
# De-randomization (By Pairwise Independence)

- Pairwise independent uniform $X_v \in \{0,1\}$ for all $v \in V$

$$X_v = 0 \implies v \text{ joins } S$$
$$X_v = 1 \implies v \text{ joins } T$$

$$|E(S,T)| = \sum_{uv \in E} I[X_u \neq X_v]$$

indicator of $X_u \neq X_v$




- Linearity of expectation:

$$\mathbf{E}[\,|E(S,T)|\,] = \sum_{uv \in E} \Pr[X_u \neq X_v] = \frac{|E|}{2} \geq \frac{OPT}{2}$$

**Theorem:** For $(S,T)$ generated from pairwise independent uniform random bits, $\mathbf{E}[\,|E(S,T)|\,] = \frac{|E|}{2}$.

# De-randomization (By Pairwise Independence)

- Let $b_1, \ldots, b_{\lceil \log_2(n+1) \rceil} \in \{0,1\}$ be mutually independent uniform bits.

parity construction

- Pairwise independent uniform $X_v \in \{0,1\}$ for all $v \in V$

$$X_v = 0 \implies v \text{ joins } S$$
$$X_v = 1 \implies v \text{ joins } T$$

**Theorem:** For $(S, T)$ generated from pairwise independent uniform random bits, $\mathbf{E}[\,|E(S,T)|\,] = \frac{|E|}{2}$.

- Enumerate all $b_1, \ldots, b_{\lceil \log_2(n+1) \rceil} \in \{0,1\}$: (only $O(n)$ in total)
  - There must exist an assignment of $b_1, \ldots, b_{\lceil \log_2(n+1) \rceil} \in \{0,1\}$ which corresponds to a cut with $|E(S,T)| \geq \frac{|E|}{2}$.

# De-randomization (By Pairwise Independence)

**Parity Search:**

for all $\boldsymbol{b} \in \{0,1\}^{\lceil \log_2(n+1) \rceil}$:

  initialize $S_{\boldsymbol{b}} = T_{\boldsymbol{b}} = \emptyset$;

  for $i = 1,2,\ldots,n$:

    if $\bigoplus_{j:\lfloor i/2^j \rfloor \bmod 2=1} b_j = 1$ then $v_i$ joins $S_{\boldsymbol{b}}$;

    else $v_i$ joins $T_{\boldsymbol{b}}$;

return the $(S_{\boldsymbol{b}}, T_{\boldsymbol{b}})$ with the largest $|E(S_{\boldsymbol{b}}, T_{\boldsymbol{b}})|$;

- **Approximation ratio:** $1/2$
- **Time cost:** $O(n^2 \log n)$

# Max-Cut

- Find a cut $E(S, T)$ of largest size
- **NP-hard:**
  - one of Karp's $21$ NP-complete problems
- Greedy algorithm: $0.5$-approximation
- Best known approx. ratio for poly-time algorithm: 0.878~
- Unique games conjecture $\Longrightarrow$ computationally hard to do better

$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$