Advanced Algorithms

Lovász Local Lemma

尹一通 Nanjing University, 2022 Fall

k-SAT

- Conjunctive Normal Form (**CNF**):

literals

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee \neg x_4 \vee \neg x_5)$$

clause

Boolean variables: $x_1, x_2, \ldots, x_n \in \{\texttt{True}, \texttt{False}\}$

- k-CNF: each clause contains **exactly** k variables

> **Problem (k-SAT)**
>
> **Input**: k-CNF formula Φ.
>
> **Output**: determine whether Φ is satisfiable.

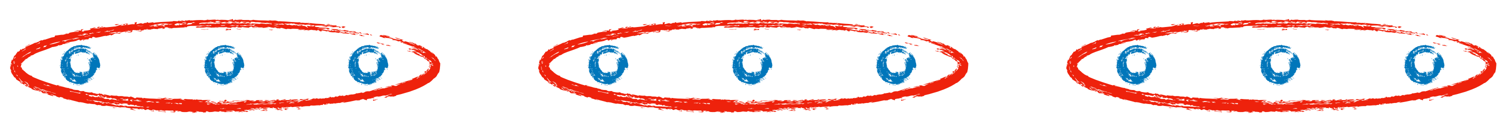
- [Cook-Levin] **NP-hard** if $k \geq 3$

Trivial Cases

Problem (k-SAT)
Input: k-CNF formula Φ.
Output: determine whether Φ is satisfiable.

- Clauses are ***disjoint***:

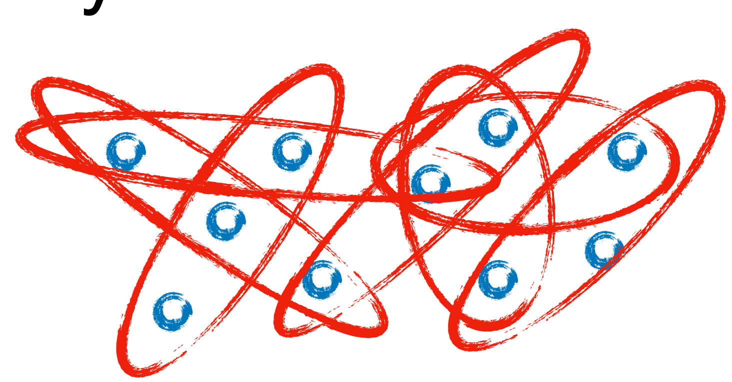
$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee \neg x_8 \vee \neg x_9)$$



resolve each clause ***independently*** (Φ is always satisfiable!)

- $m < 2^k \implies \Phi$ is always satisfiable.

m: # of clauses



The Probabilistic Method

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

k variables

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \underbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}$$

- Draw uniform random $x_1, x_2, \ldots, x_n \in \{\texttt{True}, \texttt{False}\}$

- Bad event A_i : clause C_i is violated

$$\forall 1 \le i \le m, \quad \Pr[A_i] = 2^{-k}$$

- Union bound: $\Pr\left[\bigvee_{i=1}^{m} A_j\right] \le \sum_{i=1}^{m} \Pr[A_i] = m2^{-k}$

$$m < 2^k \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0 \implies \Phi \text{ is satisfiable!}$$

disjoint clauses $\implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] = \prod_{i=1}^{m} \left(1 - \Pr[A_i]\right) > 0$ $\Uparrow$

(independent bad events)

The Probabilistic Method

Problem (k-SAT)

Input: k-CNF formula Φ.

Output: determine whether Φ is satisfiable.

- uniform random $x_1, \ldots, x_n \in \{\mathtt{True}, \mathtt{False}\}$

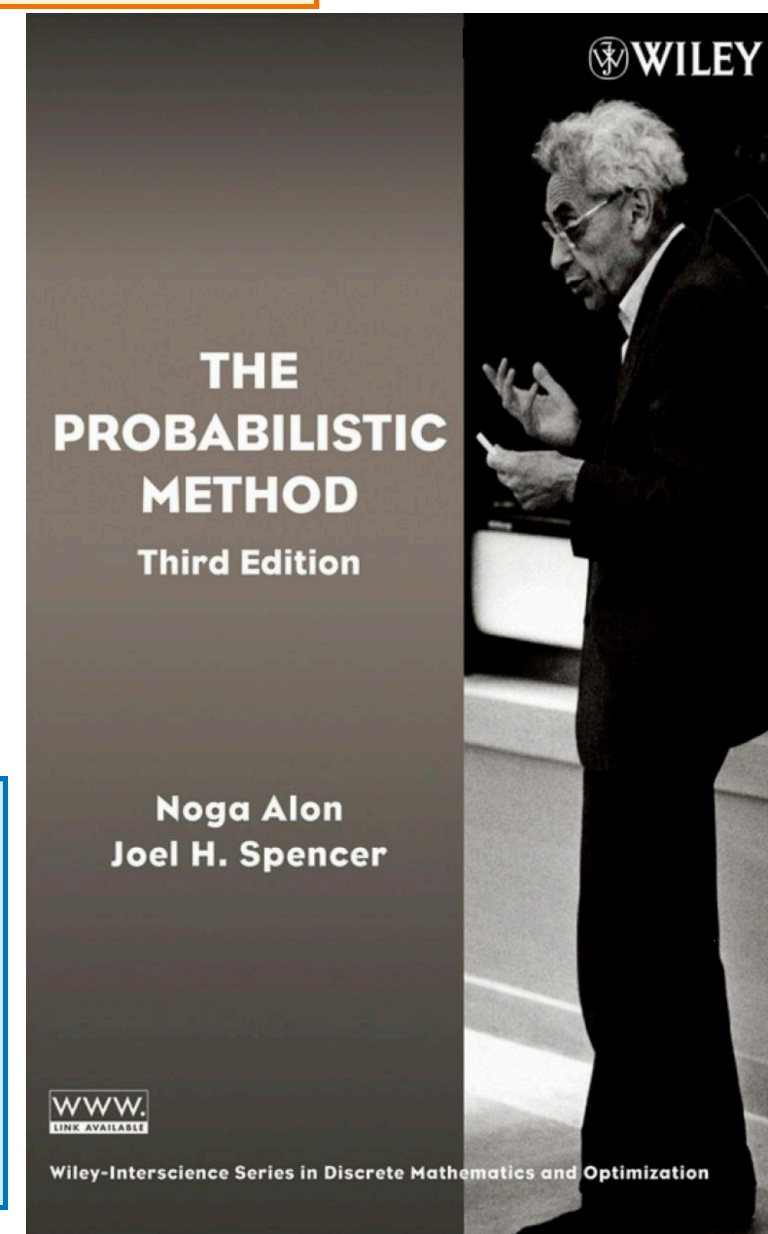
disjoint clauses or $m < 2^k$ $\Bigg\} \implies \Pr[\,\Phi(\boldsymbol{x}) = \mathtt{True}\,] > 0$

$\implies \exists \boldsymbol{x} \in \{\mathtt{T}, \mathtt{F}\}^n \quad \Phi(\boldsymbol{x}) = \mathtt{True}$

The Probabilistic Method:

Draw x from prob. space Ω: for property $\mathscr{P}$,

$$\Pr[\mathscr{P}(x)] > 0 \implies \exists x \in \Omega, \mathscr{P}(x)$$



Limited Dependency

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

k variables

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overbrace{x_3 \vee \neg x_4 \vee \neg x_5})$$

Dependency degree d:

each clause intersects $\leq d$ other clauses

- uniform random $x_1, \ldots, x_n \in \{\mathtt{T}, \mathtt{F}\}$, each clause is violated w.p. 2^{-k}

(union bound) $m2^{-k} < 1$

~~("local" union bound?) $d2^{-k} < 1$~~

(LLL) $\mathrm{e}(d+1)2^{-k} \leq 1$

$4d2^{-k} \leq 1$

$\Longrightarrow \Pr[\text{ no clause is violated }] > 0$

Lovász Local Lemma (LLL)

- "Bad" events $A_1, \ldots, A_m$, where all $\Pr[A_j] \le p$

Dependency degree d:

each A_i is "*dependent*" of $\le d$ other events

Lovász Local Lemma [Lovász and Erdős 1973; Lovász 1977]**:**

$$ep(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

- The "***LLL***" condition:
  - Bad events are not too likely to occur individually.
  - Bad events are not too dependent with each other.

Dependency Graph

- "Bad" events $A_1, \ldots, A_m$,

Dependency degree d:

each A_i is mutually independent of all except $\le d$ other events

Definition (independence):

A is independent of B if $\Pr[A \mid B] = \Pr[A]$ or B is impossible.

A_0 is mutually independent of $A_1, \ldots, A_m$ if A_0 is independent of every event $B = B_1 \wedge \cdots \wedge B_m$, where each $B_i = A_i$ or $\overline{A_i}$.

Dependency Graph

- "Bad" events $A_1, \ldots, A_m$,

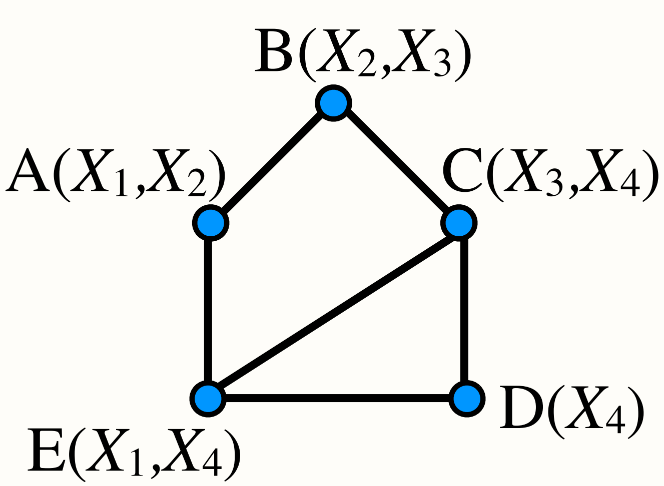
Dependency degree d: (max-degree of dependency graph)

each A_i is **mutually independent** of all except $\leq d$ other events

Dependency graph: $\Gamma(A_i)$: neighborhood of A_i.

Vertices are bad events $A_1, \ldots, A_m$.

Each A_i is mutually independent of non-adjacent events.



independent random variables:

$$X_1, X_2, X_3, X_4$$

bad events (defined on subsets of variables):

$$A(X_1, X_2), B(X_2, X_3), C(X_3, X_4)$$

$$D(X_4), E(X_1, X_4)$$

Lovász Local Lemma (LLL)

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$:

$$A_i \text{ is mutually independent of all } A_j \notin \Gamma(A_i)$$

Lovász Local Lemma:

$p \triangleq \max_i \Pr[A_i]$ and $d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

Constraint Satisfaction Problem (CSP)

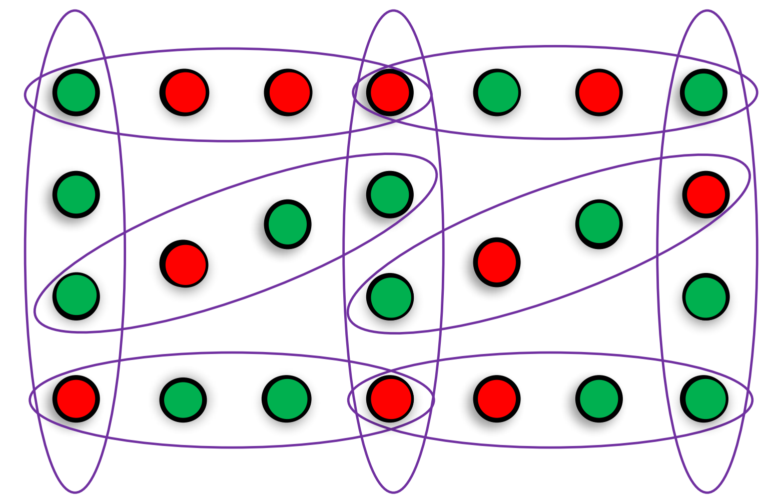
- **Variables:** $x_1, \ldots, x_n \in [q]$
- (local) **Constraints:** $C_1, \ldots, C_m$
  - each C_i is defined on a subset vbl(C_i) of variables

$$C_i : [q]^{\mathsf{vbl}(C_i)} \to \{\mathtt{True}, \mathtt{False}\}$$

- Any $\boldsymbol{x} \in [q]^n$ is a **CSP solution** if it satisfies all $C_1, \ldots, C_m$
- **Examples:**
  - k-CNF, (hyper)graph coloring, set cover, unique games…
  - vertex cover, independent set, matching, perfect matching, …

Hypergraph Coloring

- k-uniform hypergraph $H = (V, E)$:
  - V is vertex set, $E \subseteq \binom{V}{k}$ is set of hyperedges



- degree of vertex $v \in V$: # of hyperedges $e \ni v$
- proper q-coloring of H:
  - $f : V \to [q]$ such that no hyperedge is *monochromatic*

$$\forall e \in E, \quad |f(e)| > 1$$

Theorem: For any k-uniform hypergraph H of max-degree Δ,

$$\Delta \le \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

$k \ge \log_q \Delta + \log_q \log_q \Delta + O(1)$

Hypergraph Coloring

Theorem: For any k-uniform hypergraph H of max-degree Δ,

$$\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

- Uniformly and independently color each $v \in V$ a random color $\in [q]$
- Bad event A_e for each hyperedge $e \in E \subseteq \binom{V}{k}$: e is monochromatic
  - $\Pr[A_e] \leq p = q^{1-k}$
- Dependency degree for bad events $d \leq k(\Delta - 1)$
  - $\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies \mathrm{e}p(d+1) \leq 1$ Apply LLL

Lovász Local Lemma (LLL)

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$

Lovász Local Lemma (symmetric case)**:**

$p \triangleq \max_i \Pr[A_i]$ and $d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

$$\alpha_1 = \cdots = \alpha_m = \frac{1}{d+1}$$

Lovász Local Lemma (asymmetric case)**:**

$$\exists \alpha_1, \ldots, \alpha_m \in [0,1):$$

$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1-\alpha_j) \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] \ge \prod_{i=1}^{m} (1-\alpha_i)$$

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chain rule

$$\Pr\left[\bigwedge_{i=1}^m \overline{A_i}\right] = \prod_{i=1}^m \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right] = \prod_{i=1}^m \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right]\right) \ge \prod_{i=1}^m (1-\alpha_i)$$

Induction Hypothesis (I.H.):

$\forall$ distinct $A_i, A_{j_1}, A_{j_2}, \ldots, A_{j_k}$: $\quad \Pr\left[A_i \mid \overline{A_{j_1}} \cdots \overline{A_{j_k}}\right] \le \alpha_i$

Basis: when $k = 0$, trivial

LLL cond.:

$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$ s.t.

$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

I.H.: $\Pr\left[A_i \mid \overline{A}_{j_1} \cdots \overline{A}_{j_k}\right] \le \alpha_i$ holds for all smaller k

Say $A_{j_1}, \ldots, A_{j_l} \in \Gamma(A_i)$, $A_{j_{l+1}}, \ldots, A_{j_k} \notin \Gamma(A_i)$ ($l \ge 1$ or else trivial)

$$\Pr\left[A_i \mid \overline{A}_{j_1} \cdots \overline{A}_{j_l} \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right] = \frac{\Pr\left[A_i \overline{A}_{j_1} \cdots \overline{A}_{j_l} \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right]}{\Pr\left[\overline{A}_{j_1} \cdots \overline{A}_{j_l} \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right]} \le \alpha_i$$

neighbors | non-neighbors

(numerator) $\le \Pr\left[A_i \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right] = \Pr\left[A_i\right] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$ (LLL cond.)

(denominator) $= \prod_{r=1}^{l} \Pr\left[\overline{A}_{j_r} \mid \overline{A}_{j_{r+1}} \cdots \overline{A}_{j_k}\right] = \prod_{r=1}^{l} \left(1 - \Pr\left[A_{j_r} \mid \overline{A}_{j_{r+1}} \cdots \overline{A}_{j_k}\right]\right)$

(I.H.) $\ge \prod_{r=1}^{l} \left(1 - \alpha_{j_r}\right) \ge \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$

Lovász Local Lemma (asymmetric case)**:**

$$\exists \alpha_1, \ldots, \alpha_m \in [0,1):$$

$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1-\alpha_j) \implies \Pr\left[\bigwedge_{i=1}^m \overline{A_i}\right] \ge \prod_{i=1}^m (1-\alpha_i)$$

chain rule

$$\Pr\left[\bigwedge_{i=1}^m \overline{A_i}\right] = \prod_{i=1}^m \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right] = \prod_{i=1}^m \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right]\right) \ge \prod_{i=1}^m (1-\alpha_i)$$

Induction Hypothesis (I.H.):

$\forall$ distinct $A_i, A_{j_1}, A_{j_2}, \ldots, A_{j_k}$: $\quad \Pr\left[A_i \mid \overline{A_{j_1}} \cdots \overline{A_{j_k}}\right] \le \alpha_i$

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$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j) \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

$$\alpha_1 = \cdots = \alpha_m = \frac{1}{d+1} \qquad \alpha_1 = \cdots = \alpha_m = \frac{1}{2d}$$

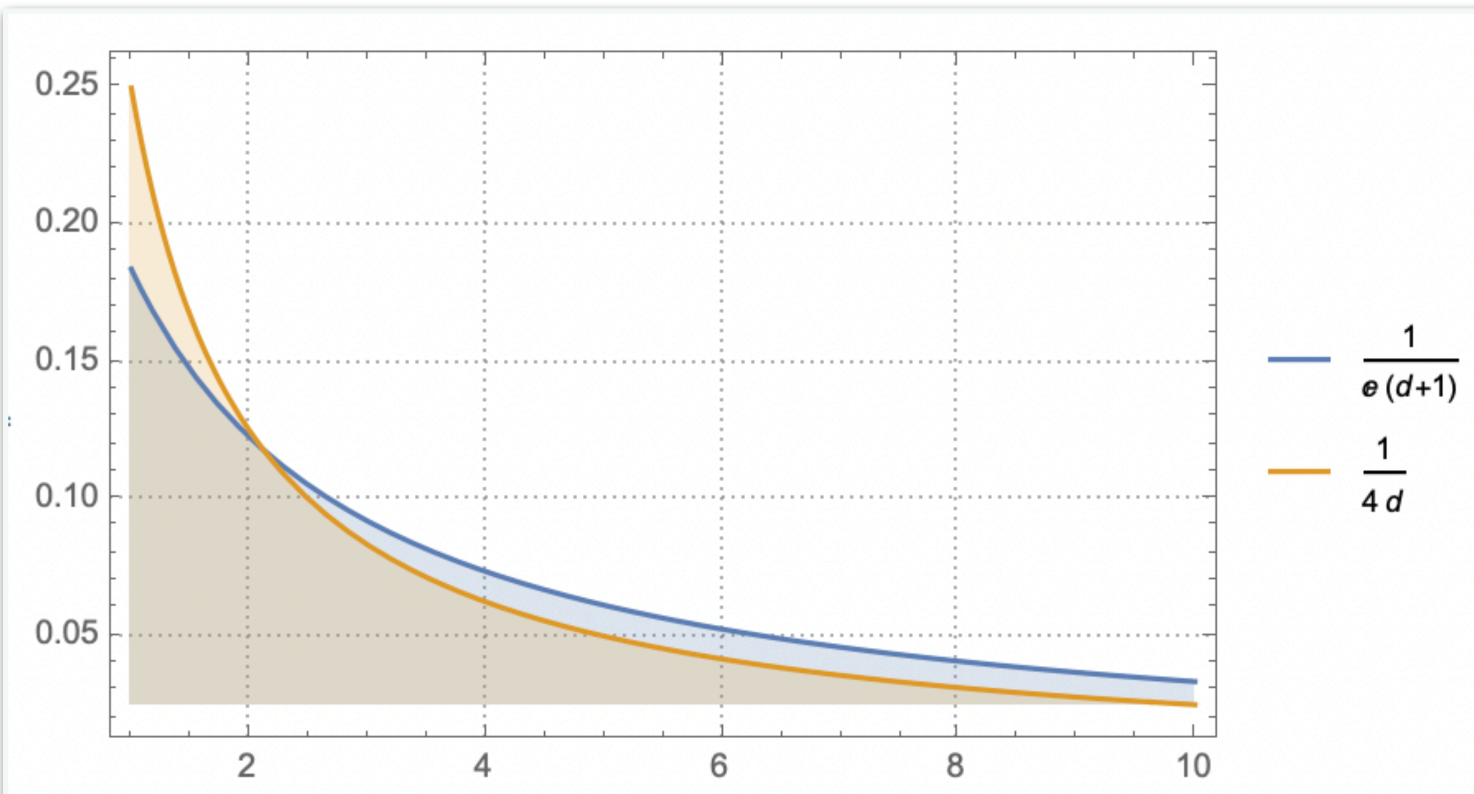
symmetric case:

$$p \triangleq \max_i \Pr[A_i]$$
$$d \triangleq \max_i |\Gamma(i)|$$

$$\left.\begin{array}{r} \mathrm{e}p(d+1) \le 1 \\ \text{or} \\ 4pd \le 1 \end{array}\right\} \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

Lovász Local Lemma (LLL)

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$



case):

$$\Rightarrow \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

symmetric case:

$$p \triangleq \max_i \Pr[A_i]$$

$$d \triangleq \max_i |\Gamma(i)|$$

$$\left.\begin{array}{r} ep(d+1) \le 1 \\ \text{or} \\ 4pd \le 1 \end{array}\right\} \Longrightarrow \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

Summary

- LLL establishes the following implication:

  LLL condition ⇒ It's possible that none of $A_1, \ldots, A_m$ occurs

  knowing only the probabilities and dependency graph of $A_1, \ldots, A_m$

- What's next:
  - tight(er) LLL condition: **Shearer's bound**
  - tighter bounds when more (than just local dependency structure) are known: the probabilistic method beyond LLL
  - beyond existence: **algorithmic/constructive LLL**

Algorithmic Lovász Local Lemma:

The Moser-Tardos Algorithm

Algorithmic LLL (abstract version)

- "Bad" events $A_1, \ldots, A_m$ in a probability space $(\Omega, \Sigma, \Pr)$

Lovász Local Lemma:

$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$:

$$\forall i, \quad \Pr[A_i] \leq \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

⇒

$\exists \sigma \in \Omega$,

avoid all $A_1, \ldots, A_m$

- ***Algorithmic*** (constructive) **Lovász Local Lemma:**

Give an efficient algorithm:

find such a good sample $\sigma \in \Omega$
avoiding all bad events $A_1, \ldots, A_m$

Algorithmic LLL (variable version)

Variable framework for LLL (**CSP with independent variables**):

- mutually independent random variables $\mathscr{X} = \{X_1, \ldots, X_n\}$
- bad events $\mathscr{A} = \{A_1, \ldots, A_m\}$, where $A_i \in \mathscr{A}$ is determined by $\mathsf{vbl}(A_i) \subseteq \mathscr{X}$

- dependency graph is given by neighborhoods $\Gamma(\cdot)$:

$$\Gamma(A_i) = \left\{ A_j \neq A_i \mid \mathsf{vbl}(A_i) \cap \mathsf{vbl}(A_j) \neq \varnothing \right\}$$

- ***Algorithmic*** (constructive) **Lovász Local Lemma:**

Give an efficient algorithm (**CSP solver**):

find such a good evaluation of $X_1, \ldots, X_n$
avoiding all bad events $A_1, \ldots, A_m$

The Moser-Tardos Algorithm

Variable framework for LLL (CSP with independent variables):

- mutually independent random variables $\mathcal{X} = \{X_1, \ldots, X_n\}$
- bad events $\mathcal{A} = \{A_1, \ldots, A_m\}$, where $A_i \in \mathcal{A}$ is determined by $\mathsf{vbl}(A_i) \subseteq \mathcal{X}$

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

resample all $X_j \in \mathsf{vbl}(A_i)$;

Assume oracles for:

- draw ind. samples of X_j
- check if A_i occurs

Theorem [Moser-Tardos 2010]:

$$\exists \alpha_1, \ldots, \alpha_m \in [0,1): \quad \forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

$\Rightarrow$

The Moser-Tardos algorithm terminates within $\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_i}$ resamples in expectation

The Moser-Tardos Algorithm

Variable framework for LLL (**CSP with independent variables**):

- mutually independent random variables $\mathscr{X} = \{X_1, \ldots, X_n\}$
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Assume **oracles** for:

- draw ind. samples of X_j
- check if A_i occurs

Theorem [Moser-Tardos 2010]:

$p \triangleq \max_i \Pr[A_i],\ d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \leq 1$$

⇒ The Moser-Tardos algorithm terminates within m/d resamples in expectation

Execution Log

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

resample all $X_j \in \text{vbl}(A_i)$;

Execution Log (exe-log) Λ of the M-T algorithm:

$$\Lambda_1, \Lambda_2, \Lambda_3, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

random sequence of resampled bad events

Theorem [Moser-Tardos 2010]**:**

$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$:

$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$ $\Rightarrow$ $\forall i, \quad \mathbb{E}_{\Lambda}\left[\#\text{ of } A_i \text{ in } \Lambda\right] \le \dfrac{\alpha_i}{1 - \alpha_i}$

Resampling Table

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

resample all $X_j \in \mathrm{vbl}(A_i)$;

Exe-log Λ:

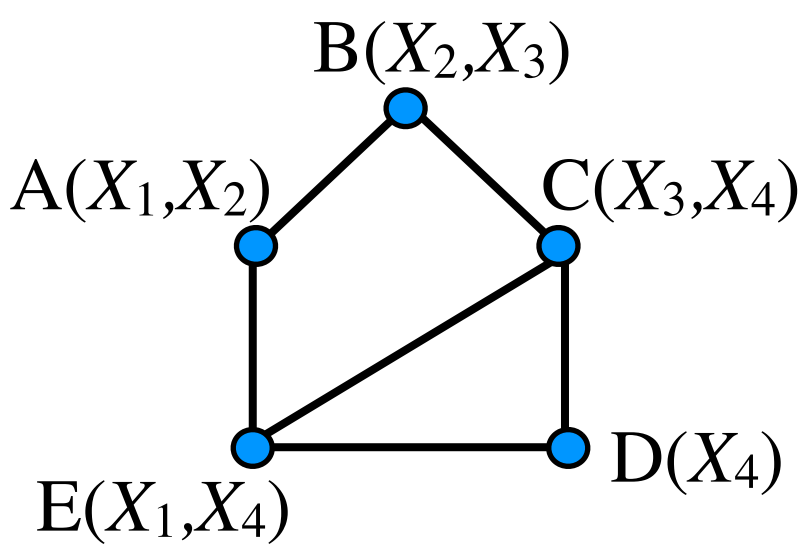
$$\Lambda_1, \Lambda_2, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

random sequence of resampled bad events

Exe-log Λ: D,C,E,D,B,A,C,A,D, ...

Resampling table:

$$X_1 : \quad X_1^{(0)}, X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)}, \ldots$$
$$X_2 : \quad X_2^{(0)}, X_2^{(1)}, X_2^{(2)}, X_2^{(3)}, X_2^{(4)}, \ldots$$
$$X_3 : \quad X_3^{(0)}, X_3^{(1)}, X_3^{(2)}, X_3^{(3)}, X_3^{(4)}, \ldots$$
$$X_4 : \quad X_4^{(0)}, X_4^{(1)}, X_4^{(2)}, X_4^{(3)}, X_4^{(4)}, \ldots$$



$X_j^{(t)}$: t-th sampling of variable X_j

Witness Tree

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;
while $\exists$ a bad event A_i that occurs:
resample all $X_j \in \mathrm{vbl}(A_i)$;

Exe-log Λ:

$$\Lambda_1, \Lambda_2, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

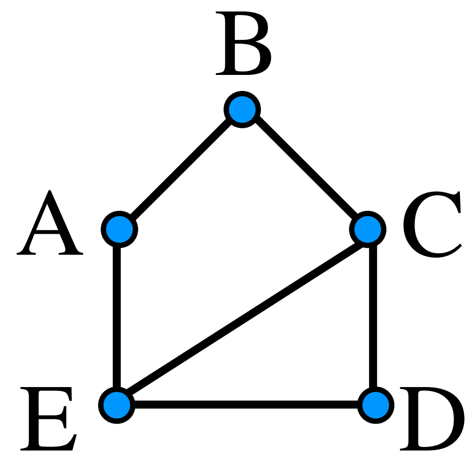
random sequence of resampled bad events

Witness tree $T(\Lambda, t)$: each node u with label $A_{[u]} \in \mathscr{A}$, siblings have *distinct* labels

- initially, T contains a single **root** r with Λ_t
- for $i = t - 1$ to 1:
  if $\Lambda_i \in \Gamma^+(A_{[u]})$ for some node $u \in T$
  add child $v \to$ *deepest* such u, labeled with Λ_i
- $T(\Lambda, t)$ is the resulting T

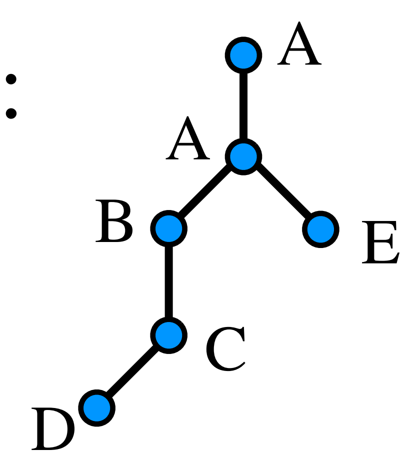
Inclusive neighborhood: $\Gamma^+(A) \triangleq \Gamma(A) \cup \{A\}$

dependency graph:



exe-log Λ: D, C, E, D, B, A, C, A, D, ...

$T(\Lambda, 8)$:



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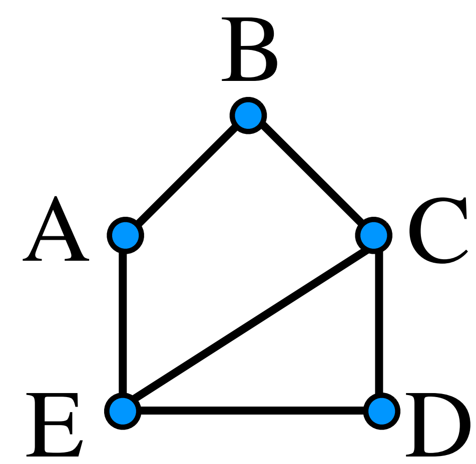

exe-log Λ: D, C, E, D, B, A, C, A, D, ...

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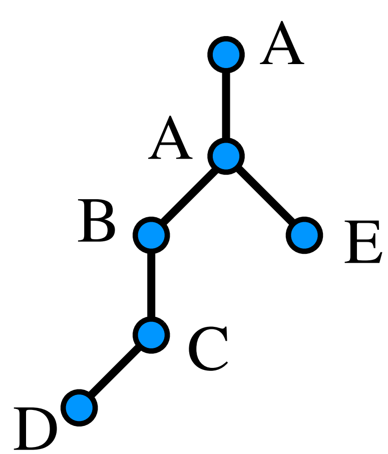
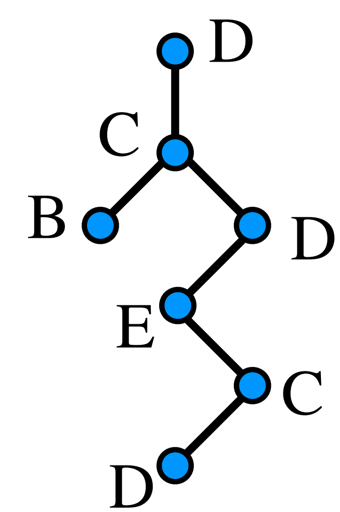


$T(\Lambda, 9)$:



Witness tree $T(\Lambda, t)$: each node u with label $A_{[u]} \in \mathscr{A}$, siblings have *distinct* labels

- initially, T contains a single **root** r with Λ_t
- for $i = t - 1$ to 1:

  if $\Lambda_i \in \Gamma^+(A_{[u]})$ for some node $u \in T$

  add child $v \to$ *deepest* such u, labeled with Λ_i
- $T(\Lambda, t)$ is the resulting T

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

resample all $X_j \in \mathsf{vbl}(A_i)$;

Exe-log Λ:

$$\Lambda_1, \Lambda_2, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

random sequence of resampled bad events

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Proposition: $\forall s \neq t, \quad T(\Lambda, s) \neq T(\Lambda, t)$

$$\text{\# of } A_i \text{ in } \Lambda = \sum_{\tau \in \mathscr{T}_{A_i}} I\left[\exists t, T(\Lambda, t) = \tau\right]$$

$\mathscr{T}_{A_i}$: set of all witness trees with root-label A_i

linearity of expectation:

$$\mathop{\mathbb{E}}_{\Lambda}\left[\text{\# of } A_i \text{ in } \Lambda\right] = \sum_{\tau \in \mathscr{T}_{A_i}} \Pr_{\Lambda}\left[\exists t, T(\Lambda, t) = \tau\right]$$

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Lemma 1 (coupling). For any particular witness tree τ:

$$\Pr_{\Lambda}\left[\,\exists t, T(\Lambda, t) = \tau\,\right] \le \prod_{u \in \tau} \Pr(A_{[u]})$$

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LLL condition: $\exists \alpha_1, \ldots, \alpha_m \in [0,1) : \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$

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Convergence: $$\le \frac{\alpha_i}{1 - \alpha_i}$$

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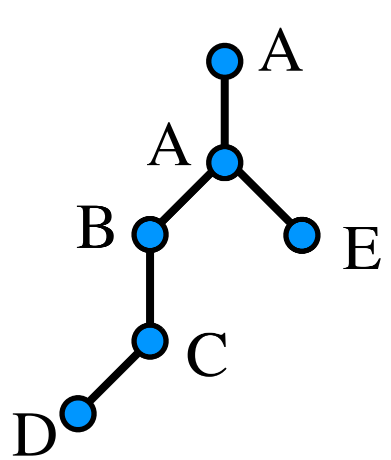
For some **coupling**: $\exists t, T(\Lambda, t) = \tau \implies$ simulation of τ succeeds

Simulation of witness tree τ:

Each node $u \in \tau$ independently does an experiment of its label-event $A_{[u]}$.

The process succeeds if all $A_{[u]}$ occurs.



$$\Pr[\text{ simulation of } \tau \text{ succeeds }]$$

$$= \Pr[D] \cdot \Pr[C] \cdot \Pr[B] \cdot \Pr[E] \cdot \Pr[A] \cdot \Pr[A]$$

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Resampling table:

$$X_1 : \quad X_1^{(0)}, X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)}, \ldots$$

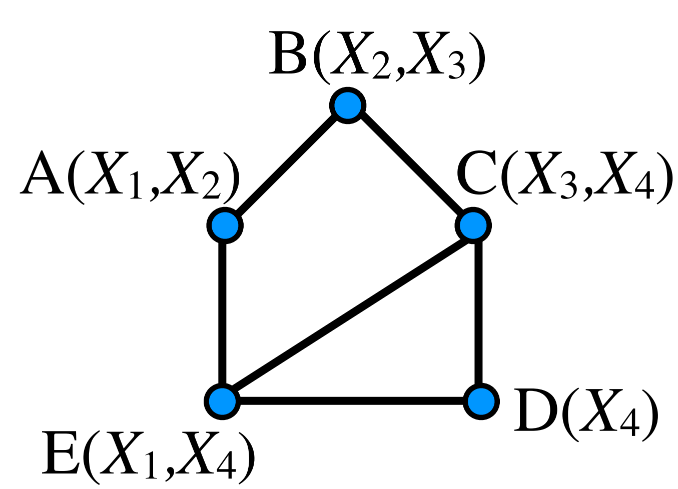
$$X_2 : \quad X_2^{(0)}, X_2^{(1)}, X_2^{(2)}, X_2^{(3)}, X_2^{(4)}, \ldots$$

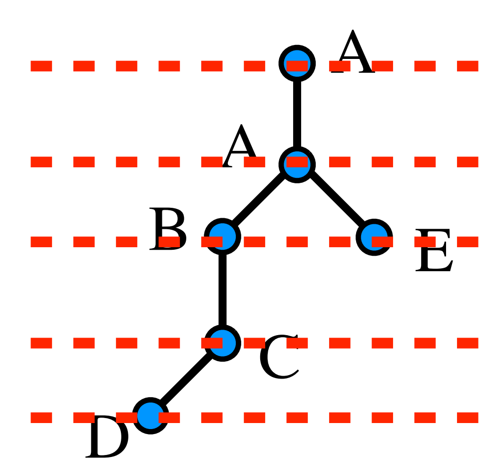
$$X_3 : \quad X_3^{(0)}, X_3^{(1)}, X_3^{(2)}, X_3^{(3)}, X_3^{(4)}, \ldots$$

$$X_4 : \quad X_4^{(0)}, X_4^{(1)}, X_4^{(2)}, X_4^{(3)}, X_4^{(4)}, \ldots$$

$X_j^{(t)}$: t-th sampling of variable X_j

exe-log Λ: D,C,E,D,B,A,C,A,D, ...





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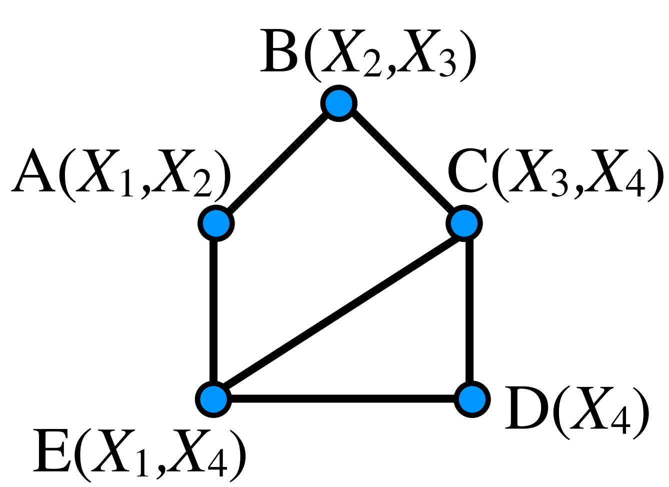
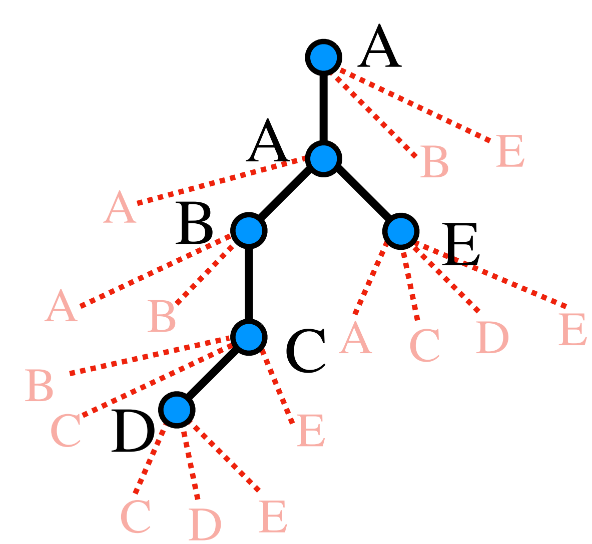
Random Tree (Galton-Watson process)

- Grow a random witness tree T_A with root-label A
  - initially, T_A is a single root with label A
  - for $i = 1, 2, \ldots$:
    - for every vertex u at depth i (root has depth 1) in T_A
    - for every $A_j \in \Gamma^+(A_{[u]})$:
      - add a new child v to u ind. with prob. α_j and label it with A_j;
  - stop if no new child added for an entire level
- Can generate all possible witness trees $\in \mathscr{T}_A$





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Lemma 2. For any particular witness tree $\tau \in \mathcal{T}_{A_i}$:

$$\Pr\left[T_{A_i} = \tau\right] = \frac{1-\alpha_i}{\alpha_i} \prod_{u \in \tau} \left[\alpha_{[u]} \prod_{A_j \in \Gamma(A_{[u]})} (1-\alpha_j)\right]$$

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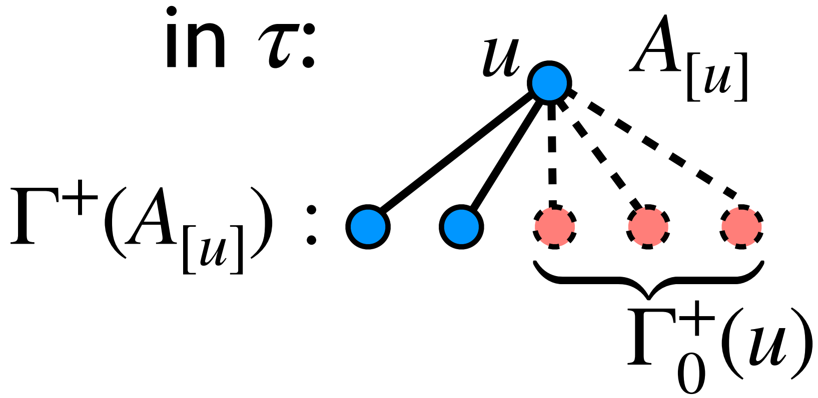
in τ:



$$\Pr\left[T_{A_i} = \tau\right] = \frac{1}{\alpha_i} \prod_{u \in \tau} \left[\alpha_{[u]} \prod_{A_j \in \Gamma^+_0(u)} (1-\alpha_j) \right] = \frac{1-\alpha_i}{\alpha_i} \prod_{u \in \tau} \left[\frac{\alpha_{[u]}}{1-\alpha_{[u]}} \prod_{A_j \in \Gamma^+(A_{[u]})} (1-\alpha_j) \right]$$

$$= \frac{1-\alpha_i}{\alpha_i} \prod_{u \in \tau} \left[\alpha_{[u]} \prod_{A_j \in \Gamma(A_{[u]})} (1-\alpha_j) \right]$$

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

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The Moser-Tardos Algorithm

Variable framework for LLL (**CSP with independent variables**):

- mutually independent random variables $\mathcal{X} = \{X_1, \ldots, X_n\}$
- bad events $\mathcal{A} = \{A_1, \ldots, A_m\}$, where $A_i \in \mathcal{A}$ is determined by $\mathsf{vbl}(A_i) \subseteq \mathcal{X}$

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

while $\exists$ a bad event A_i that occurs:

    resample all $X_j \in \mathsf{vbl}(A_i)$;

Theorem [Moser-Tardos 2010]**:**

$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$:

$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

$\Rightarrow$

The Moser-Tardos algorithm terminates within $\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_i}$ resamples in expectation

Algorithmic
Lovász Local Lemma:

Moser's Algorithm and Entropic Proof

k-SAT

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

  k variables

  $$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \underbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}$$

- dependency degree d: each clause C_i intersects $\leq d$ other clauses
- uniform independent

  $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$
- bad event A_i: $p = 2^{-k}$

  C_i is violated

Moser-Tardos Algorithm:

draw uniform independent $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

while $\exists$ a violated clause C_i :

  resample all $X_j \in \mathsf{vbl}(C_i)$;

Theorem [Moser-Tardos 2010]**:**

$$d \leq 2^{k-2} \Leftrightarrow 4pd \leq 1$$

⇒ The Moser-Tardos algorithm terminates after $O(m/d)$ iterations in expectation

Moser's *Fix-It* Algorithm

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

k variables

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \underbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}$$

Moser's Algorithm:

draw uniform $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

while $\exists$ a violated clause C_i :

  Fix(C_i);

Fix(C_i)**:**

resample all variables in vbl(C_i);

while $\exists$ violated $C_j \in \Gamma^+(C_i)$:

  Fix(C_j);

- Inclusive neighborhood:

$$\Gamma^+(C_i) \triangleq \Gamma(C_i) \cup \{C_i\} = \left\{ C_j \mid \text{vbl}(C_j) \cap \text{vbl}(C_i) \neq \varnothing \right\}$$

- **Correctness**: any clause is fixed at most once in top-level

top-level **Fix**(C_i) returned $\Longrightarrow$ C_i remains satisfied

Moser's *Fix-It* Algorithm

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

k variables

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_3 \vee \neg x_4 \vee \neg x_5)$$

Moser's Algorithm:

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Theorem [Moser 2009]**:**

$d < 2^{k-3} \Longrightarrow$ total # of calls to Fix() is $O(m \log m + \log n)$
with high probability

with probability 1-O(1/n)

Incompressibility Principle

"**Lossless compression of random data is impossible.**"

Incompressibility Principle:

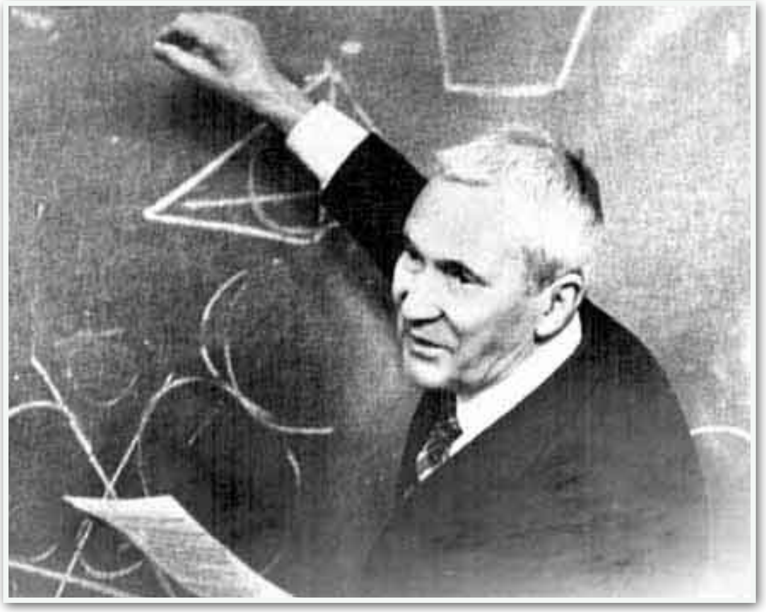
For any injective function $\texttt{Enc} : \{0,1\}^N \xrightarrow{\text{1-1}} \{0,1\}^*$, for uniform random $s \in \{0,1\}^N$, for any integer $l > 0$,

$$\Pr\left[\text{length of } \texttt{Enc}(s) \le N - l\right] < 2^{1-l}$$

$$\left|\{0,1\}^N\right| = 2^N$$

\# of strings of $\le N - l$ bits

$$= \sum_{i \le N-l} 2^i < 2^{N-l+1}$$



Andrey Kolmogorov
(1903–1987)

Entropic Proof

- k-CNF $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ with dependency degree $d < 2^{k-3}$

Moser's Algorithm:

draw uniform $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

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  Fix(C_i);

Fix(C_i)**:**

resample all variables in vbl(C_i);

while $\exists$ violated $C_j \in \Gamma^+(C_i)$:

  Fix(C_j);

- **Random bits**: n initial bits + t calls to **Fix**() $\times$ k bits per each call

- t calls to **Fix**() $\Longrightarrow$ we can compress random bits:

$n + tk$ random bits → $\mathtt{Enc}_\Phi$ → $\leq n + O(m \log m) + t(\lceil \log_2(d+1) \rceil + 2)$ bits

Simulate(Φ, t):

Run Moser's Algorithm on k-CNF Φ for up to t calls to Fix();

printf("$X_1 \ldots X_n$"); `//the current` $X_1, \ldots, X_n$ `in Moser's algorithm`

Moser's Algorithm:

draw uniform $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

while $\exists$ a violated clause C_i :

    printf("(i");

    Fix(C_i);

Fix(C_i):

resample all variables in vbl(C_i);

while $\exists$ violated $C_j \in \Gamma^+(C_i)$:

    let r = rank of C_j in $\Gamma^+(C_i)$;

    printf("(r") and **Fix**(C_j);

printf(")")

- **Output string**:

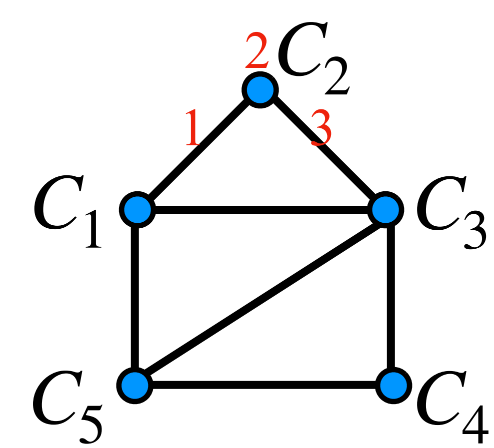
  (98(2(1(1)(3))(3))(4(2)))(126(3(2⋯

  ⋯⋯⋯(11010010101010110111

  t calls to Fix() $X_1, \ldots, X_n$

all $C_j \in \Gamma^+(C_i)$ are sorted in an order

$|\Gamma^+(C_i)| \leq d + 1$



- **Recursion trees** + **final assignment**

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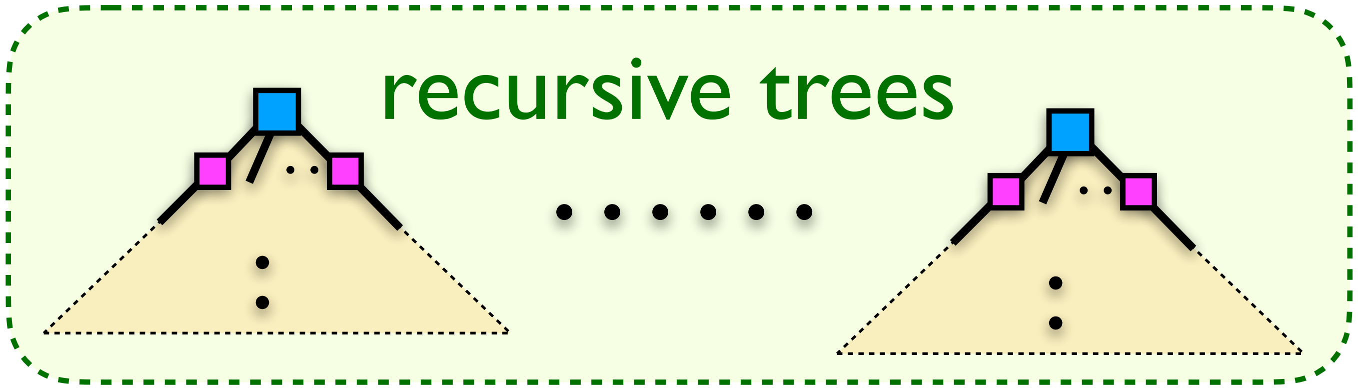


+

final assignment
$X_1, \ldots X_n$

Observation: Fix(C_i) is called $\Longrightarrow$ C_i is violated at the moment

$\Longrightarrow$ current values of all $X_j \in$ vbl(C_i) is uniquely determined

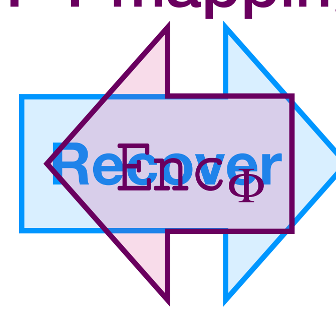
- **Output string**:

(98(2(1(1)(3))(3))(4(2)))(126(3(2···

·········(11010010101010110111

t calls to Fix() $X_1, \ldots, X_n$

1-1 mapping



$n + tk$ random bits used by the algorithm

- **Recursion trees** + **final assignment**

Simulate(Φ, t):

Run Moser's Algorithm on k-CNF Φ for up to t calls to Fix();

printf("$X_1 \ldots X_n$");

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printf("(r") and **Fix**(C_j);

printf(")")

- **Output string**:
  - $\leq m \times$ "(i", where $i \in [m]$ at top-level
  - $\leq t \times$ "(r", where $r \in [d+1]$
  - $\leq t \times$ ")"
  - "(" ")" form prefix of legal parenthesization
  - n bits $X_1, \ldots, X_n$ in the end

$$\leq n + m(\lceil \log_2 m \rceil + 2) + t(\lceil \log_2 (d+1) \rceil + 2) \text{ bits}$$

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For any injective function $\mathtt{Enc} : \{0,1\}^N \xrightarrow{\text{1-1}} \{0,1\}^*$, for uniform random $\boldsymbol{s} \in \{0,1\}^N$, for any integer $l > 0$,

$$\Pr\left[\text{length of } \mathtt{Enc}(\boldsymbol{s}) \leq N - l\right] < 2^{1-l}$$

- Assume $\geq t$ calls made to **Fix()**:

$n + tk$ random bits used by the algorithm $\Rightarrow \mathtt{Enc}_\Phi \Rightarrow$ $\leq n + m(\lceil \log_2 m \rceil + 2) + t(\lceil \log_2 (d+1) \rceil + 2)$ bits

(LLL condition $d < 2^{k-3}$) $\leq n + m(\log_2 m + 3) + t(k-1)$

- For any $t \geq m(\log_2 m + 3) + \log_2 n + 1$:

this can only occur with probability $< \dfrac{1}{n}$

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k variables

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with high probability

with probability 1-O(1/n)

Lovász Local Lemma:

The Probabilistic Method

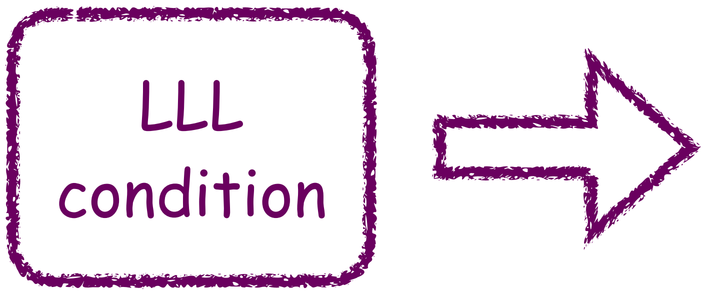
Algorithmic Lovász Local Lemma:

Moser-Tardos Algorithm

Moser's Algorithm and Entropic Proof

Summary

- The **Moser-Tardos** algorithm:



Efficiently find $X_1, \ldots, X_n$
that avoids all $A_1, \ldots, A_m$

$A_1, \ldots, A_m$ are defined on independent random variables $X_1, \ldots, X_n$

- The proof based on ***resampling table*** and ***witness trees***

- What's next:
  - tighter LLL condition for algorithms
  - beyond independent variables: **non-variable framework**
  - more than construction: **sampling LLL**