Advanced Algorithms (Fall 2024)
Linear Programming Duality

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Outline

1. Duality of Linear Programming
   - Linear Programming Duality
2. Examples
   - Max-Flow Min-Cut Theorem Using LP Duality
   - 0-Sum Game and Nash Equilibrium

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1. Duality of Linear Programming
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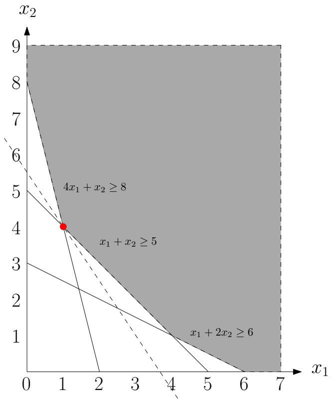
$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Q: How can we prove a lower bound for the value?

- $7x_1 + 4x_2 \geq 2(x_1 + x_2) + (x_1 + 2x_2) \geq 2 \times 5 + 6 = 16$
- $7x_1 + 4x_2 \geq (x_1 + x_2) + (x_1 + 2x_2) + (4x_1 + x_2) \geq 5 + 6 + 8 = 19$
- $7x_1 + 4x_2 \geq 4(x_1 + x_2) \geq 4 \times 5 = 20$
- $7x_1 + 4x_2 \geq 3(x_1 + x_2) + (4x_1 + x_2) \geq 3 \times 5 + 8 = 23$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

A way to prove lower bound on the value of primal LP

$$\begin{aligned} & 7x_1 + 4x_2 && (\text{if } 7 \geq y_1 + y_2 + 4y_3 \text{ and } 4 \geq y_1 + 2y_2 + y_3) \\ \geq\ & y_1(x_1 + x_2) + y_2(x_1 + 2x_2) + y_3(4x_1 + x_2) && (\text{if } y_1, y_2, y_3 \geq 0) \\ \geq\ & 5y_1 + 6y_2 + 8y_3. \end{aligned}$$

- Goal: need to maximize $5y_1 + 6y_2 + 8y_3$

Primal LP

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & 5y_1 + 6y_2 + 8y_3 \\ & y_1 + y_2 + 4y_3 \leq 7 \\ & y_1 + 2y_2 + y_3 \leq 4 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 4 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 6 \\ 8 \end{pmatrix} \quad c = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$\min \quad c^T x \qquad \text{s.t.}$$

$$Ax \geq b$$

$$x \geq 0$$

$$\max \quad b^T y \qquad \text{s.t.}$$

$$A^T y \leq c$$

$$y \geq 0$$

Primal LP

$$\min \quad c^T x \quad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \quad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

- $P =$ value of primal LP
- $D =$ value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Theorem (strong duality theorem) $D = P$.

- Can always prove the optimality of the primal solution, by adding up primal constraints.

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

- $P =$ value of primal LP
- $D =$ value of dual LP

Theorem (weak duality theorem) $D \leq P$.

Proof.

- x^*: optimal primal solution
- y^*: optimal dual solution

$$D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P. \qquad \square$$

Fact If a point x does not belong to a polytope $\mathcal{P}$, then there is a hyperplane separating x and $\mathcal{P}$.

Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Proof.

- b does not belong to $\{Ax : x \geq 0\}$, so $\exists$ some hyperplane separating b and $\{Ax : x \geq 0\}$.
- $y^{\mathrm{T}}b < g$ and $y^{\mathrm{T}}Ax > g$ for every $x \geq 0$
- $g < 0$ and $y^{\mathrm{T}}A \geq 0$
- $y^{\mathrm{T}}b < g < 0$ $\square$

Lemma (Farkas Lemma) $Ax = b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0$ is feasible.

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0, y \geq 0$ is feasible.

Proof.

- system equivalent to $Ax + x' = b, x, x' \geq 0$

$$(A, I)\begin{pmatrix} x \\ x' \end{pmatrix} = b, \qquad \begin{pmatrix} x \\ x' \end{pmatrix} \geq 0$$

- By Farkas Lemma, $\exists y$ such that $y^{\mathrm{T}}(A, I) \geq 0, y^{\mathrm{T}}b < 0$
- $\iff y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}} \geq 0, y^{\mathrm{T}}b < 0$ $\square$

Primal LP

$$\min \quad c^T x \qquad \text{s.t.}$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^T y \qquad \text{s.t.}$$
$$A^T y \leq c$$
$$y \geq 0$$

Lemma (Variant of Farkas Lemma) $Ax \leq b, x \geq 0$ is infeasible, if and only if $y^{\mathrm{T}}A \geq 0, y^{\mathrm{T}}b < 0, y \geq 0$ is feasible.

Proof of Strong Duality Theorem

- $\forall \epsilon > 0$, $\begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} x \leq \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix}, x \geq 0$ is infeasible
- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$,

  $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- we can prove $\alpha > 0$, since the primal LP is feasible.

Proof of Strong Duality Theorem

- There exists $y \in \mathbb{R}^m_{\geq 0}, \alpha \geq 0$, such that $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -A \\ c^{\mathrm{T}} \end{pmatrix} \geq 0$, $(y^{\mathrm{T}}, \alpha) \begin{pmatrix} -b \\ P - \epsilon \end{pmatrix} < 0$
- assume $\alpha = 1$
- $-y^{\mathrm{T}}A + c^{\mathrm{T}} \geq 0, -y^{\mathrm{T}}b + P - \epsilon < 0 \Longleftrightarrow A^{\mathrm{T}}y \leq c, b^{\mathrm{T}}y > P - \epsilon$
- $\forall \epsilon > 0, D > P - \epsilon \quad \Longrightarrow \quad D = P$ (since $D \leq P$) $\square$

Primal LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}}x \\ Ax &\geq b \\ x &\geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}}y \\ A^{\mathrm{T}}y &\leq c \\ y &\geq 0 \end{aligned}$$

Relationships

Primal LP	dual LP
variables	constraints
constraints	variables
obj. coefficients	RHS constants
RHS constants	obj. coefficients

More Relationships

Primal LP	Dual LP
variable in $\mathbb{R}$	equlities
equlities	variable in $\mathbb{R}$

- duality is mutual: the dual of the dual of an LP is the LP itself.

Primal LP

$$\begin{aligned} \max \quad & b^{\mathrm{T}} y \\ A^{\mathrm{T}} y & \leq c \\ y & \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min \quad & c^{\mathrm{T}} x \\ A x & \geq b \\ x & \geq 0 \end{aligned}$$

- Duality theorem holds when one LP is infeasible:
- Minimization LP is infeasible $\implies$ value $= \infty$ $\iff$ dual LP value $= \infty$ $\implies$ feasible region of dual LP is unbounded

Complementary Slackness

Primal LP

$$\min \quad c^{\mathrm{T}} x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual LP

$$\max \quad b^{\mathrm{T}} y$$
$$A^{\mathrm{T}} y \leq c$$
$$y \geq 0$$

- x^* and y^*: optimum primal and dual solutions
- $D = b^{\mathrm{T}} y^* \leq (Ax^*)^{\mathrm{T}} y^* = (x^*)^{\mathrm{T}} A^{\mathrm{T}} y^* \leq (x^*)^{\mathrm{T}} c = c^{\mathrm{T}} x^* = P$.
- $P = D$: all the inequlaities hold with equalities.

Complementary Slackness

- $y_i^* > 0 \implies \sum_j a_{ij} x_j^* = b_i$.
- $x_j^* > 0 \implies \sum_i a_{ij} y_i^* = c_j$.

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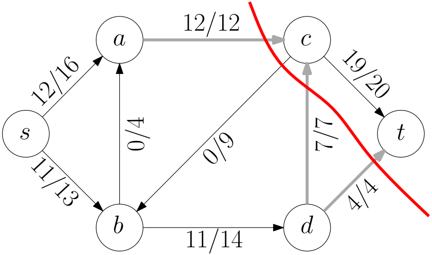
Maximum Flow Problem

Input: flow network $(G = (V, E), c, s, t)$

Output: maximum value of a s-t flow f



LP for Maximum Flow

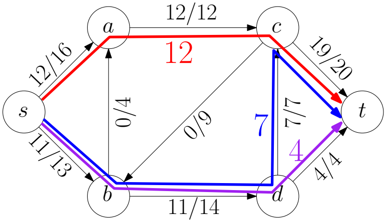
$$\max \sum_{e \in \delta^{\text{in}}(t)} x_e$$

$$x_e \leq c_e \qquad \forall e \in E$$

$$\sum_{e \in \delta^{\text{out}}(v)} x_e - \sum_{e \in \delta^{\text{in}}(v)} x_e = 0 \qquad \forall v \in V \setminus \{s, t\}$$

$$x_e \geq 0 \qquad \forall e \in E$$

An Equivalent Packing LP



- $\mathcal{P}$: the set of all simple paths from s to t
- $f_P, P \in \mathcal{P}$: the flow on P

$$\max \quad \sum_{P \in \mathcal{P}} f_P \qquad\qquad \min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{P \in \mathcal{P}: e \in P} f_P \le c_e \quad \forall e \in E \qquad\qquad \sum_{e \in P} y_e \ge 1 \quad \forall P \in \mathcal{P}$$

$$f_P \ge 0 \quad \forall P \in \mathcal{P} \qquad\qquad y_e \ge 0 \quad \forall e \in E$$

- dual constraints: the shortest s-t path w.r.t weights y has length ≥ 1

Dual LP

$$\min \quad \sum_{e \in E} c_e y_e$$

$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$

$$y_e \geq 0 \qquad \forall e \in E$$

Theorem The optimum value can be attained at an integral point y.

Maximum Flow Minimum Cut Theorem The value of the maximum flow equals the value of the minimum cut.

Proof of Theorem.

- Given any optimum y, let d_v be the length of shortest path from s to v, for every $v \in V$. $\quad d_s = 0, d_t = 1$
- Randomly choose $\theta \in (0, 1)$, and output cut $(S := \{v : d_v \leq \theta\}, T := \{v : d_v > \theta\})$
- Lemma: $\mathbb{E}[\text{cut value of}(S, T)] \leq \sum_{e \in E} c_e y_e$
- Any cut (S, T) in the support is optimum $\square$

$$\max \sum_{P \in \mathcal{P}} f_P$$
$$\sum_{P \in \mathcal{P}: e \in P} f_P \leq c_e \qquad \forall e \in E$$
$$f_P \geq 0 \qquad \forall P \in \mathcal{P}$$

$$\min \sum_{e \in E} c_e y_e$$
$$\sum_{e \in P} y_e \geq 1 \qquad \forall P \in \mathcal{P}$$
$$y_e \geq 0 \qquad \forall e \in E$$

- pros of new LP: it is a packing LP, dual is a covering LP, easier to understand and analyze
- cons of new LP: exponential size, can not be solved directly
  - when we only need to do non-algorithmic analysis
  - ellipsoid method with separation oracle can solve some exponential size LP

Outline

0-Sum Game

Input: a payoff matrix $M \in \mathbb{R}^{m \times n}, m, n \geq 1$,

two players: row player R, column player C

Output: R plays a row $i \in [m]$, C plays a column $j \in [n]$

payoff of game is M_{ij}

R wants to minimize M_{ij}, C wants to maximize M_{ij}

Rock-Scissor-Paper Game

payoff	R	S	P
R	0	-1	1
S	1	0	- 1
P	-1	1	0

- game depends on who plays first

By allowing mixed strategies, each player has a best strategy, regardless of who plays first

	row player R	column player C
pure strategy	row $i \in [m]$	column $j \in [n]$
mixed strategy	distribution x over $[m]$ $x \in [0,1]^m, \sum_{i=1}^m x_i = 1$	distribution y over $[n]$ $y \in [0,1]^n, \sum_{j=1}^n y_j = 1$

$$M(x,y) := \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j M_{ij}$$

$$M(x,j) := \sum_{i=1}^{m} x_i M_{ij}, \qquad M(i,y) := \sum_{j=1}^{n} y_j M_{ij}$$

- If R plays a mixed strategy y first, then it is the best for C to play a pure strategy j. Value of game is $\inf_x \max_{j \in [n]} M(x,j)$.
- If C plays a mixed strategy x first, then it is the best for R to play a pure strategy i. Value of game is $\sup_y \min_{i \in [m]} M(i,y)$.

Theorem (Von Neumann (1928), Nash's Equilibrium)

$$\inf_x \max_{j\in[n]} M(x,j) = \sup_y \min_{i\in[m]} M(i,y).$$

Coro. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y).$

Coro. There are mixed strategies x^* and y^* satisfying $M(x,y^*) \geq M(x^*,y^*), \forall x$ and $M(x^*,y) \leq M(x^*,y^*), \forall y$.

Proof.

- $V := \inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$
- x^*: the strategy x that minimizes $\sup_y M(x,y)$
- y^*: the strategy y that maximizes $\inf_x M(x,y)$
- $M(x^*,y^*) \leq V, M(x^*,y^*) \geq V \implies M(x^*,y^*) = V$
- $M(x^*,y) \leq V, \forall y$ and $M(x,y^*) \geq V, \forall x.$ $\square$

- As long as the first player can play a mixed strategy, then he will not be at a disadvantage.
- If both players can play mixed strategies, then they do not need to know the strategy of the other player.

Def. $\inf_x \sup_y M(x,y) = \sup_y \inf_x M(x,y)$ is called the value of the game. The two strategies x^* and y^* in the corollary are called the optimum strategies for R and C respectively.

Theorem (Von Neumann (1928), Nash's Equilibrium)

$$\inf_x \max_{j \in [n]} M(x,j) = \sup_y \min_{i \in [m]} M(i,y).$$

- Can be proved by LP duality.

LP for Row Player

$$\begin{aligned} \min \quad & R \\ \textstyle\sum_{i=1}^m x_i &= 1 \\ R - \textstyle\sum_{i=1}^m M_{ij} x_i &\geq 0 \quad \forall j \in [n] \\ x_i &\geq 0 \quad \forall i \in [m] \end{aligned}$$

LP for Column Player

$$\begin{aligned} \max \quad & C \\ \textstyle\sum_{j=1}^n y_j &= 1 \\ C - \textstyle\sum_{j=1}^n M_{ij} y_j &\leq 0 \quad \forall i \in [m] \\ y_j &\geq 0 \quad \forall j \in [n] \end{aligned}$$

- The two LPs are dual to each other.

$$\begin{array}{c|c|c} 1 & 0,0,0,\cdots,0 & \\ \hline 0 & 1,1,1,\cdots,1 & 1 \\ \hdashline 1 & & 0 \\ 1 & -M & 0 \\ \vdots & & \vdots \\ 1 & & 0 \\ R & x_i & \end{array}$$

LP for Row Player

$$\begin{aligned} \min \quad & R \\ \textstyle\sum_{i=1}^m x_i &= 1 \\ R - \textstyle\sum_{i=1}^m M_{ij} x_i &\geq 0 \quad \forall j \in [n] \\ x_i &\geq 0 \quad \forall i \in [m] \end{aligned}$$

LP for Column Player

$$\begin{aligned} \max \quad & C \\ \textstyle\sum_{j=1}^n y_j &= 1 \\ C - \textstyle\sum_{j=1}^n M_{ij} y_j &\leq 0 \quad \forall i \in [m] \\ y_j &\geq 0 \quad \forall j \in [n] \end{aligned}$$

- The two LPs are dual to each other.

$x_i, i \in [m]$	primal variable ($\in \mathbb{R}_{\geq 0}$)	dual constraint ($\leq$)
$y_j, j \in [n]$	dual variable ($\in \mathbb{R}_{\geq 0}$)	primal constraint ($\geq$)
R	primal variable ($\in \mathbb{R}$)	dual constraint ($=$)
C	dual variable ($\in \mathbb{R}$)	primal constraint ($=$)

- Let V be the value of the game, x^* and y^* be the two optimum strategies. Complementrary slackness implies:
  - If $x_i^* > 0$, then $M(i, y^*) = V$.
  - If $y_j^* > 0$, then $M(x^*, j) = V$.
- The game is called 0-sum game as the payoff for R is the negative of the payoff for C.