Advanced Algorithms (Fall 2024)

Extension Complexity of Polytopes

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Outline

1. Motivation and Definition
   - Example: Permutation Polytope
   - Extension Complexity of Spanning Tree Polytope
2. Connection Between Extension Complexity and Non-Negative Rank
3. Polytopes with Exponential Extension Complexity

Motivation

Typical Combinatorial Optimization Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

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Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

$\mathcal{P} :=$ conv$(\{\chi^S : S \in \mathcal{S}\})$: convex hull of all valid solutions

LP to Solve Problem Exactly

$$\min/\max \quad \sum_{i=1}^{n} w_i x_i \quad \text{s.t.} \qquad x \in \mathcal{P}$$

- inequality constraints needed to describe $x \in \mathcal{P}$ (or $\mathcal{P}$ in short) is facets($\mathcal{P}$) := the number of facets of $\mathcal{P}$

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Def. An extension of a polytope $\mathcal{P} \in \mathbb{R}^n$ is a polyhedron $\mathcal{Q} \subseteq \mathbb{R}^{n+r}$ for some $r \geq 0$, such that $\mathcal{P}$ is the projection of $\mathcal{Q}$ to $\mathbb{R}^n$:

$$\mathcal{P} = \left\{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^r, (x, y) \in \mathcal{Q}\right\}$$

LP to Solve Problem Exactly with Auxiliary Variables

$$\min / \max \quad \sum_{i=1}^{n} c_i x_i \quad \text{s.t.} \qquad (x, y) \in \mathcal{Q},$$

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Def. The extension complexity of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as xc($\mathcal{P}$), is defined as follows:

$$\text{xc}(\mathcal{P}) := \min\{\text{facets}(\mathcal{Q}) : \mathcal{Q} \text{ is an extension of } \mathcal{P}\}.$$

Def. An extended formulation of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is a set of linear constraints:

$$(E, F)\begin{pmatrix} x \\ y \end{pmatrix} = g$$
$$y \geq 0$$

where $E \in \mathbb{R}^{N\times n}, F \in \mathbb{R}^{N\times r}, g \in \mathbb{R}^N$ are given, and $x \in \mathbb{R}^n$ is the vector of main variables, $y \in \mathbb{R}^r$ is the vector of auxiliary variables.

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Def. (An alternative definition) The extension complexity of a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, denoted as $\mathrm{xc}(\mathcal{P})$, is defined as the minimum complexity of an extended formulation of $\mathcal{P}$.

The Equivalence Between the Two Definitions

- $\mathrm{xc}_1(\mathcal{P}), \mathrm{xc}_2(\mathcal{P})$: $\mathrm{xc}(\mathcal{P})$ according to the first/second definition

$\mathrm{xc}_2(\mathcal{P}) \leq \mathrm{xc}_1(\mathcal{P})$

- Given an extension $\mathcal{Q}$ of $\mathcal{P}$, we can use facets$(\mathcal{Q})$ inequalities (and some equalities, if the dimension of $\mathcal{Q}$ is smaller than the dimension of its host space) to describe $\mathcal{Q}$, one for each facet.
- For the i-th inequality $a_i x \geq b_i$, we introduce a variable y_i, and replace the inequality by $y_i = a_i x - b_i, y_i \geq 0$.
- This gives an extended formulation of $\mathcal{P}$ with facets$(\mathcal{Q})$ y-variables.
- Remark: there might be some auxiliary variables with no non-negativity constraints; but they can be removed.

$\mathrm{xc}_1(\mathcal{P}) \leq \mathrm{xc}_2(\mathcal{P})$

- An extended formulation with m y-variables defines a polyhedron with at most m facets.

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Lemma For any $S \subsetneq [n], S \neq \emptyset$, $\sum_{i\in S} x_i \geq \frac{|S|(|S|+1)}{2}$ is a facet of $\mathcal{P}$.

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- To show it's a facet, need to prove its dimension is $n-2$
- We can find $x^0, x^1, \cdots, x^{n-2}$ on the face such that $x^1 - x^0, x^2 - x^0, \cdots, x^{n-2} - x^0$ are linearly independent. □

Representation using Permutation Matrices

- Represent a permutation $x \in [n]^{[n]}$ by the permutation matrix $M \in \{0,1\}^{n\times n}$ so that $M_{ij} = 1$ iff $x_i = j$.

$$\text{Example :} \qquad (3,1,2) \Longleftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

- Crucial property: x is a linear function of entries in M
- $\mathcal{P}' := \text{conv}\big(\{M : M \text{ is a permutation matrix}\}\big)$

Lemma $\mathcal{P}' = \left\{y \in [0,1]^{n\times n} : \sum_i y_{i,j} = 1, \forall j; \sum_j y_{i,j} = 1, \forall i\right\}$.

Proof.

- permutation $\Longleftrightarrow$ perfect matching in complete bipartite graph over $2n$ vertices
- permutation matrix polytope $\Longleftrightarrow$ perfect matching polytope

□

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Lemma The permutation polytope $\mathcal{P}$ has extension complexity $O(n^2)$.

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Spanning Tree Polytope

Recall:

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

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Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \leq |S| - 1 \qquad \forall S \subseteq V, 2 \leq |S| \leq n - 1 \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

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$$
\begin{aligned}
\sum_{(u,v) \in E} y_{u \to v} &= 1 && \forall v \in V \setminus \{r\} \\
y_{v \to r} &= 0 && \forall (v, r) \in E \\
y_{u \to v} &\geq 0 && \forall u, v \text{ with } (u, v) \in E \\
x_{\{u,v\}} &= y_{u \to v} + y_{v \to u} && \forall (u, v) \in E \\
y \text{ supports 1 unit flow from } r \text{ to } v & && \forall v \in V \setminus \{r\} \qquad (\dagger)
\end{aligned}
$$

- $(\dagger)$ for every v can be captured using a maximum-flow LP, with $O(|E|)$ variables and constraints.

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- Remaining goal: prove that every valid (x, y) satisfies:

$$\sum_{e \in E} x_e = n - 1 \tag{1}$$

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- $\implies$ total fractional number of edges is $n - 1 \implies$ (1)

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  $\implies \geq 1$ fractional edge from $V \setminus S$ to S
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- When G is complete graph, $\mathcal{P}_{\mathrm{ST}}$ has $O(n^3)$ extension complexity
- The lower bound is $\Omega(n^2)$
- Big open problem to close the gap.

Outline

Def. The non-negative rank of a matrix $M \in \mathbb{R}_{\geq 0}^{m \times n}$ is the minimum $r \geq 0$ such that there are matrices $L \in \mathbb{R}_{\geq 0}^{m \times r}$ and $R \in \mathbb{R}_{\geq 0}^{r \times n}$ such that $M = LR$. We use $\mathrm{rank}_+(M)$ to denote the non-negative rank of M.

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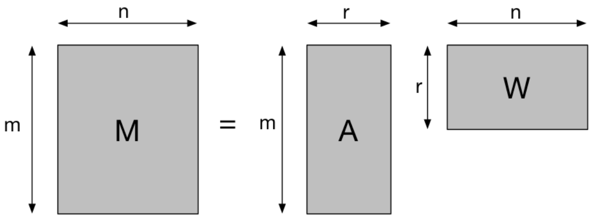

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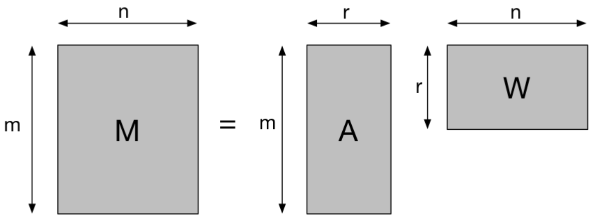

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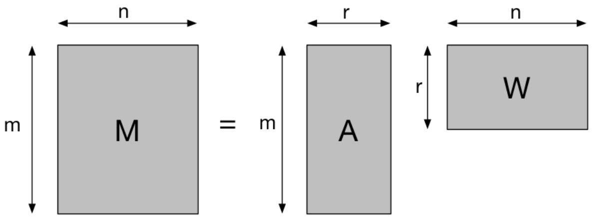
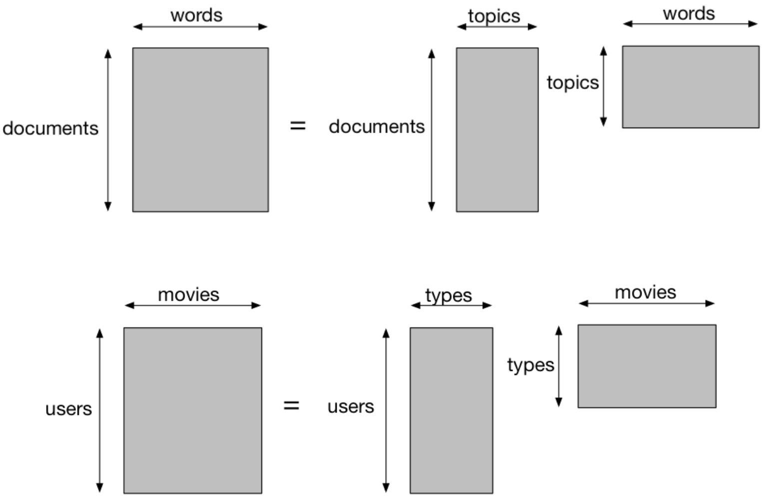


- if we allow $L \in \mathbb{R}^{m \times r}$ and $R \in \mathbb{R}^{r \times n}$, then the non-negative rank becomes the rank
- the rank of a matrix can be computed efficiently
- it is NP-hard to compute the non-negative rank of a matrix

Application of Non-Negative Rank



Def. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be defined as

$$\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b; Ex = f\},$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $E \in \mathbb{R}^{m' \times n}$, $f \in \mathbb{R}^{m'}$. Assume the equations $Ex = f$ are linearly independent, and there is a 1-1 correspondence between inequalities in $Ax \leq b$ and facets of $\mathcal{P}$. Let $x^1, x^2, \cdots, x^v$ be all the vertices of $\mathcal{P}$. The slack matrix $\mathbf{SM}^{\mathcal{P}}$ of $\mathcal{P}$ w.r.t this description is a matrix in $\mathbb{R}^{m \times v}_{\geq 0}$ such that

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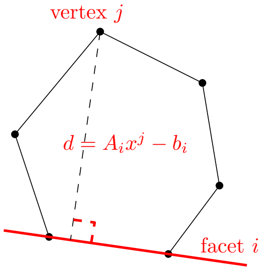

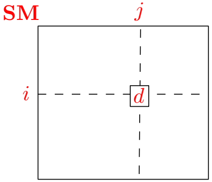

Slack Matrix Theorem

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Notes

- Considering non-vertex points in $\mathcal{P}$ for the columns of $\mathbf{SM}^{\mathcal{P}}$ does not increase is non-negative rank
- Considering non-facet faces of $\mathcal{P}$ for rows of $\mathbf{SM}^{\mathcal{P}}$ does not increase its non-negative rank

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- if $\exists y \geq 0$ with $Ax + Fy = b$, then $Ax \leq b$ $\qquad \mathcal{P}' \subseteq \mathcal{P}$

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- it is a non-negative combination of columns of F
- so, $\exists y \ge 0$ with $b - Ax^j = Fy$ — $\mathcal{P} \subseteq \mathcal{P}'$ $\square$

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- This is a decomposition with rank r. $\square$

Outline

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- [Samuel Fiorini, Serge Massar, Sebastian Pokutta, Hans Raj Tiwary and Ronald de Wolf]: "Exponential Lower Bounds for Polytopes in Combinatorial Optimization": All the above polytopes have exponential extension complexity.
- 2023 Godel Prize Winner Paper

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- [Rothvoss 2017]: "The Matching Polytope has Exponential Extension Complexity." 2023 Godel Prize Winner Paper