Advanced Algorithms (Fall 2024)

Greedy Algorithms

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Outline

1. Greedy Algorithms and Matroids
   - Recap: Maximum-Weight Spanning Tree Problem
   - Maximum-Weight Independent Set in Matroids
   - Examples of Matroids

2. Greedy Approximation Algorithms
   - $(\ln n + 1)$-Approximation for Set-Cover
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Maximum Coverage
   - Submodular Functions
   - $\left(1 - \frac{1}{e}\right)$-Approximation for Cardinality-Constraied Submodular Maximization

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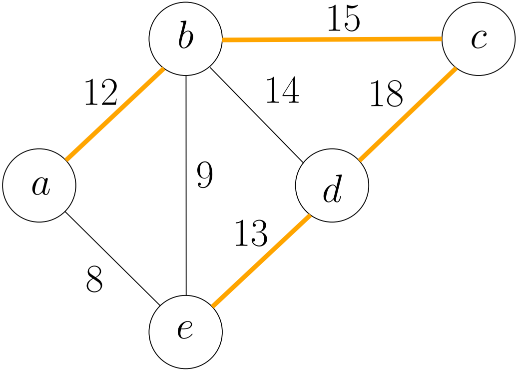
Maximum-Weight Spanning Tree Problem

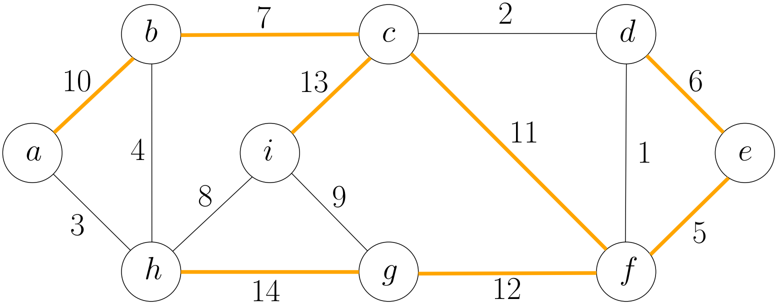
Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}^E_{>0}$

Output: the spanning tree T of G with the maximum total weight



Kruskal's Algorithm for Maximum-Weight Spanning Tree

$F \leftarrow \emptyset$
sort edges in E in non-increasing order of weights w
for each edge (u, v) in the order **do**
    if u and v are not connected by a path of edges in F **then**
        $F \leftarrow F \cup \{(u, v)\}$
return (V, F)



Proof of Correctness of Kruskal's Algorithm

Maximum-Weight Spanning Tree (MST) with Pre-Selected Edges

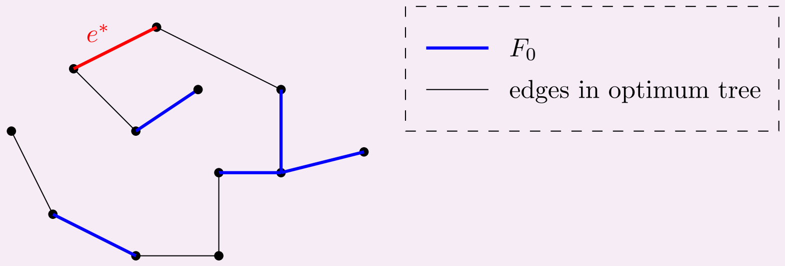
Input: Graph $G = (V, E)$ and edge weights $w \in \mathbb{Z}_{>0}^{E}$
a set $F_0 \subseteq E$ of edges, that does not contain a cycle

Output: the maximum-weight spanning tree $T = (V, E_T)$ of G satisfying $F_0 \subseteq E_T$

Lemma (Key Lemma) Given an instance $(G = (V, E), w, F_0)$ of the MST with pre-selected edges problem, let e^* be the maximum weight edge in $E \setminus F_0$ such that $F_0 \cup \{e^*\}$ does not contain a cycle. Then there is an optimum solution $T = (V, E_T)$ to the instance with $e^* \in E_T$.

Proof of Correctness of Kruskal's Algorithm

Proof of Key Lemma.



$\square$

Outline

Q: Does the greedy algorithm work for more general problems?

A General Maximization Problem

Input: E: the ground set of elements

$w \in \mathbb{Z}_{>0}^E$: weight vector on elements

$\mathcal{S}$: an (implicitly given) family of subsets of E

- $\emptyset \in \mathcal{S}$
- $\mathcal{S}$ is downward closed: if $A \in \mathcal{S}, B \subsetneq A$, then $B \in \mathcal{S}$.

Output: $A \in \mathcal{S}$ that maximizes $\sum_{e \in A} w_e$

- maximum-weight spanning tree: $\mathcal{S} =$ family of forests

Greedy Algorithm

1: $A \leftarrow \emptyset$
2: sort elements in E in non-decreasing order of weights w
3: **for** each element e in the order **do**
4:     **if** $A \cup \{e\} \in \mathcal{S}$ **then** $A \leftarrow A \cup \{e\}$
5: **return** A

Examples where Greedy Algorithm is Not Optimum

- Knapsack Packing: given elements E, where every element has a value and a cost, and a cost budget C, the goal is to find a maximum value subset of items with cost at most C
- Maximum Weight Bipartite Graph Matching

- Matroids: cases where greedy algorithm is optimum

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
3. (augmentation/exchange property) If $A, B \in \mathcal{I}$ and $|B| < |A|$, then there exists $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$.

Lemma Let $G = (V, E)$. $F \subseteq E$ is in $\mathcal{I}$ iff (V, F) is a forest. Then $(E, \mathcal{I})$ is a matroid, and it is called a graphic matroid.

Proof of Exchange Property.

- $|B| < |A| \Rightarrow (V, B)$ has more CC than (V, A).
- Some edge in A connects two different CC of (V, B). $\square$

Feasible Family for Knapsack Packing Does Not Satisfy Augmentation Property

- $c_1 = c_2 = 10, c_3 = 20, C = 20$.
- $\{1, 2\}, \{3\} \in \mathcal{I}$, but $\{1, 3\}, \{2, 3\} \notin \mathcal{I}$.

Feasible Family for Bipartite Matching Does Not Satisfy Augmentation Property

- Complete bipartite graph between $\{a_1, a_2\}$ and $\{b_1, b_2\}$.
- $\{(a_1, b_1), (a_2, b_2)\}, \{(a_1, b_2)\} \in \mathcal{I}$.

Theorem The greedy algorithm gives optimum solution for the maximum-weight independent set problem in a matroid.

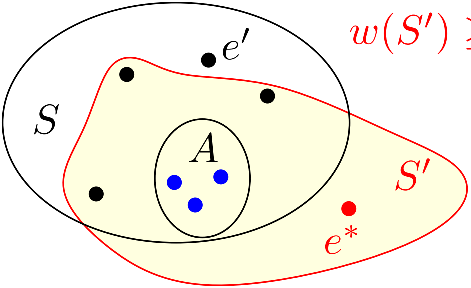
Lemma (Key Lemma)

- given: matroid $\mathcal{M} = (E, \mathcal{I})$, weights $w \in \mathbb{Z}_{>0}^E$, $A \in \mathcal{I}$,
- goal: find a maximum weight independent set containing A
- $e^* = \arg\max_{e \in E \setminus A: A \cup \{e\} \in \mathcal{I}} w_e$, assuming e^* exists
- Then, some optimum solution contains e^*

- let $S \supseteq A, S \in \mathcal{I}$ be an optimum solution, $e^* \notin S$

$$w_{e^*} \geq w_{e'}$$

$$w(S') \geq w(S)$$



Lemma (Key Lemma)

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- Then, some optimum solution contains e^*

Proof.

- let $S \supseteq A, S \in \mathcal{I}$ be an optimum solution, $e^* \notin S$

  $S' \leftarrow A \cup \{e^*\}$
  while $|S'| < |S|$ **do**
      let e be any element in $S \setminus S'$ with $S' \cup \{e\} \in \mathcal{I}$
      $\triangleright$ e exists due to exchange property
      $S' \leftarrow S' \cup \{e\}$

- S' and S differ by exactly one element
- $w(S') := \sum_{e \in S'} w_e \geq w(S) \implies S'$ is also optimum □

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Examples of Matroids

- E: the ground set $\qquad$ $\mathcal{I}$: the family of independent sets
- Uniform Matroid: $k \in \mathbb{Z}_{>0}$.

  $$\mathcal{I} = \{A \subseteq E : |A| \leq k\}.$$

- Partition Matroid: partition $(E_1, E_2, \cdots, E_t)$ of E, positive integers $k_1, k_2, \cdots, k_t$

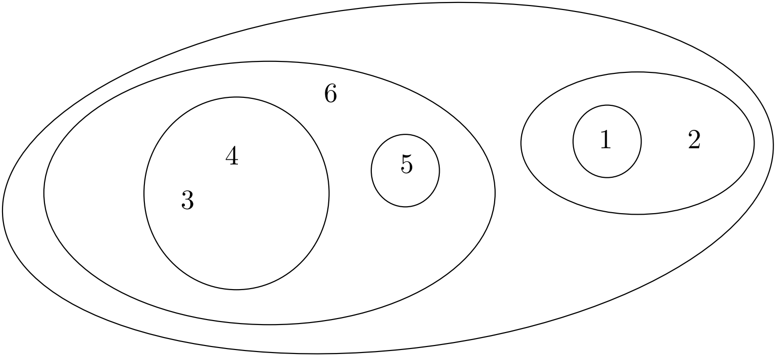
  $$\mathcal{I} = \{A \subseteq E : |A \cap E_i| \leq k_i, \forall i \in [t]\}.$$

- Laminar Matroid: laminar family of subsets of E $\{E_1, E_2, \cdots, E_t\}$, positive integers $k_1, k_2, \cdots, k_t$

  $$\mathcal{I} = \{A \subseteq E : |A \cap E_i| \leq k_i, \forall i \in [t]\}.$$

Def. A family $\{E_1, E_2, \cdots, E_t\}$ of subsets of E is said to be laminar if for every two distinct subsets E_i, E_j in the family, we have $E_i \cap E_j = \emptyset$ or $E_i \subsetneq E_j$ or $E_j \subsetneq E_i$.

- $\Big\{\{1\},\{1,2\},\{3,4\},\{5\},\{3,4,5,6\},\{1,2,3,4,5,6\}\Big\}$ is a laminar family.



Examples of Matroids

- E: the ground set $\qquad$ $\mathcal{I}$: the family of independent sets
- Graphic Matroid: graph $G = (V, E)$

  $$\mathcal{I} = \{A \subseteq E : (V, A) \text{ is a forest}\}$$

- Transversal Matroid: a bipartite graph $G = (E \uplus B, \mathcal{E})$

  $$\mathcal{I} = \{A \subseteq E : \text{there is a matching in } G \text{ covering } A\}$$

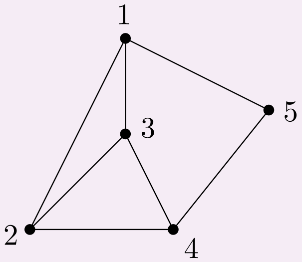
- Linear Matroid: a vector $\vec{v}_e \in \mathbb{R}^d$ for every $e \in E$

  $$\mathcal{I} = \{A \subseteq E : \text{vectors } \{\vec{v}_e\}_{e \in A} \text{ are linearly independent}\}$$

Relationship between matroids

$$\begin{tikzcd}
 & & \text{Laminar} \arrow[dr] & \\
\text{Uniform} \arrow[r] & \text{Partition} \arrow[ur] \arrow[r] & \text{Transversal} \arrow[r] & \text{Linear} \\
 & & \text{Graphic} \arrow[ur] &
\end{tikzcd}$$

A Graphic Matroid is A Linear Matroid



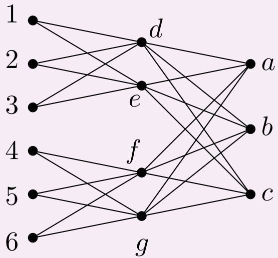
edges	vectors
$(1,2)$	$(1,-1,0,0,0)$
$(1,3)$	$(1,0,-1,0,0)$
$(1,5)$	$(1,0,0,0,-1)$
$(2,3)$	$(0,1,-1,0,0)$
$(2,4)$	$(0,1,0,-1,0)$
$(3,4)$	$(0,0,1,-1,0)$
$(4,5)$	$(0,0,0,1,-1)$

A Laminar Matroid is A Linear Matroid

Example

sets	upper bounds
$\{1,2,3\}$	2
$\{3,4,5\}$	2
$\{1,2,3,4,5,6\}$	3

A DAG (left to right)



- $x^a, x^b, x^c \in \mathbb{R}^3$ are linearly independent rational vectors
- x^d, x^e, x^f, x^g: $\text{rand}(0,1) \cdot x^a + \text{rand}(0,1) \cdot x^b + \text{rand}(0,1) \cdot x^c$
- x^1, x^2, x^3: $\text{rand}(0,1) \cdot x^d + \text{rand}(0,1) \cdot x^e$
- x^4, x^5, x^6: $\text{rand}(0,1) \cdot x^f + \text{rand}(0,1) \cdot x^g$
- each $\text{rand}(0,1)$ gives an independent random real in $[0,1]$
- almost surely, all the random numbers are algebraically independent

Outline

Recap: Approximation Algorithms

- For minimization problems:

$$\text{approximation ratio} := \frac{\text{cost of our solution}}{\text{cost of optimum solution}} \geq 1$$

- For maximization problems:

$$\text{approximation ratio} := \frac{\text{value of our solution}}{\text{value of optimum solution}} \leq 1$$

or

$$\text{approximation ratio} := \frac{\text{value of optimum solution}}{\text{value of our solution}} \geq 1$$

Outline

Set Cover

Input: $U, |U| = n$: ground set

$S_1, S_2, \cdots, S_m \subseteq U$

Output: minimum size set $C \subseteq [m]$ such that $\bigcup_{i \in C} S_i = U$

Greedy Algorithm for Set Cover

$C \leftarrow \emptyset, U' \leftarrow U$
while $U' \neq \emptyset$ **do**
    choose the i that maximizes $|U' \cap S_i|$
    $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
return C

- g: minimum number of sets needed to cover U

Lemma Let $u_t, t \in \mathbb{Z}_{\geq 0}$ be the number of uncovered elements after t steps. Then for every $t \geq 1$, we have

$$u_t \leq \left(1 - \frac{1}{g}\right) \cdot u_{t-1}.$$

Proof.

- Consider the g sets $S_1^*, S_2^*, \cdots, S_g^*$ in optimum solution
- $S_1^* \cup S_2^* \cup \cdots \cup S_g^* = U$
- at beginning of step t, some set in $S_1^*, S_2^*, \cdots, S_g^*$ must contain $\geq \frac{u_{t-1}}{g}$ uncovered elements
- $u_t \leq u_{t-1} - \frac{u_{t-1}}{g} = \left(1 - \frac{1}{g}\right) u_{t-1}$. □

Proof of $(\ln n + 1)$-approximation.

- Let $t = \lceil g \cdot \ln n \rceil$. $u_0 = n$. Then
$$u_t \le \left(1 - \frac{1}{g}\right)^{g \cdot \ln n} \cdot n < e^{-\ln n} \cdot n = n \cdot \frac{1}{n} = 1.$$
- So $u_t = 0$, approximation ratio $\le \frac{\lceil g \cdot \ln n \rceil}{g} \le \ln n + 1$. □

- A more careful analysis gives a H_n-approximation, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n-th harmonic number.
- $\ln(n+1) < H_n < \ln n + 1$.

$(1-c)\ln n$-hardness for any $c = \Omega(1)$

Let $c > 0$ be any constant. There is no polynomial-time $(1-c)\ln n$-approximation algorithm for set-cover, unless

- NP $\subseteq$ quasi-poly-time, [Lund, Yannakakis 1994; Feige 1998]
- P = NP. [Dinur, Steuer 2014]

Outline

- set cover: use smallest number of sets to cover all elements.
- maximum coverage: use k sets to cover maximum number of elements

Maximum Coverage

Input: $U, |U| = n$: ground set,
$S_1, S_2, \cdots, S_m \subseteq U$, $\quad k \in [m]$

Output: $C \subseteq [m], |C| = k$ with the maximum $\bigcup_{i \in C} S_i$

Greedy Algorithm for Maximum Coverage

1: $C \leftarrow \emptyset, U' \leftarrow U$
2: **for** $t \leftarrow 1$ to k **do**
3: $\quad$ choose the i that maximizes $|U' \cap S_i|$
4: $\quad$ $C \leftarrow C \cup \{i\}, U' \leftarrow U' \setminus S_i$
5: **return** C

Theorem Greedy algorithm gives $(1 - \frac{1}{e})$-approximation for maximum coverage.

Proof.

- o: max. number of elements that can be covered by k sets.
- p_t: #(covered elements) by greedy algorithm after step t
- $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o - p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ $\square$

- The $\left(1 - \frac{1}{e}\right)$-approximation extends to a more general problem.

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   - **Submodular Functions**
   - $(1 - \frac{1}{e})$-Approximation for Cardinality-Constraied Submodular Maximization

Def. Let $n \in \mathbb{Z}_{>0}$. A set function $f : 2^{[n]} \to \mathbb{R}$ is called submodular if it satisfies one of the following three equivalent conditions:

(1) $\forall A, B \subseteq [n]$:

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B).$$

(2) $\forall A \subseteq B \subsetneq [n], i \in [n] \setminus B$:

$$f(B \cup \{i\}) - f(B) \leq f(A \cup \{i\}) - f(A).$$

(3) $\forall A \subseteq [n], i, j \in [n] \setminus A, i \neq j$:

$$f(A \cup \{i, j\}) + f(A) \leq f(A \cup \{i\}) + f(A \cup \{j\}).$$

- (2): diminishing marginal values: the marginal value by getting i when I have B is at most that when I have $A \subseteq B$.
- $(1) \Rightarrow (2) \Rightarrow (3)$, $\quad (3) \Rightarrow (2) \Rightarrow (1)$

Examples of Sumodular Functions

- linear function: $f(S) = \sum_{i \in S} w_i, \forall S \subseteq [n]$
- budget-additive function: $f(S) = \min \left\{ \sum_{i \in S} w_i, B \right\}, \forall S \subseteq [n]$
- coverage function: given sets $S_1, S_2, \cdots, S_n \subseteq \Omega$,

$$f(C) := \left| \bigcup_{i \in C} S_i \right|, \forall C \subseteq [n]$$

- matroid rank function:

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of any $A \subseteq E$ is defined as

$$r_{\mathcal{M}}(A) = \max \left\{ |A'| : A' \subseteq A, A' \in \mathcal{I} \right\}.$$

The function $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

- cut function: given graph $G = ([n], E)$

$$f(A) = \left| E(A, [n] \setminus A) \right|, \forall A \subseteq [n]$$

Examples of Sumodular Functions

- linear function, budget-additive function, coverage function,
- matroid rank function, cut function
- entropy function: given random variables $X_1, X_2, \cdots, X_n$

$$f(S) := H(X_i : i \in S), \forall S \subseteq [n]$$

Def. A submodular function $f : 2^{[n]} \to \mathbb{R}$ is said to be monotone if $f(A) \leq f(B)$ for every $A \subseteq B \subseteq [n]$.

Def. A submodular function $f : 2^{[n]} \to \mathbb{R}$ is said to be symmetric if $f(A) = f([n] \setminus A)$ for every $A \subseteq [n]$.

- coverage, matroid rank and entropy functions are monotone
- cut function is symmetric

Outline

$(1-\frac{1}{e})$-Approximation for Submodular Maximization with Cardinality Constraint

Submodular Maximization under a Cardinality Constraint

Input: An oracle to a non-negative monotone submodular function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$, $k \in [n]$

Output: A subset $S \subseteq [n]$ with $|S| = k$, so as to maximize $f(S)$

- We can assume $f(\emptyset) = 0$

Greedy Algorithm for the Problem

1: $S \leftarrow \emptyset$
2: **for** $t \leftarrow 1$ to k **do**
3: choose the i that maximizes $f(S \cup \{i\})$
4: $S \leftarrow S \cup \{i\}$
5: **return** S

Theorem Greedy algorithm gives $(1-\frac{1}{e})$-approximation for submodular-maximization under a cardinality constraint.

Proof.

- o: optimum value
- p_t: value obtained by greedy algorithm after step t
- need to prove: $p_t \geq p_{t-1} + \dfrac{o - p_{t-1}}{k}$
- $o - p_t \leq o - p_{t-1} - \frac{o-p_{t-1}}{k} = \left(1 - \frac{1}{k}\right)(o - p_{t-1})$
- $o - p_k \leq \left(1 - \frac{1}{k}\right)^k (o - p_0) \leq \frac{1}{e} \cdot o$
- $p_k \geq \left(1 - \frac{1}{e}\right) \cdot o$ $\square$

Def. A set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is sub-additive if for every two sets $A, B \subseteq [n]$, we have $f(A \cup B) \leq f(A) + f(B)$.

Lemma A non-negative submodular set function $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ is sub-additive.

Proof.

For $A, B \subseteq [n]$, we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$.
So, $f(A \cup B) \leq f(A) + f(B)$ as $f(A \cap B) \geq 0$. □

Lemma Let $f : 2^{[n]} \to \mathbb{R}$ be submodular. Let $S \subseteq [n]$, and $f_S(A) = f(S \cup A) - f(S)$ for every $A \subseteq [n]$. (f_S is the marginal value function for set S.) Then f_S is also submodular.

Proof.

- Let $A, B \subseteq [n] \setminus S$; it suffices to consider ground set $[n] \setminus S$.

$$\begin{aligned} & f_S(A \cup B) + f_S(A \cap B) - (f_S(A) + f_S(B)) \\ &= f(S \cup A \cup B) - f(S) + f(S \cup (A \cap B)) - f(S) \\ &\quad - \Big(f(S \cup A) - f(S) + f(S \cup B) - f(S) \Big) \\ &= f(S \cup A \cup B) + f(S \cup (A \cap B)) - f(S \cup A) - f(S \cup B) \\ &\leq 0 \end{aligned}$$

- The last inequality is by $S \cup A \cup B = (S \cup A) \cup (S \cup B)$, $S \cup (A \cap B) = (S \cup A) \cap (S \cup B)$ and submodularity of f. □

Proof of $p_t \geq p_{t-1} + \frac{o - p_{t-1}}{k}$.

- $S^* \subseteq [n]$: optimum set, $|S^*| = k$, $o = f(S^*)$
- S: set chosen by the algorithm at beginning of time step t
  $|S| = t - 1$, $p_{t-1} = f(S)$
- f_S is submodular and thus sub-additive

$$f_S(S^*) \leq \sum_{i \in S^*} f_S(i) \quad \Rightarrow \quad \exists i \in S^*, f_S(i) \geq \frac{1}{k} f_S(S^*)$$

- for the i, we have

$$f(S \cup \{i\}) - f(S) \geq \frac{1}{k}(f(S^*) - f(S))$$

$$p_t \geq f(S \cup \{i\}) \geq p_{t-1} + \frac{1}{k}(o - p_{t-1}) \qquad \square$$

Submodular Maximization for Monotone Functions:

Constraint	Approx.	Hardness	Technique
$\lvert S\rvert \leq k$	$1 - 1/e$	$1 - 1/e$	greedy
matroid	$1 - 1/e$	$1 - 1/e$	multilinear ext.
$O(1)$ knapsacks	$1 - 1/e$	$1 - 1/e$	multilinear ext.
k matroids	$k + \epsilon$	$\Omega(k/\log k)$	local search
k matroids $O(1)$ knapsacks	$O(k)$	$\Omega(k/\log k)$	multilinear ext.

Submodular Maximization for Non-Monotone Functions:

Constraint	Approx.	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k + O(1)$	$\Omega(k/\log k)$	local search
k matroids $O(1)$ knapsacks	$O(k)$	$\Omega(k/\log k)$	multilinear ext.

From Prof. Jan Vodrak's slides "Optimization of Submodular Functions"

Submodular Minimization

Constraint	Approx.	Hardness	Technique
Unconstrained	1	1	combinatorial
$\|S\| \geq k$, Monotone	$\tilde{O}(\sqrt{n})$ *	$\Omega(\sqrt{n})$ *	combinatorial

- * bounds are for query complexity under oracle model.