Advanced Algorithms

Lovász Local Lemma

尹一通 Nanjing University, 2024 Fall

k-SAT

- Conjunctive Normal Form (**CNF**):

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge \underbrace{(x_1 \vee x_2 \vee x_4)}_{\text{clause}} \wedge (x_3 \vee \neg x_4 \vee \neg x_5)$$

literals

Boolean variables: $x_1, x_2, \ldots, x_n \in \{\texttt{True}, \texttt{False}\}$

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- [Cook-Levin] **NP-hard** if $k \geq 3$

Trivial Cases

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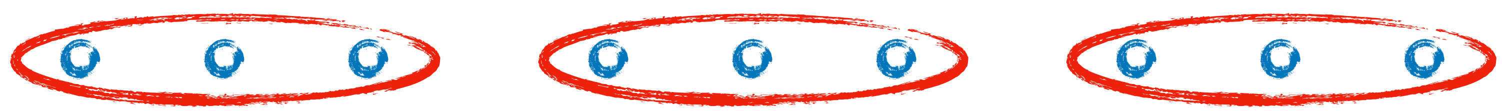
- Clauses are ***disjoint***:

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6) \wedge (x_7 \vee \neg x_8 \vee \neg x_9)$$



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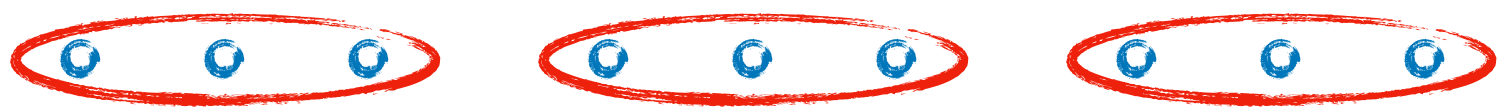

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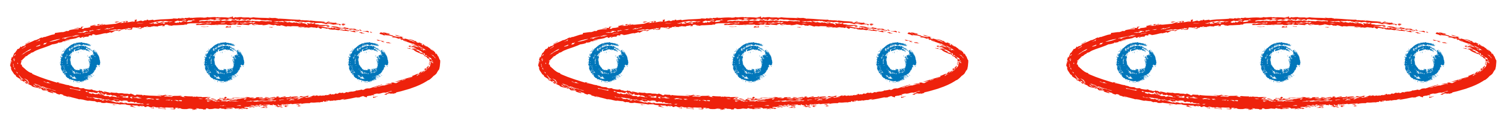

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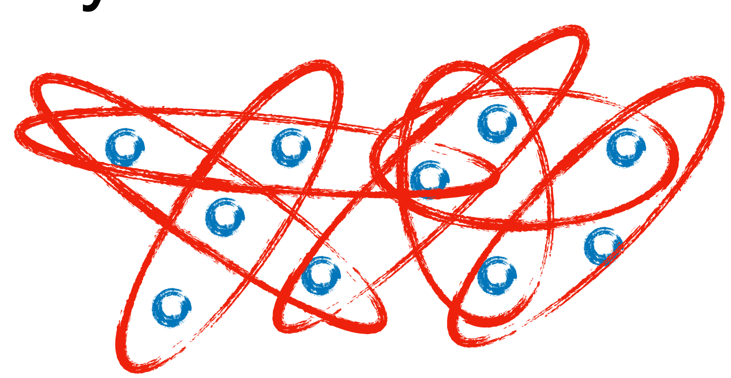
- $m < 2^k \implies \Phi$ is always satisfiable.

m: # of clauses



The Probabilistic Method

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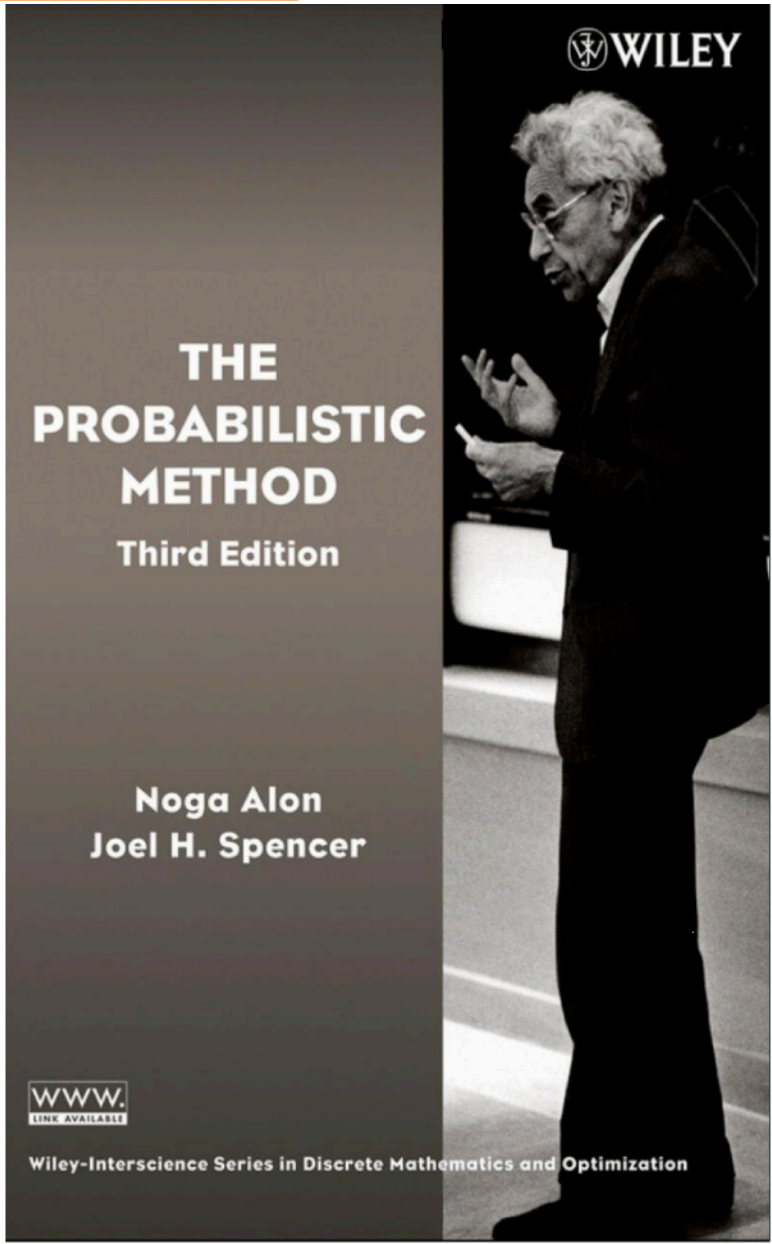

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The Probabilistic Method:

Draw x from prob. space Ω: for property $\mathscr{P}$,

$$\Pr[\mathscr{P}(x)] > 0 \implies \exists x \in \Omega, \mathscr{P}(x)$$

Limited Dependency

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each clause intersects $\leq d$ other clauses

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$$4d2^{-k} \leq 1$$

Lovász Local Lemma (LLL)

- "Bad" events $A_1, \ldots, A_m$, where all $\Pr[A_j] \leq p$

Dependency degree d:

each A_i is "*dependent*" of $\leq d$ other events

Lovász Local Lemma [Lovász and Erdős 1973; Lovász 1977]**:**

$$\mathrm{e}p(d+1) \leq 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

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- The "***LLL***" condition:
  - Bad events are not too likely to occur individually.
  - Bad events are not too dependent with each other.

Dependency Graph

- “Bad” events $A_1, \ldots, A_m$,

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Definition (independence):

A is independent of B if $\Pr[A \mid B] = \Pr[A]$ or B is impossible.

A_0 is mutually independent of $A_1, \ldots, A_m$ if A_0 is independent of every event $B = B_1 \wedge \cdots \wedge B_m$, where each $B_i = A_i$ or $\overline{A}_i$.

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Vertices are bad events $A_1, \ldots, A_m$.

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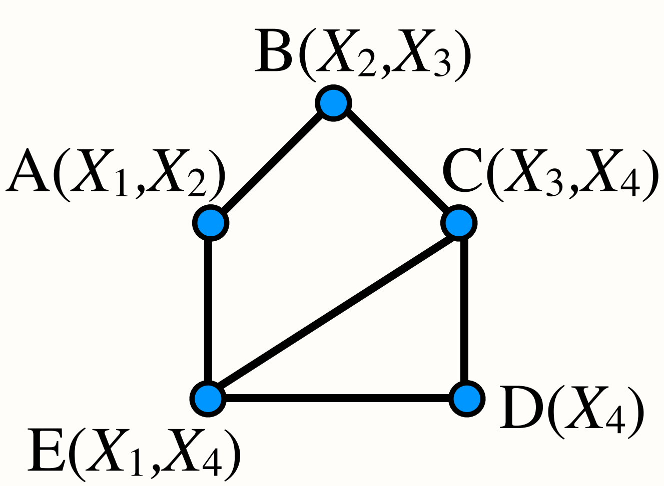

independent random variables:

$$X_1, X_2, X_3, X_4$$

bad events (defined on subsets of variables):

$$A(X_1, X_2), B(X_2, X_3), C(X_3, X_4)$$
$$D(X_4), E(X_1, X_4)$$

Dependency Graph

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Dependency graph: $\Gamma(A_i)$: neighborhood of A_i.

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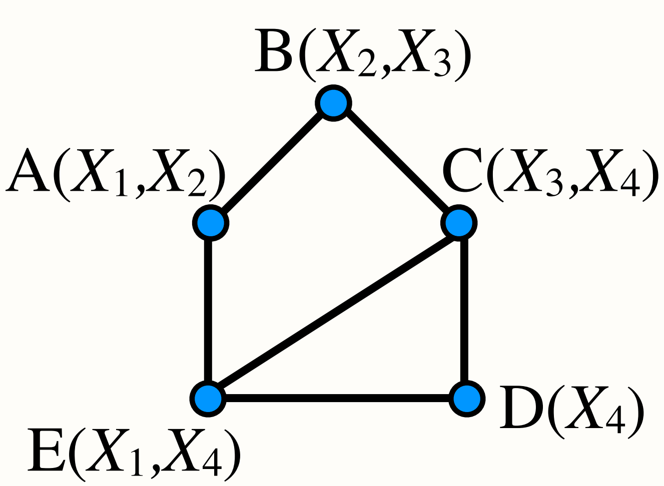

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Dependency degree d: (max-degree of dependency graph)

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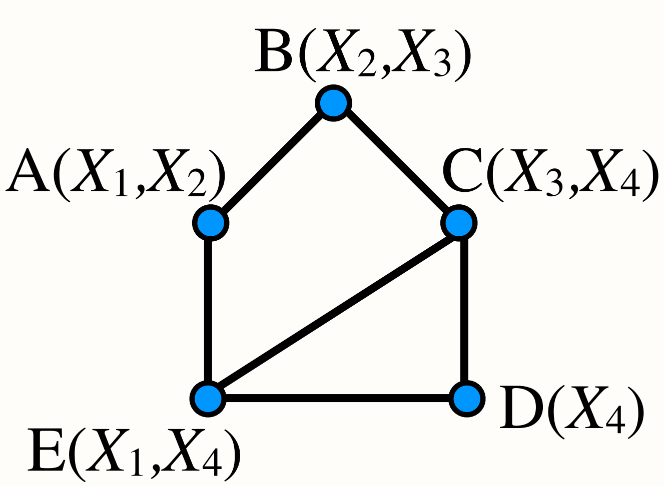

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- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$:

A_i is mutually independent of all $A_j \notin \Gamma(A_i)$

Lovász Local Lemma:

$p \triangleq \max_i \Pr[A_i]$ and $d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^m \overline{A_i}\right] > 0$$

Constraint Satisfaction Problem (CSP)

- **Variables:** $x_1, \ldots, x_n \in [q]$
- (local) **Constraints:** $C_1, \ldots, C_m$
  - each C_i is defined on a subset $\mathsf{vbl}(C_i)$ of variables

$$C_i : [q]^{\mathsf{vbl}(C_i)} \to \{\mathtt{True}, \mathtt{False}\}$$

- Any $\boldsymbol{x} \in [q]^n$ is a **CSP solution** if it satisfies all $C_1, \ldots, C_m$

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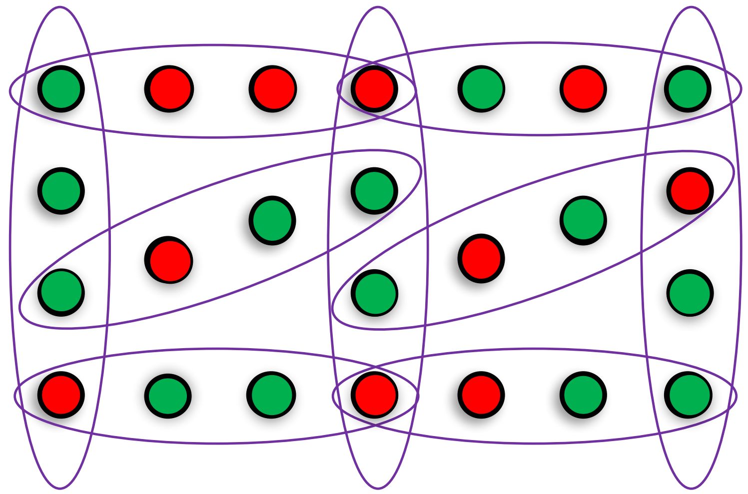
- **Examples:**
  - k-CNF, (hyper)graph coloring, set cover, unique games…
  - vertex cover, independent set, matching, perfect matching, …

Hypergraph Coloring

- k-uniform hypergraph $H = (V, E)$:
  - V is vertex set, $E \subseteq \binom{V}{k}$ is set of hyperedges
- degree of vertex $v \in V$: # of hyperedges $e \ni v$



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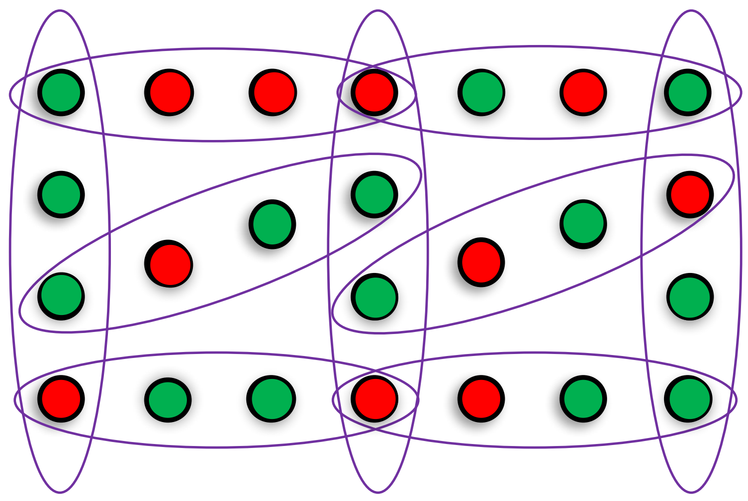

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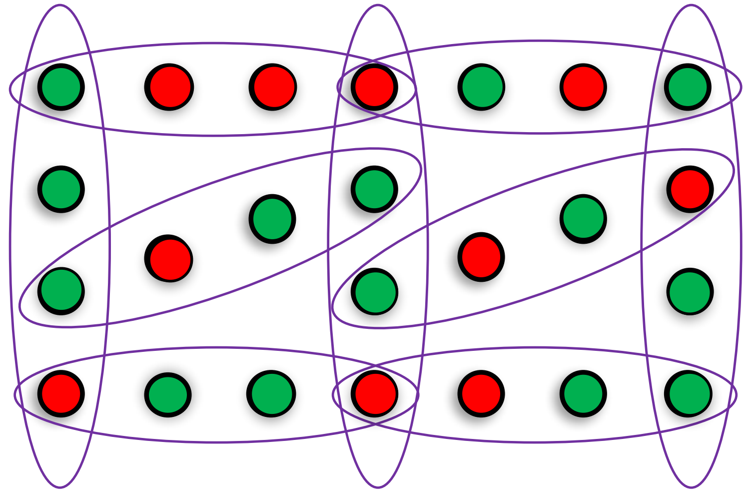

Theorem: For any k-uniform hypergraph H of max-degree Δ,

$$\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

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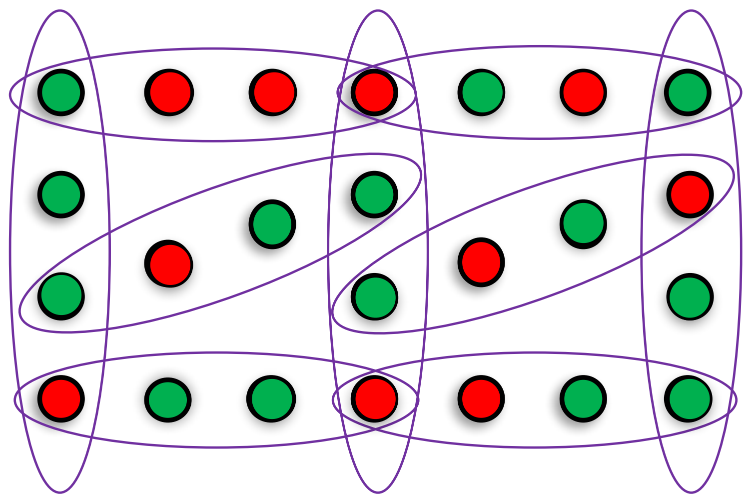

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$k \ge \log_q \Delta + \log_q \log_q \Delta + O(1)$

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$$\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

- Uniformly and independently color each $v \in V$ a random color $\in [q]$
- Bad event A_e for each hyperedge $e \in E \subseteq \binom{V}{k}$: e is monochromatic
  - $\Pr[A_e] \leq p = q^{1-k}$
- Dependency degree for bad events $d \leq k(\Delta - 1)$
  - $\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies \mathrm{e}p(d+1) \leq 1$

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Lovász Local Lemma (LLL)

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$

Lovász Local Lemma (symmetric case)**:**

$p \triangleq \max_i \Pr[A_i]$ and $d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

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$$\exists \alpha_1, \ldots, \alpha_m \in [0,1):$$

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Induction Hypothesis (I.H.):

$$\forall \text{ distinct } A_i, A_{j_1}, A_{j_2}, \ldots, A_{j_k}: \qquad \Pr\left[A_i \mid \overline{A_{j_1}} \cdots \overline{A_{j_k}}\right] \le \alpha_i$$

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Basis: when $k = 0$, trivial

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$$\Pr\left[A_i \mid \underbrace{\overline{A}_{j_1} \cdots \overline{A}_{j_l}}_{\text{neighbors}} \underbrace{\overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}}_{\text{non-neighbors}}\right] = \frac{\Pr\left[A_i \overline{A}_{j_1} \cdots \overline{A}_{j_l} \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right]}{\Pr\left[\overline{A}_{j_1} \cdots \overline{A}_{j_l} \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right]} \le \alpha_i$$

$$\le \Pr\left[A_i \mid \overline{A}_{j_{l+1}} \cdots \overline{A}_{j_k}\right] = \Pr\left[A_i\right] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1-\alpha_j) \quad \text{(LLL cond.)}$$

$$= \prod_{r=1}^{l} \Pr\left[\overline{A}_{j_r} \mid \overline{A}_{j_{r+1}} \cdots \overline{A}_{j_k}\right] = \prod_{r=1}^{l} \left(1 - \Pr\left[A_{j_r} \mid \overline{A}_{j_{r+1}} \cdots \overline{A}_{j_k}\right]\right)$$

$$\text{(I.H.)} \ge \prod_{r=1}^{l} \left(1 - \alpha_{j_r}\right) \ge \prod_{A_j \in \Gamma(A_i)} (1-\alpha_j)$$

- $A_1, \ldots, A_m$ has a dependency graph given by neighborhoods $\Gamma(\cdot)$

Lovász Local Lemma (asymmetric case)**:**

$$\exists \alpha_1, \ldots, \alpha_m \in [0,1):$$
$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1-\alpha_j) \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] \ge \prod_{i=1}^{m} (1-\alpha_i)$$

chain rule

$$\Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] = \prod_{i=1}^{m} \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right] = \prod_{i=1}^{m}\left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j<i} \overline{A_j}\right]\right) \ge \prod_{i=1}^{m}(1-\alpha_i)$$

Induction Hypothesis (I.H.):

$\forall$ distinct $A_i, A_{j_1}, A_{j_2}, \ldots, A_{j_k}$: $\quad \Pr\left[A_i \mid \overline{A}_{j_1} \cdots \overline{A}_{j_k}\right] \le \alpha_i$

Lovász Local Lemma (LLL)

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$$\alpha_1 = \cdots = \alpha_m = \frac{1}{d+1}$$

symmetric case:

$$p \triangleq \max_i \Pr[A_i]$$

$$d \triangleq \max_i |\Gamma(i)|$$

$$ep(d+1) \le 1 \implies \Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] > 0$$

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$$\alpha_1 = \cdots = \alpha_m = \frac{1}{d+1} \qquad \Downarrow \qquad \alpha_1 = \cdots = \alpha_m = \frac{1}{2d}$$

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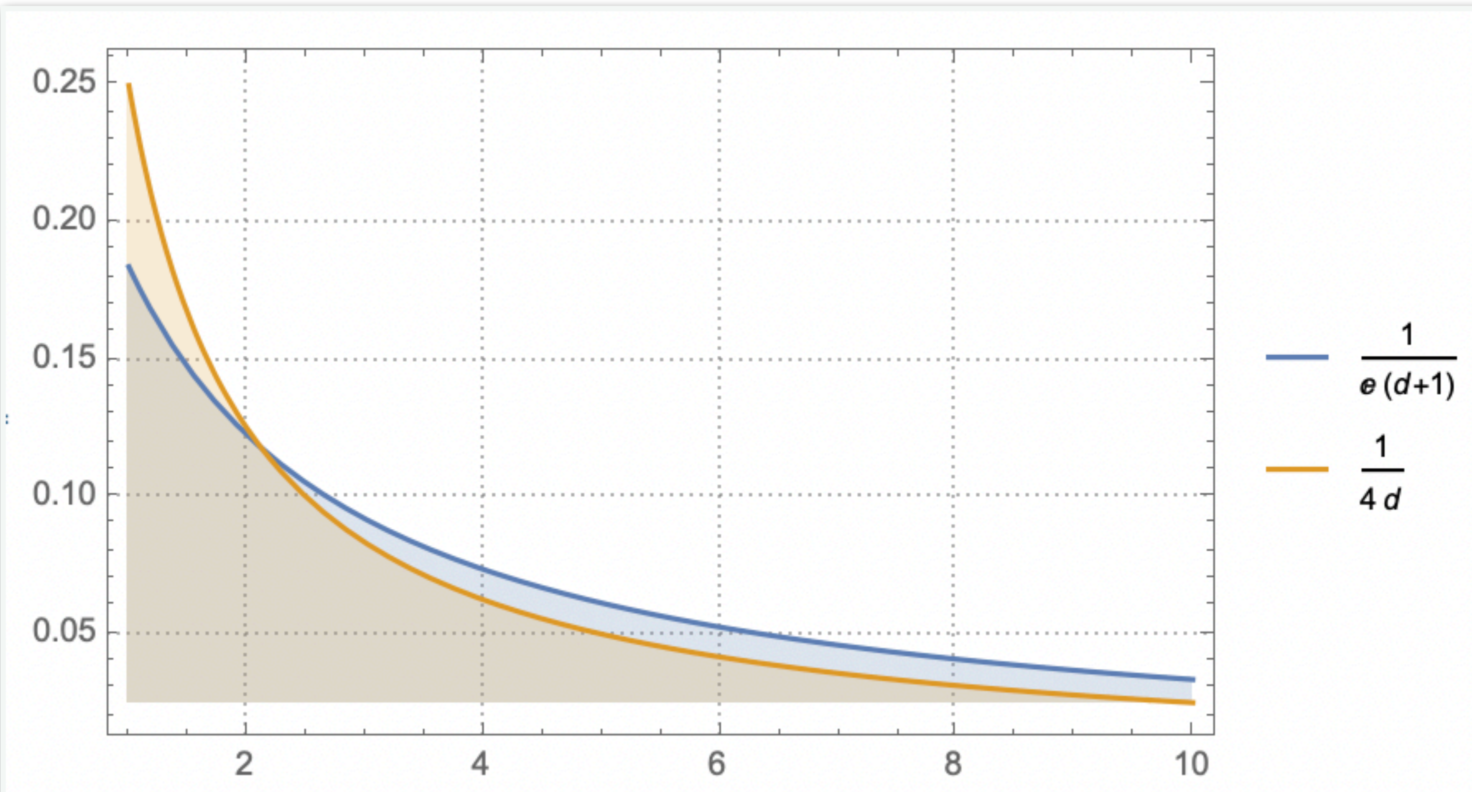

case):

$$\Pr\left[\bigwedge_{i=1}^{m} \overline{A_i}\right] \geq \prod_{i=1}^{m}(1-\alpha_i)$$

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Summary

- LLL establishes the following implication:

$\boxed{\text{LLL condition}} \Longrightarrow$ It's possible that none of $A_1, \ldots, A_m$ occurs

knowing only the probabilities and dependency graph of $A_1, \ldots, A_m$

Summary

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  knowing only the probabilities and dependency graph of $A_1, \ldots, A_m$

- What's next:
  - tight(er) LLL condition: **Shearer's bound**
  - tighter bounds when more (than just local dependency structure) are known: the probabilistic method beyond LLL
  - beyond existence: **algorithmic/constructive LLL**

Algorithmic
Lovász Local Lemma:

The Moser-Tardos Algorithm

Algorithmic LLL (abstract version)

- "Bad" events $A_1, \ldots, A_m$ in a probability space $(\Omega, \Sigma, \Pr)$

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$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$:

$\forall i, \quad \Pr[A_i] \leq \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$

$\Rightarrow$

$\exists \sigma \in \Omega$,

avoid all $A_1, \ldots, A_m$

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- ***Algorithmic*** (constructive) **Lovász Local Lemma:**

Give an efficient algorithm:

find such a good sample $\sigma \in \Omega$ avoiding all bad events $A_1, \ldots, A_m$

Algorithmic LLL (variable version)

Variable framework for LLL (**CSP with independent variables**):

- mutually independent random variables $\mathscr{X} = \{X_1, \ldots, X_n\}$
- bad events $\mathscr{A} = \{A_1, \ldots, A_m\}$, where $A_i \in \mathscr{A}$ is determined by $\mathsf{vbl}(A_i) \subseteq \mathscr{X}$

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$$\Gamma(A_i) = \left\{ A_j \neq A_i \mid \mathsf{vbl}(A_i) \cap \mathsf{vbl}(A_j) \neq \varnothing \right\}$$

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- ***Algorithmic*** (constructive) **Lovász Local Lemma:**

Give an efficient algorithm (CSP solver):

find such a good evaluation of $X_1, \ldots, X_n$ avoiding all bad events $A_1, \ldots, A_m$

The Moser-Tardos Algorithm

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Theorem [Moser-Tardos 2010]:

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The Moser-Tardos algorithm terminates within $\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_i}$ resamples in expectation

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Theorem [Moser-Tardos 2010]**:**

$p \triangleq \max_i \Pr[A_i],\ d \triangleq \max_i |\Gamma(A_i)|$

$$\mathrm{e}p(d+1) \le 1$$

$\Rightarrow$

The Moser-Tardos algorithm terminates within m/d resamples in expectation

Execution Log

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Resampling Table

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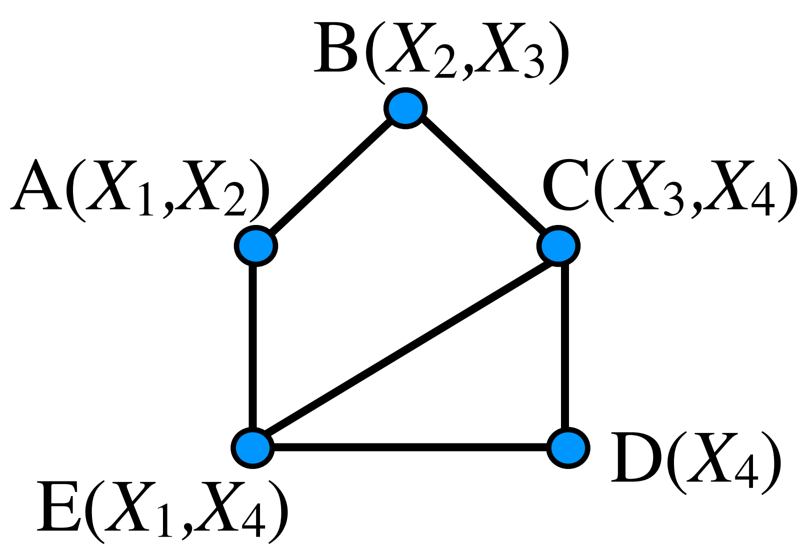

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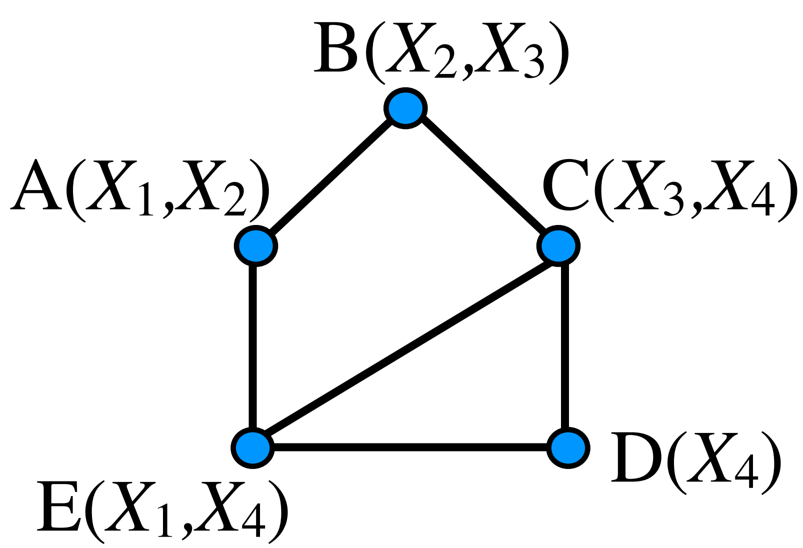

$X_j^{(t)}$: t-th sampling of variable X_j

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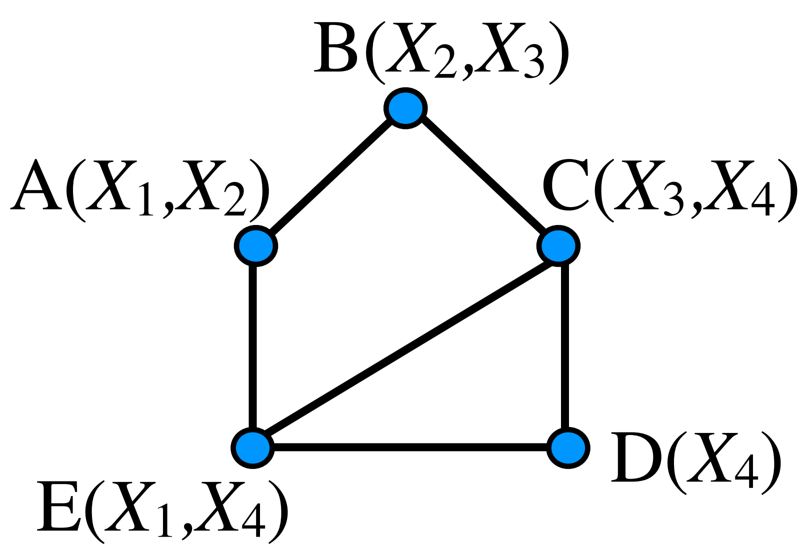

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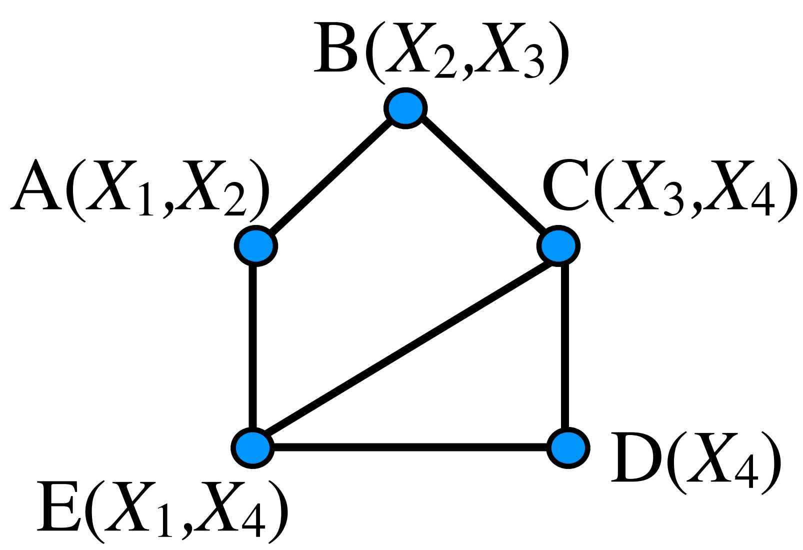

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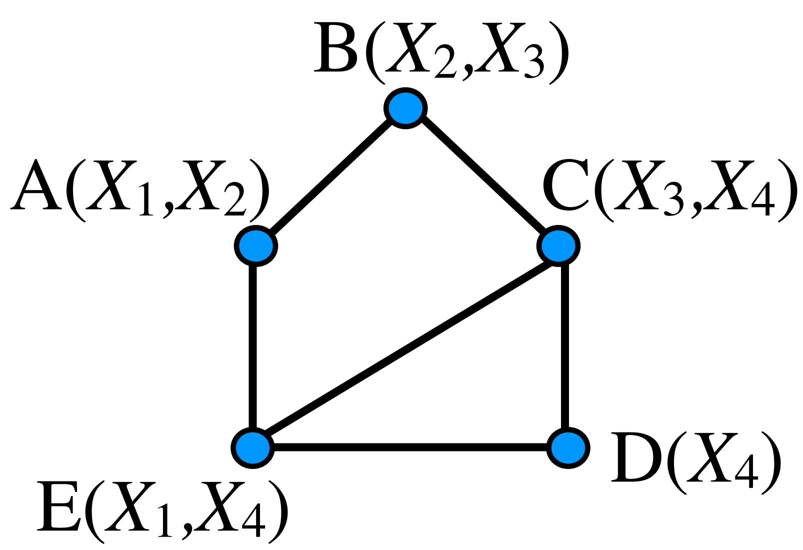

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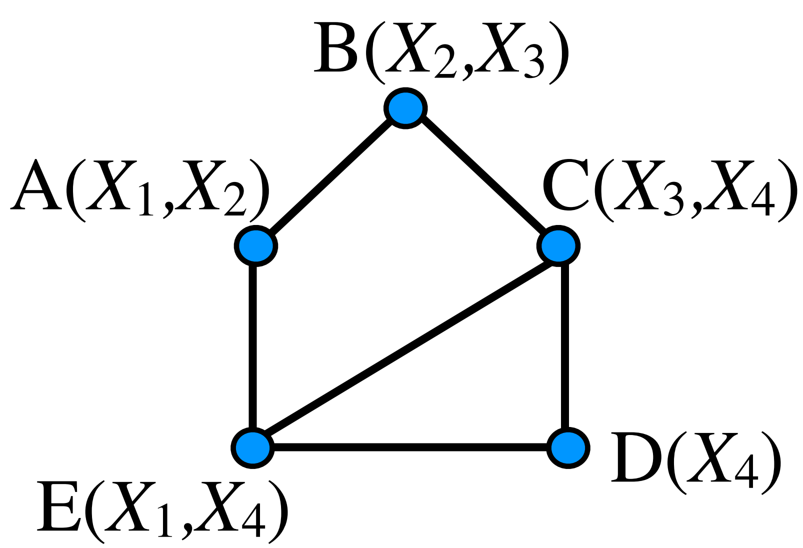

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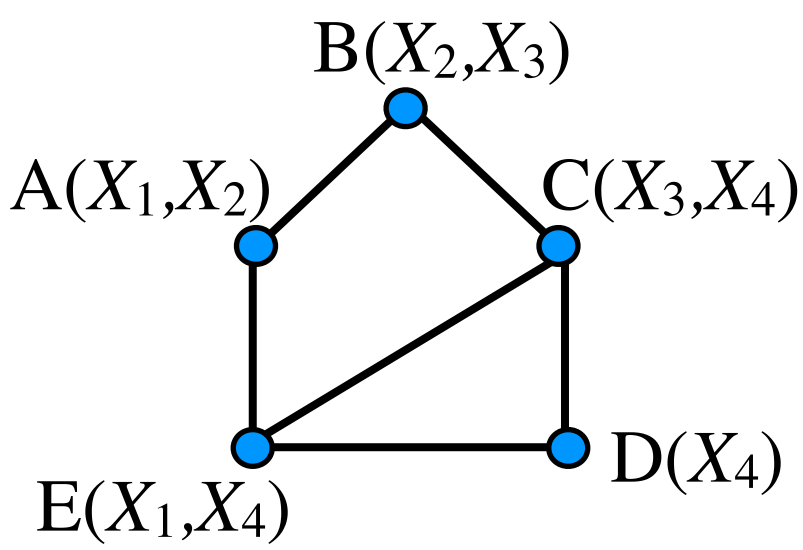

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\end{array}
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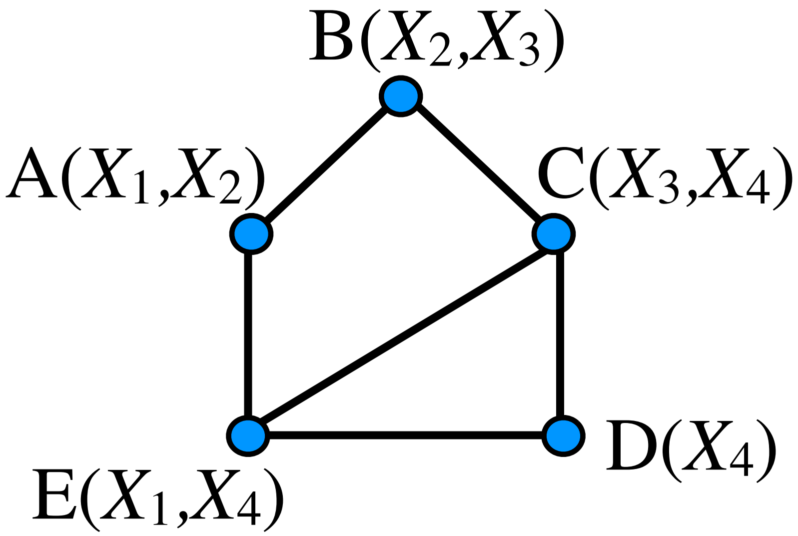

$X_j^{(t)}$: t-th sampling of variable X_j

Resampling Table

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;
while $\exists$ a bad event A_i that occurs:
resample all $X_j \in \mathsf{vbl}(A_i)$;

Exe-log Λ:

$$\Lambda_1, \Lambda_2, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

random sequence of resampled bad events

Exe-log Λ: D,C,E,D,B,A,C,A,D, ...

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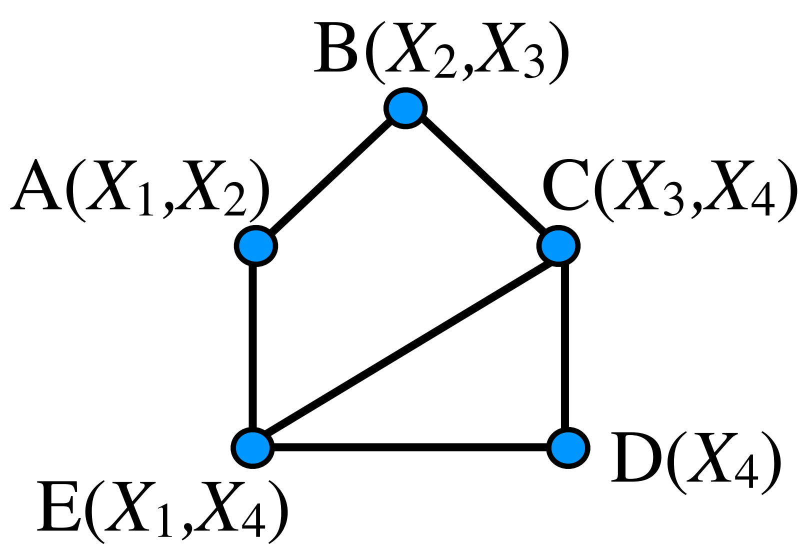

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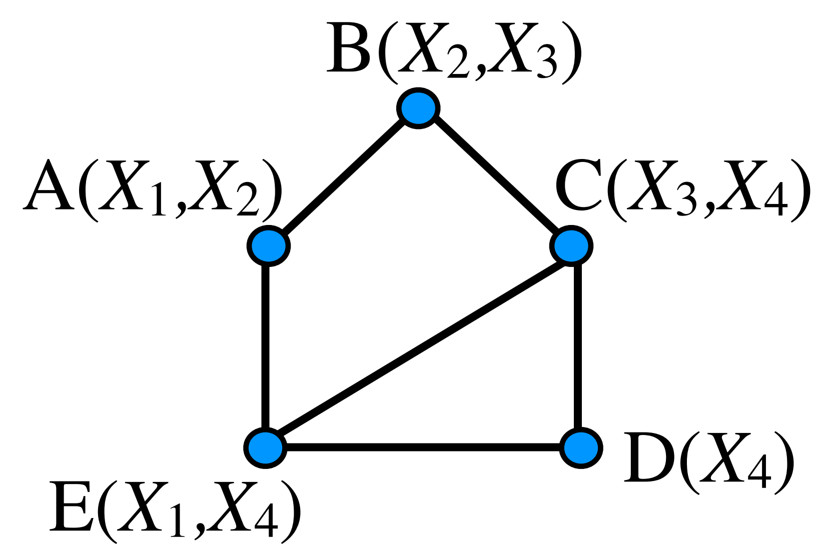

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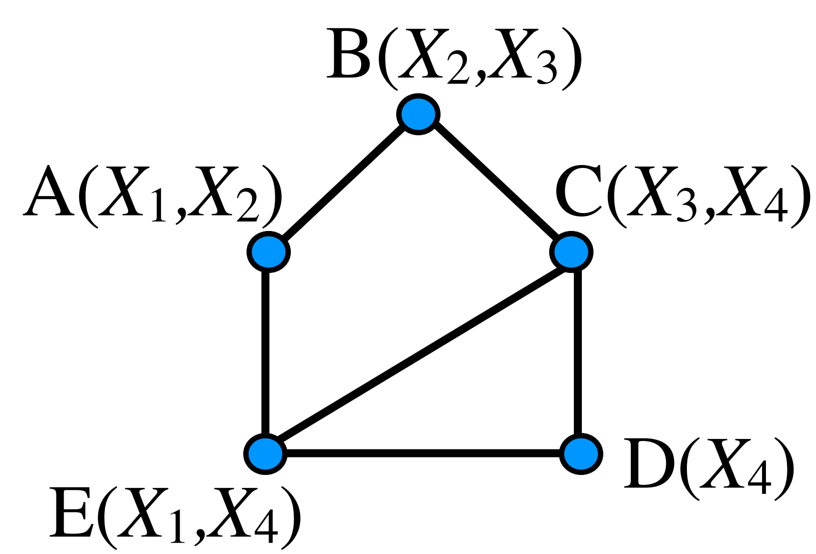

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Witness Tree

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random sequence of resampled bad events

Witness tree $T(\Lambda, t)$: each node u with label $A_{[u]} \in \mathscr{A}$, siblings have *distinct* labels

- initially, T contains a single **root** r with Λ_t
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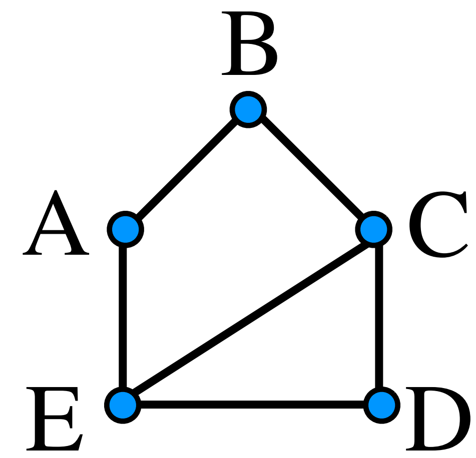
- $T(\Lambda, t)$ is the resulting T

Inclusive neighborhood: $\Gamma^+(A) \triangleq \Gamma(A) \cup \{A\}$

dependency graph:



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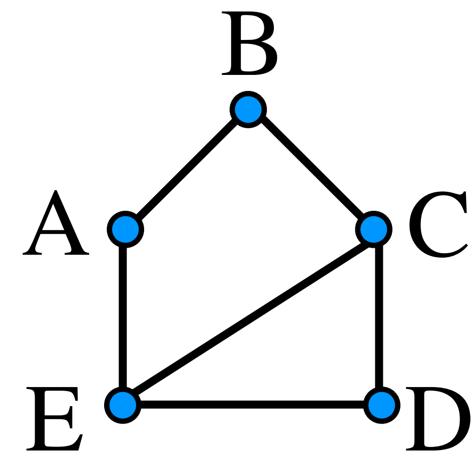

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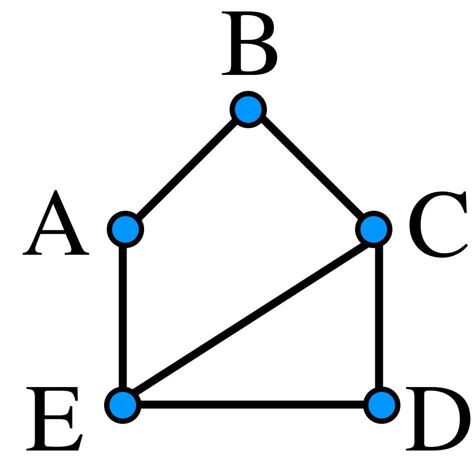

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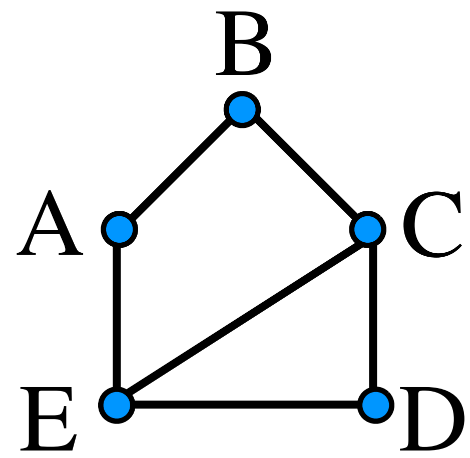

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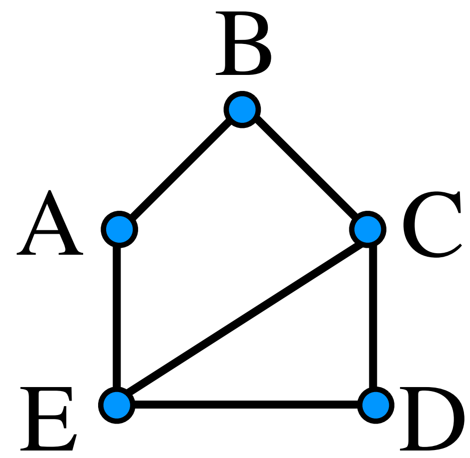

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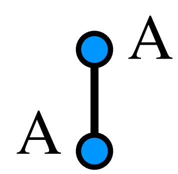

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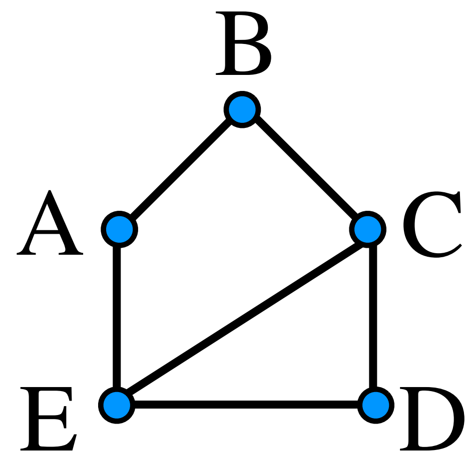

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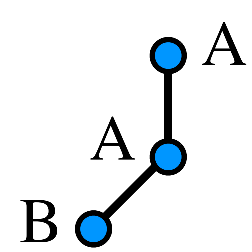

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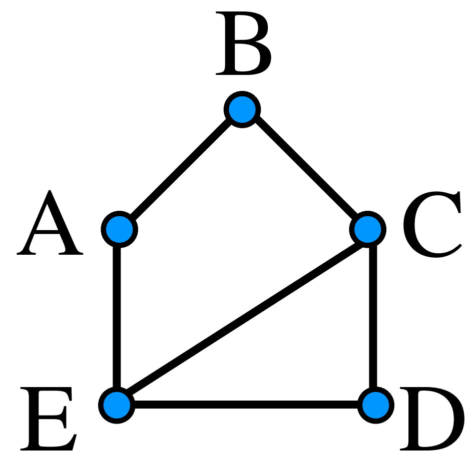

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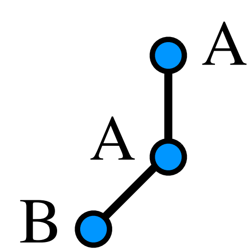

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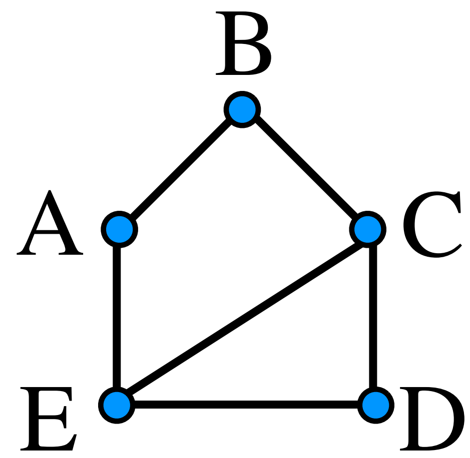

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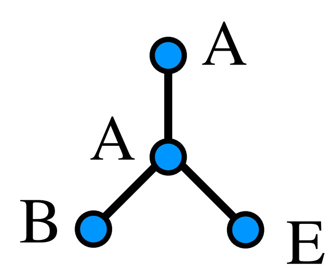

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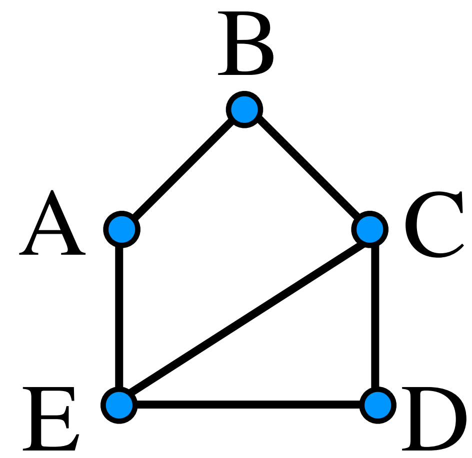

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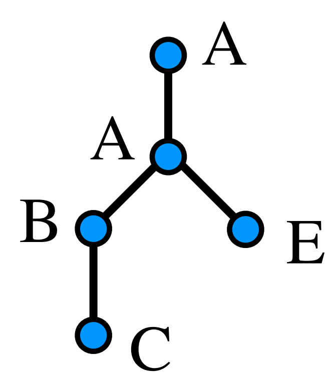

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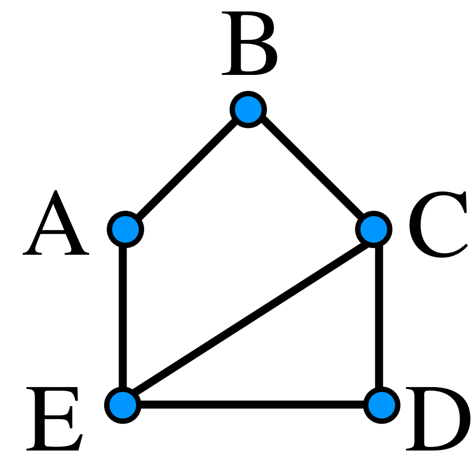

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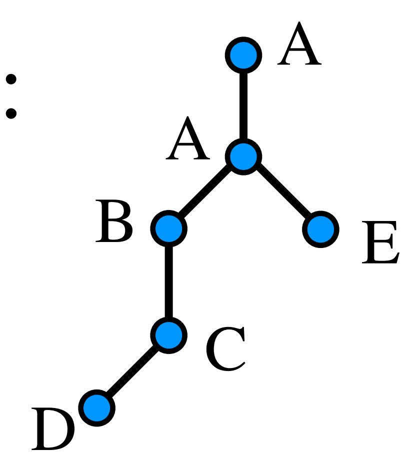

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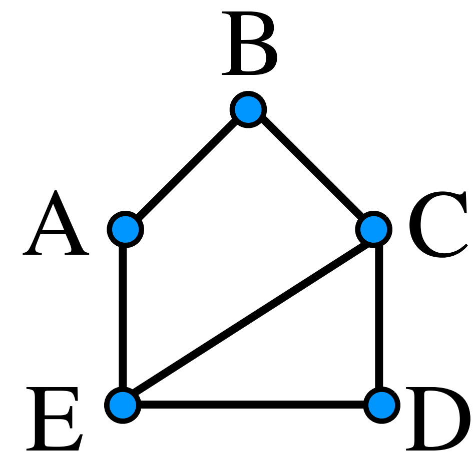

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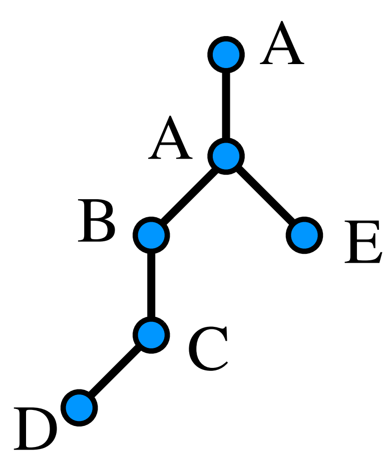

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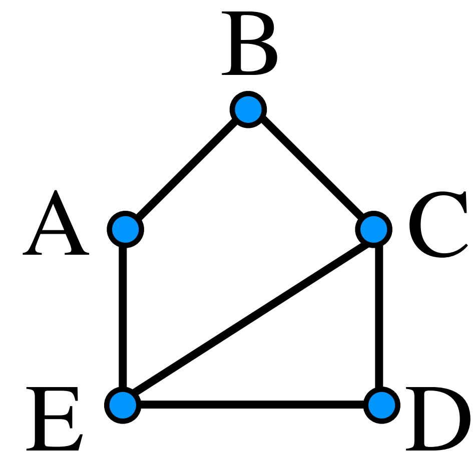

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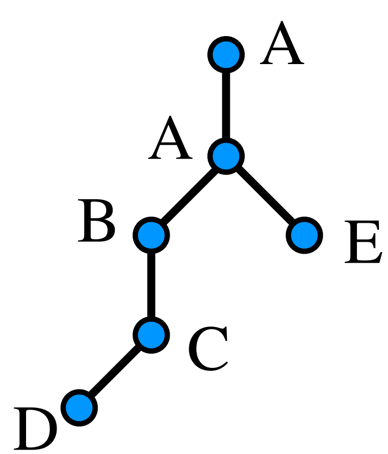

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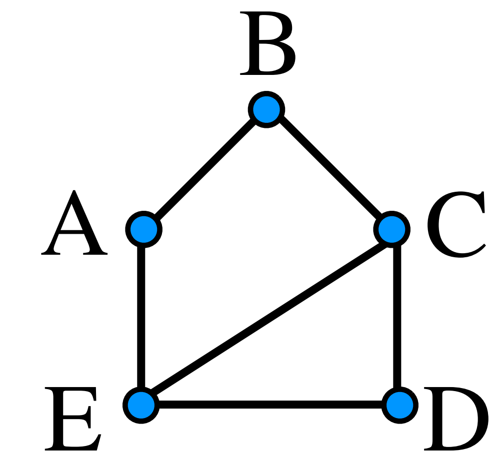

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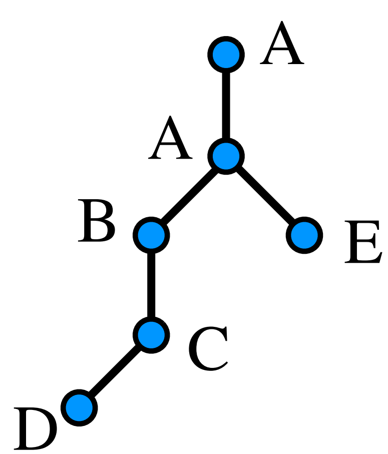

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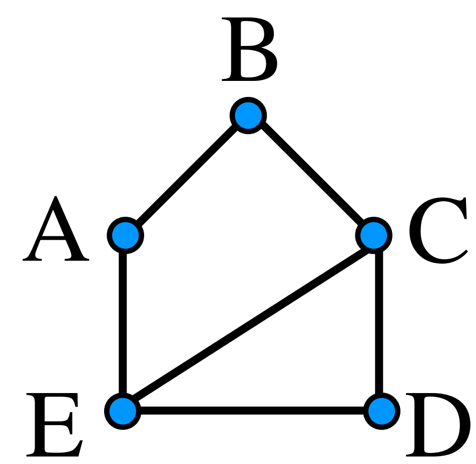

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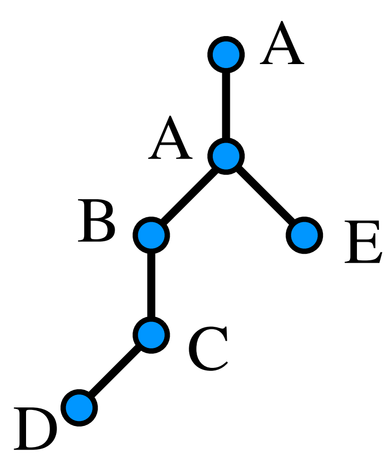

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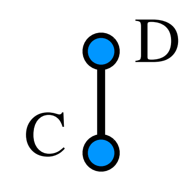

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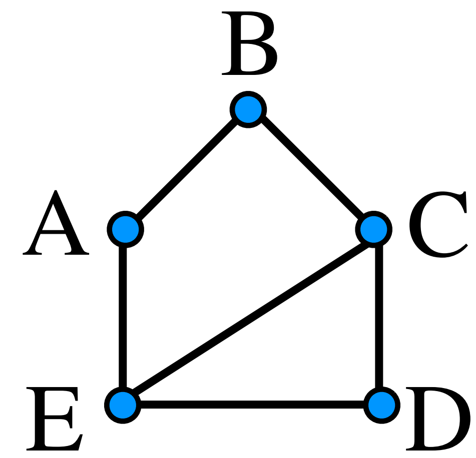

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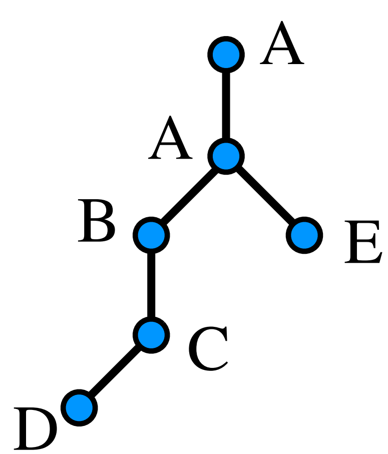

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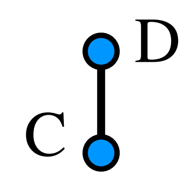

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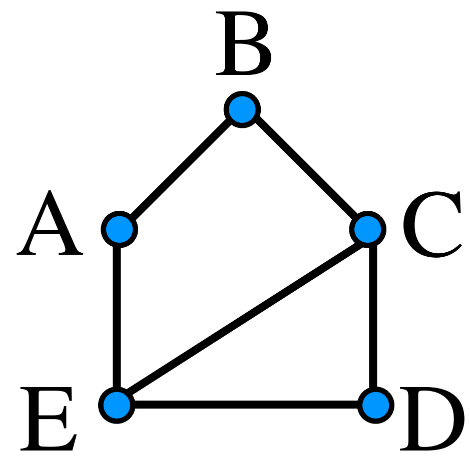

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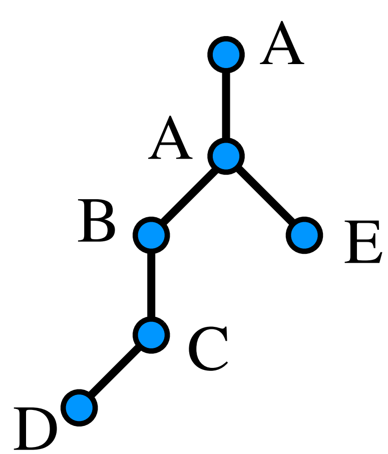

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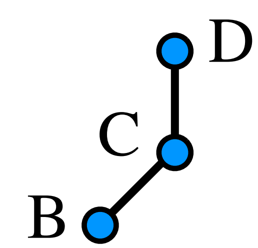

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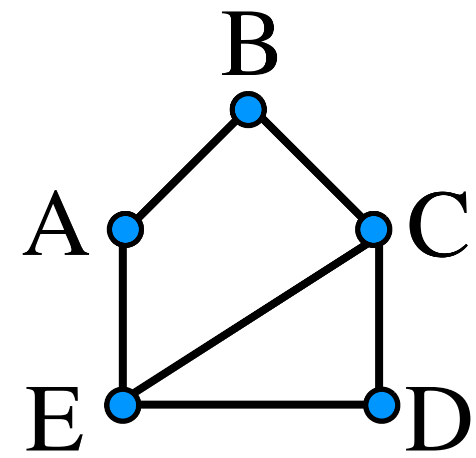

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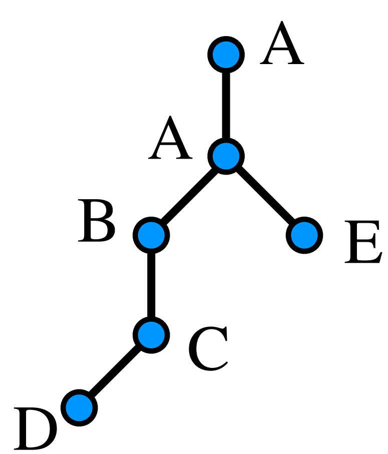
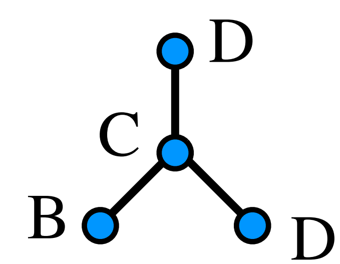


$T(\Lambda, 9)$:



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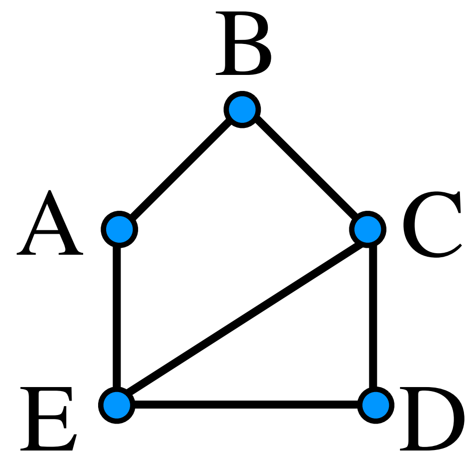

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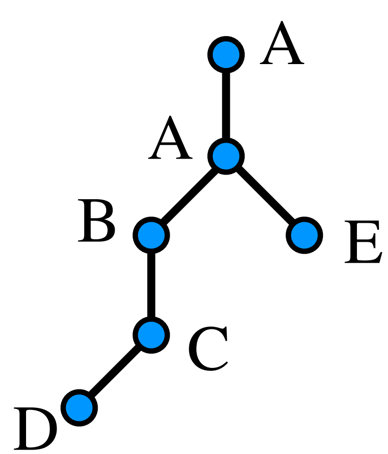

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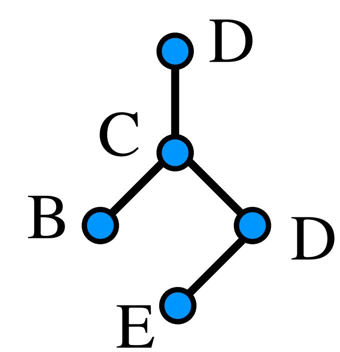

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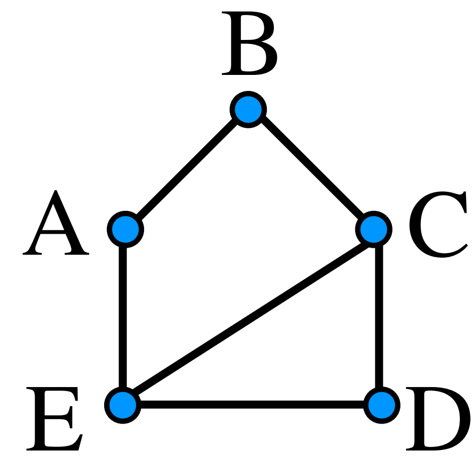

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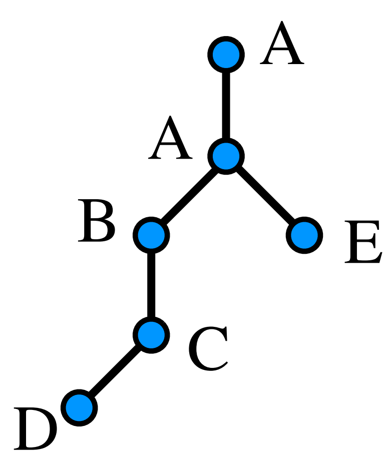

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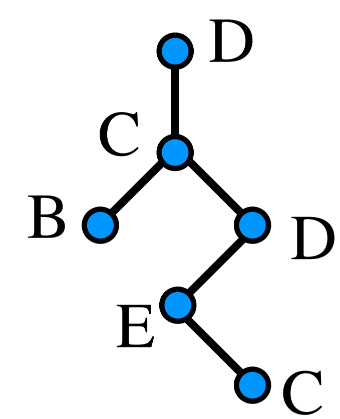

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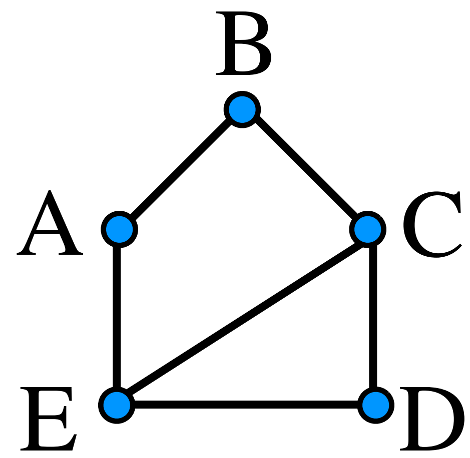

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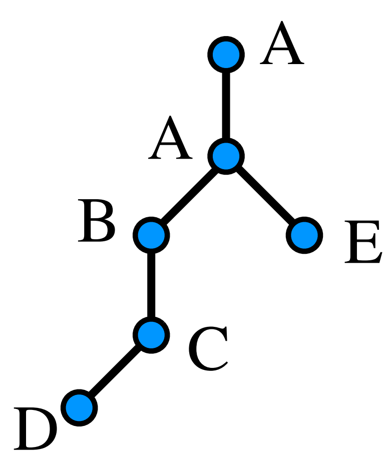

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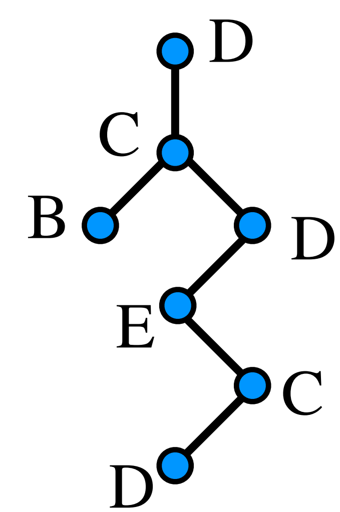

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Each node $u \in \tau$ independently does an experiment of its label-event $A_{[u]}$.
The process succeeds if all $A_{[u]}$ occurs.

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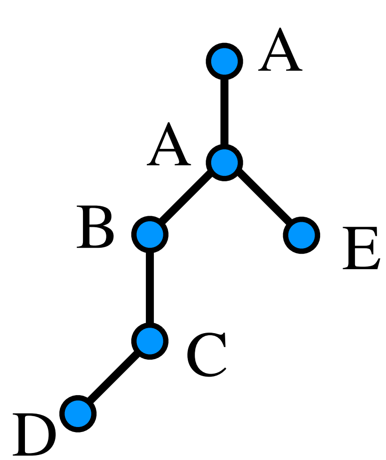

$$\Pr[\text{ simulation of } \tau \text{ succeeds }]$$
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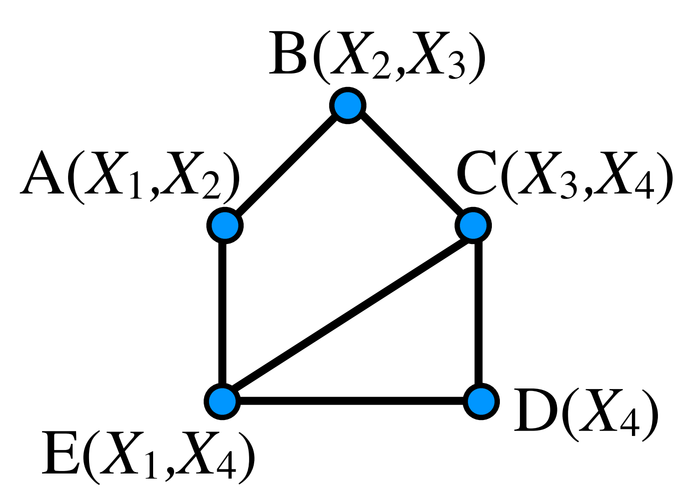

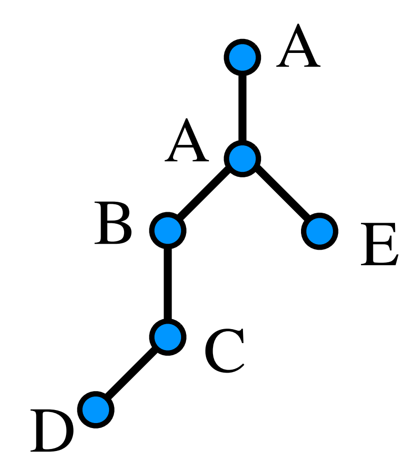

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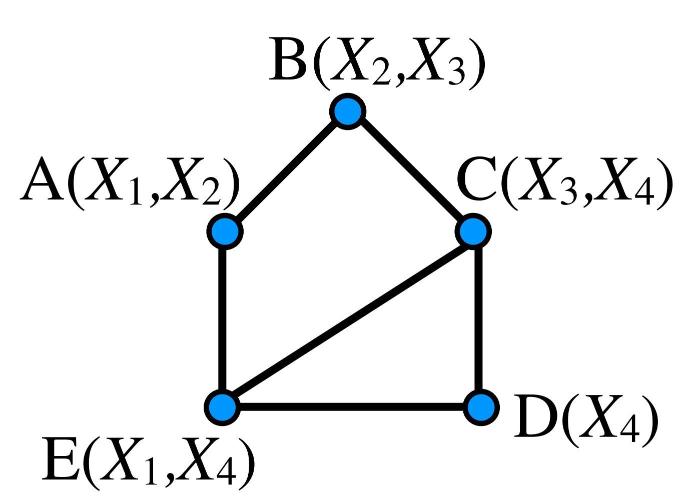

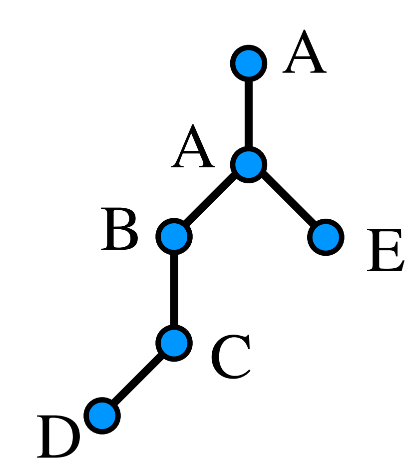

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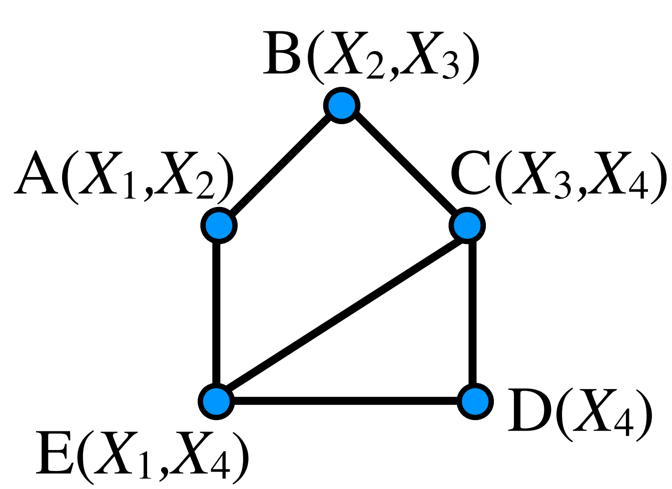

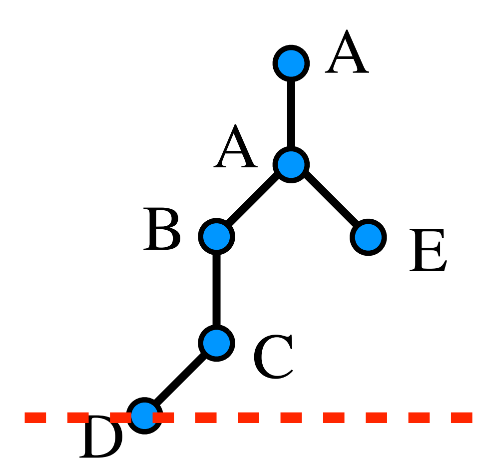

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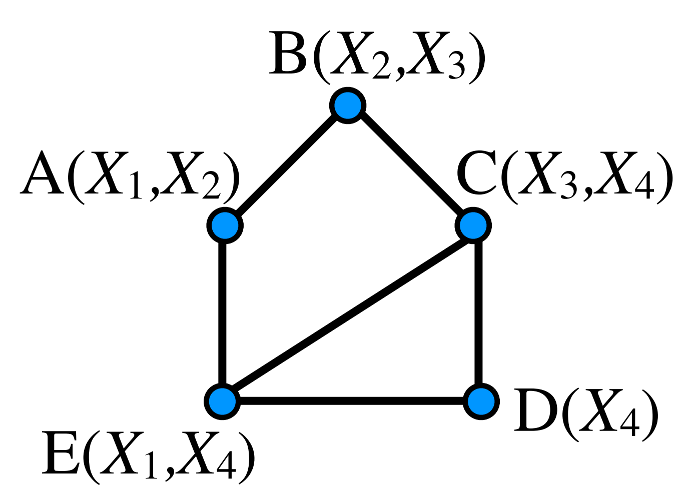

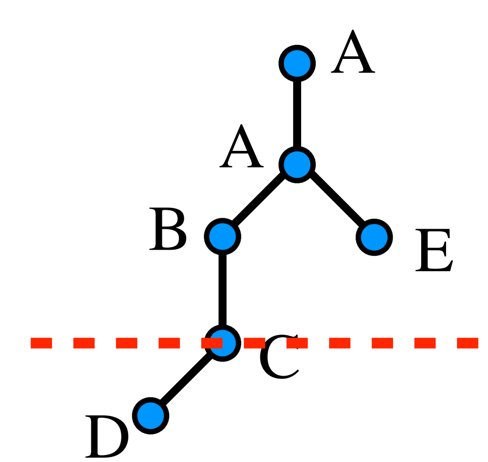

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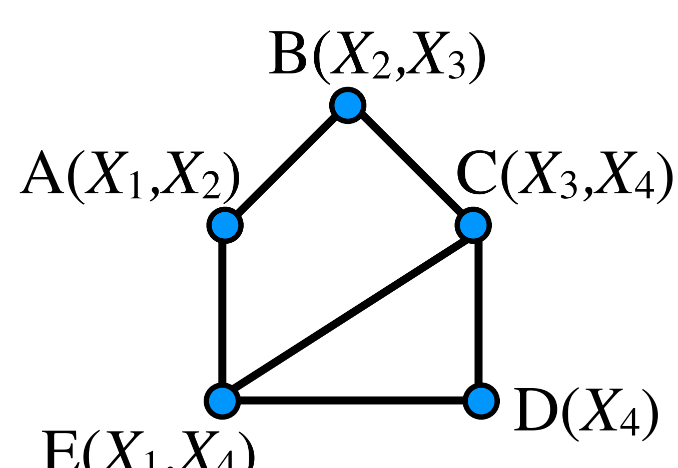

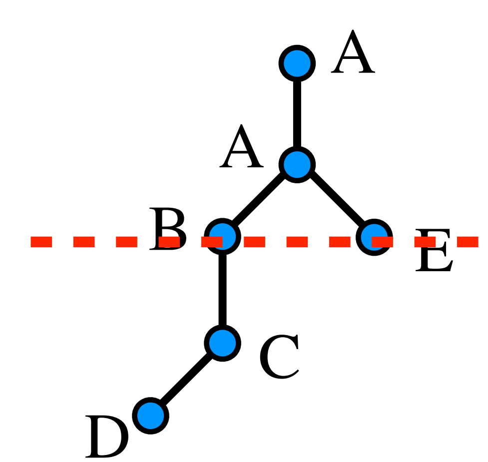

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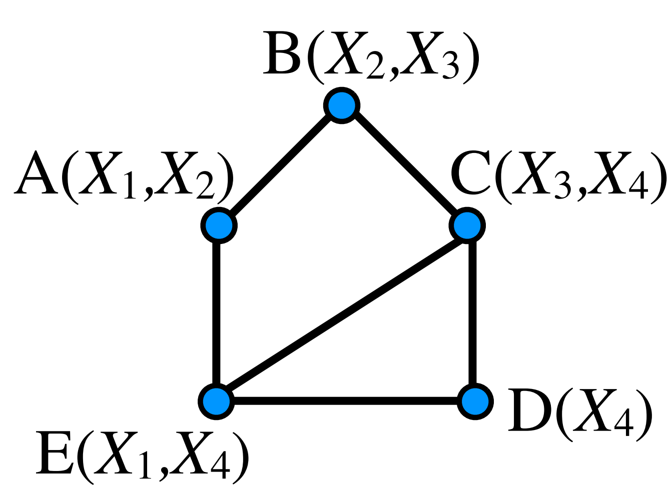

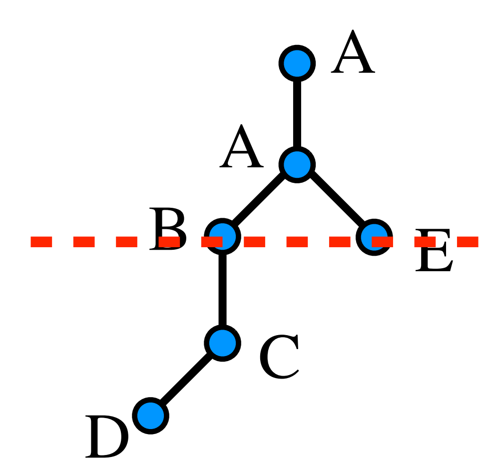

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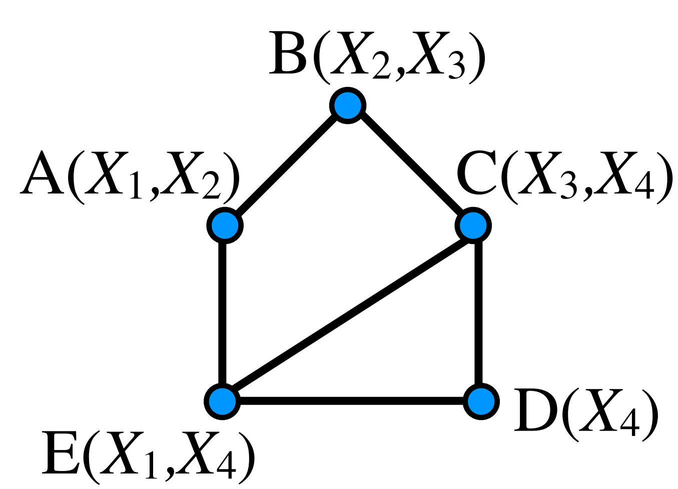

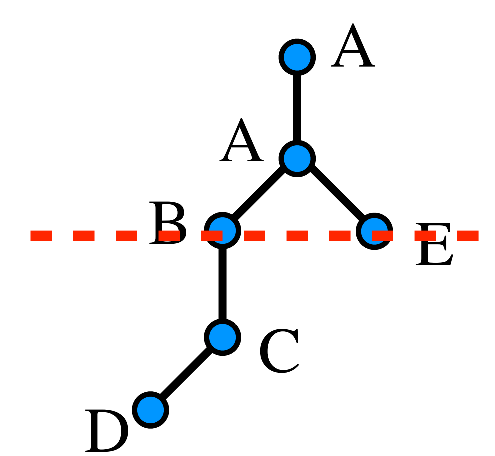

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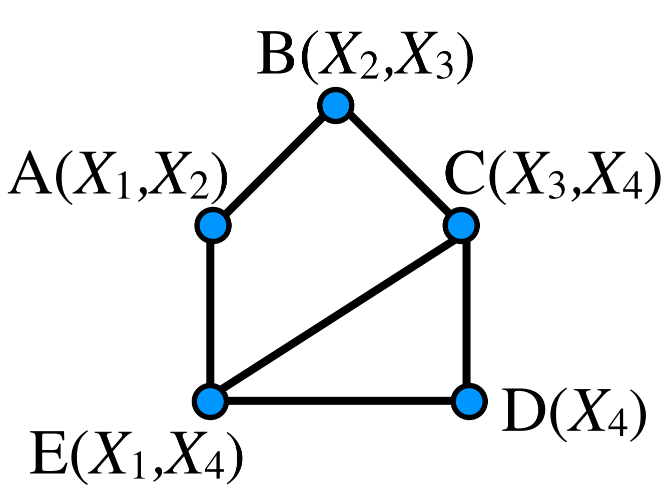

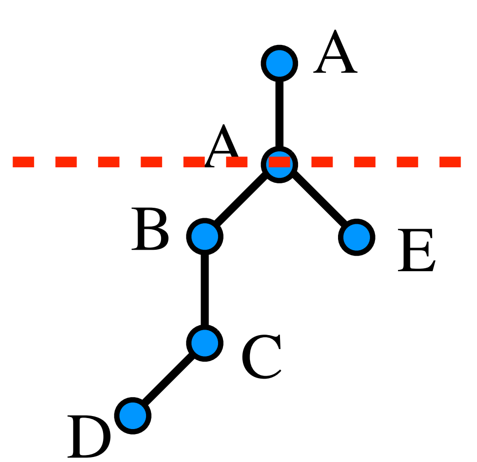

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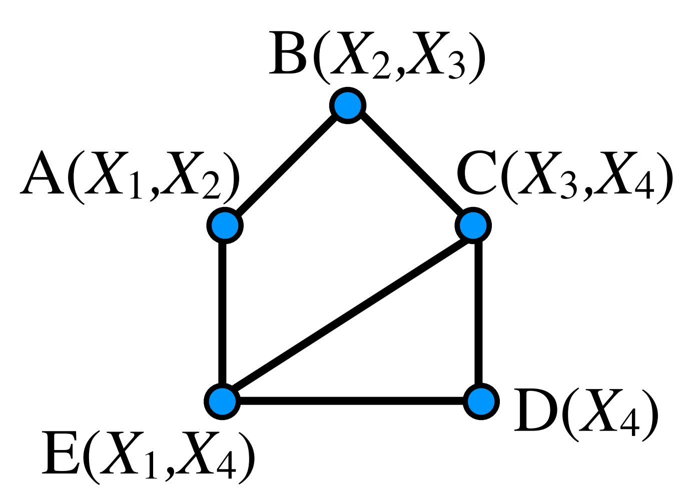

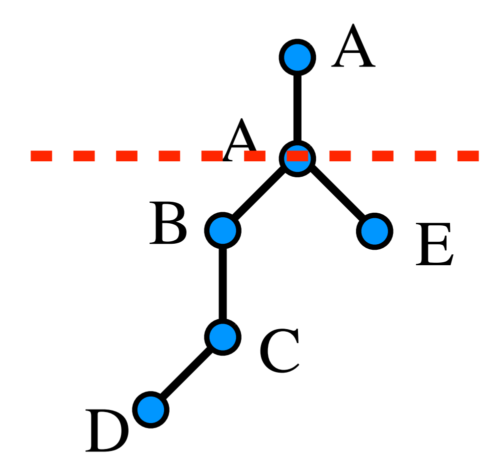

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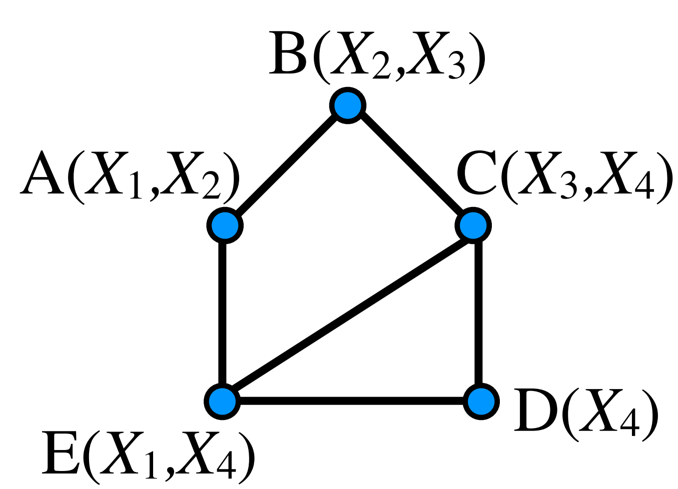

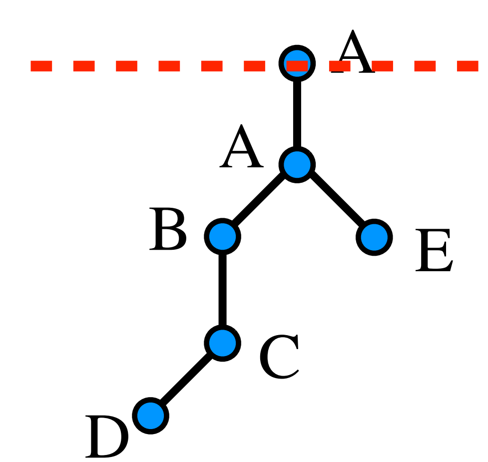

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$$\mathbb{E}_{\Lambda}\left[\#\text{ of } A_i \text{ in } \Lambda\right] = \sum_{\tau \in \mathscr{T}_{A_i}} \Pr_{\Lambda}\left[\exists t, T(\Lambda, t) = \tau\right]$$

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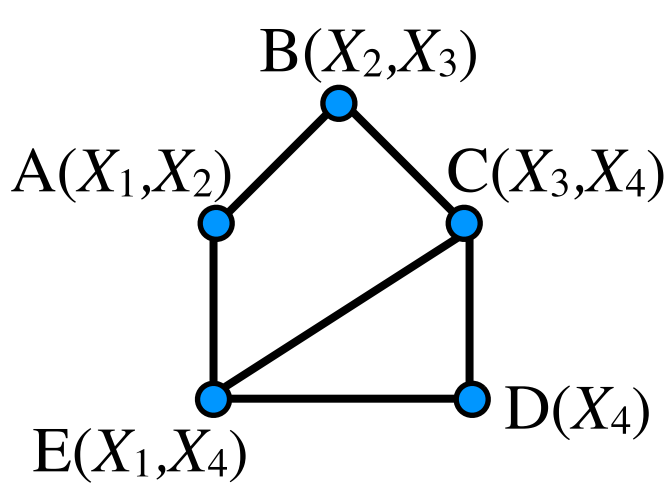
- Can generate all possible witness trees $\in \mathscr{T}_A$



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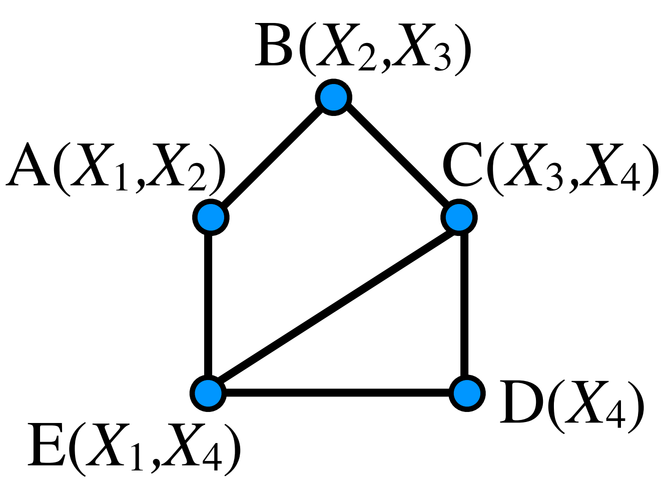

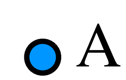

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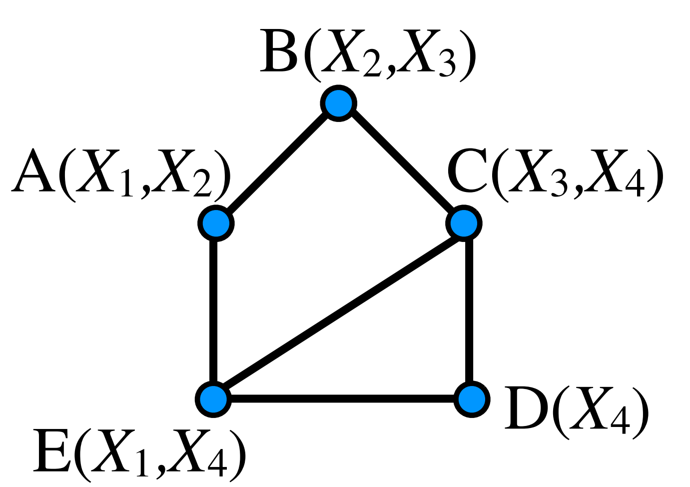

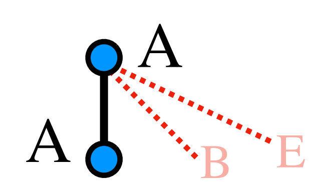

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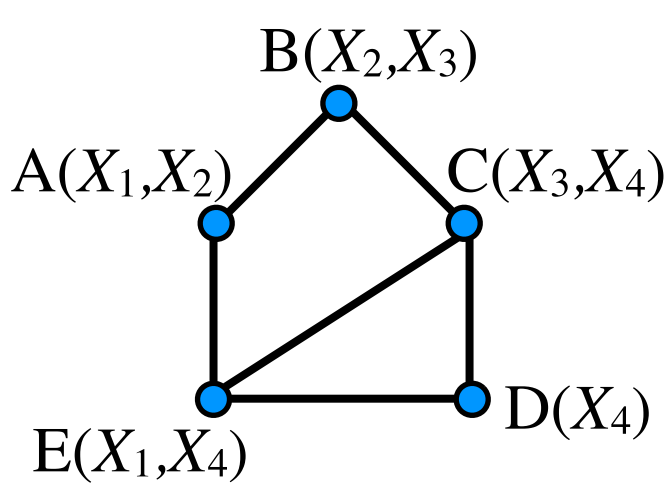

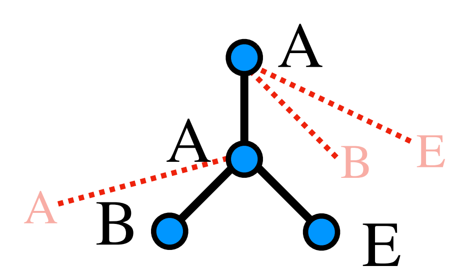

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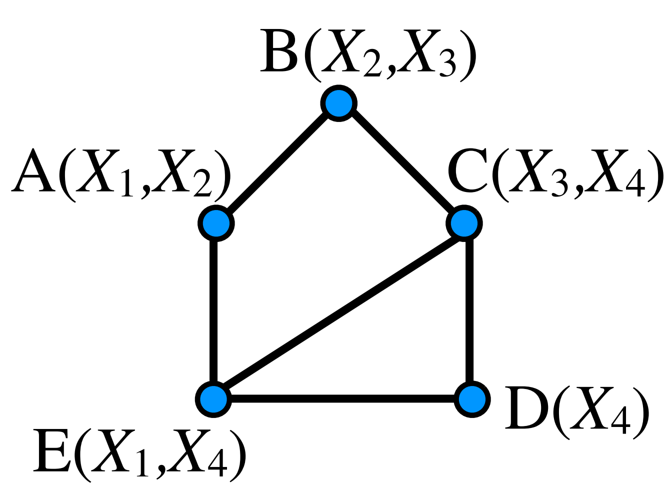

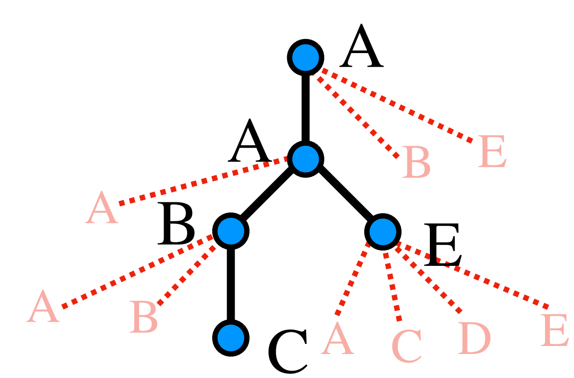

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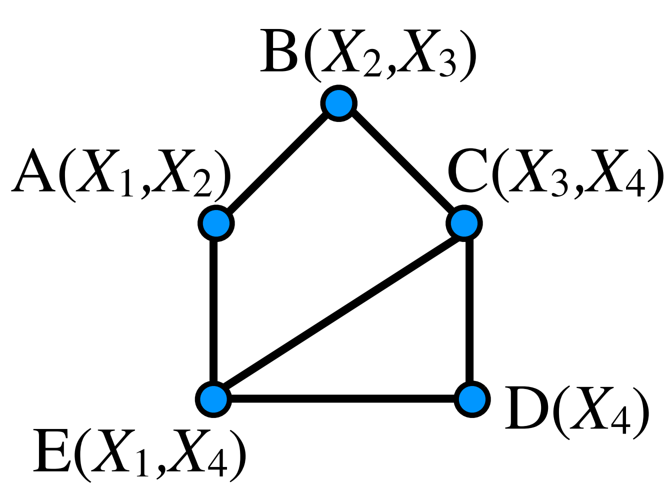

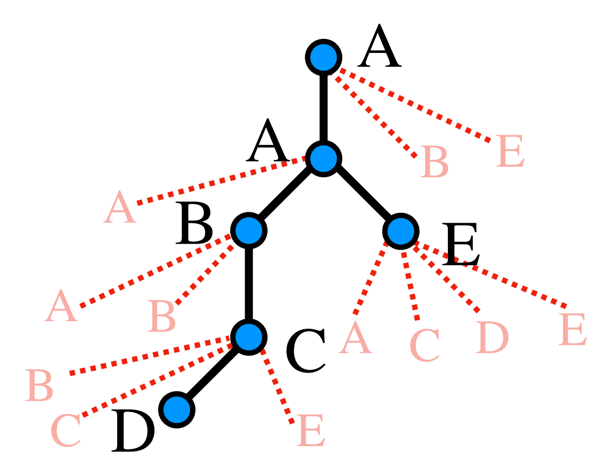

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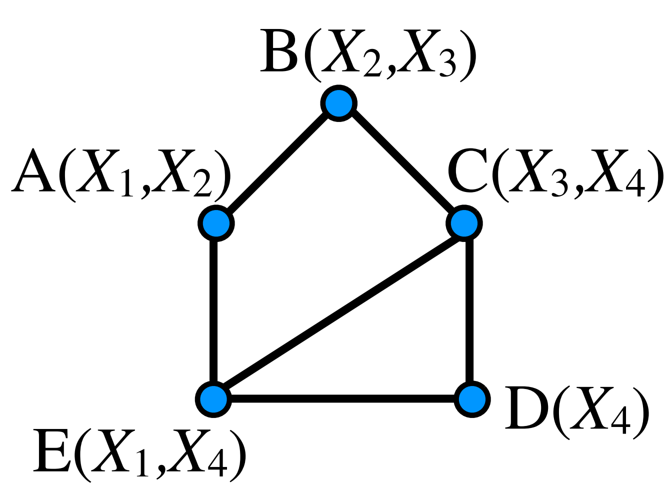

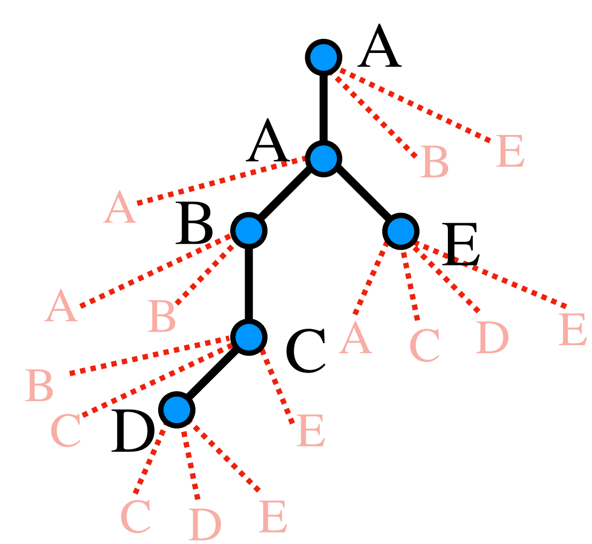

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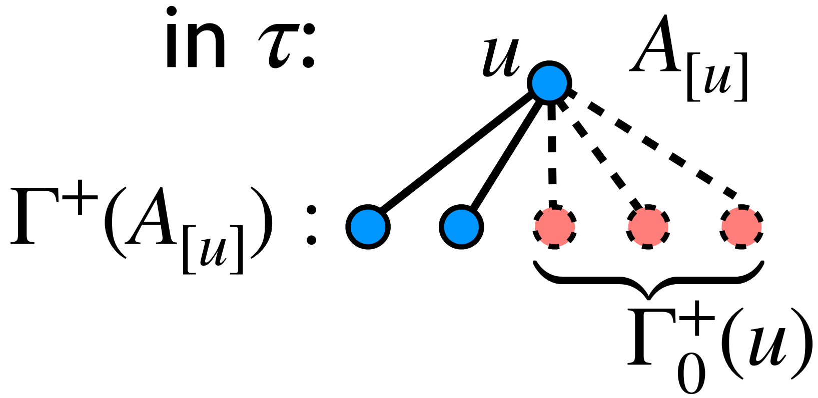

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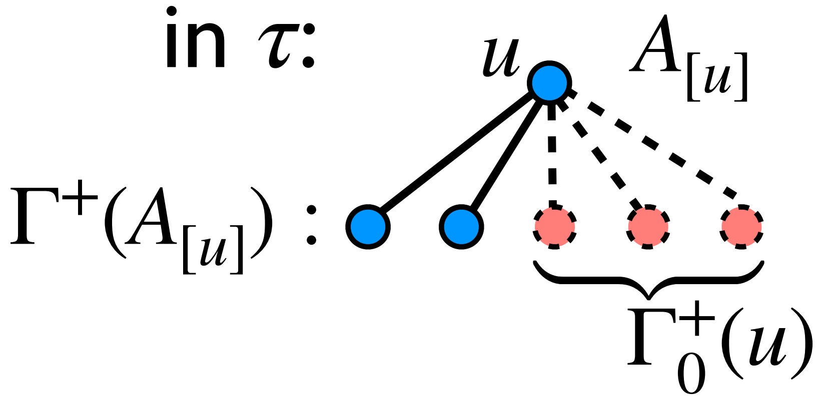

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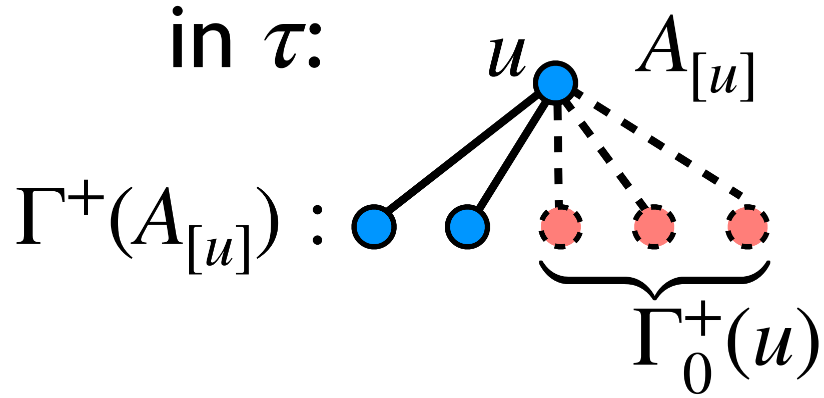

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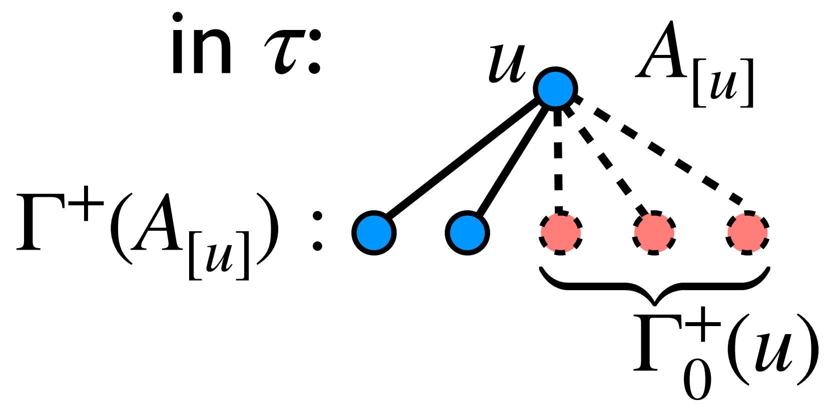

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draw independent samples of $X_1, \dots, X_n$;

while $\exists$ a bad event A_i that occurs:

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Exe-log Λ:

$$\Lambda_1, \Lambda_2, \ldots \in \mathscr{A} = \{A_1, \ldots, A_m\}$$

random sequence of resampled bad events

LLL condition: $\exists \alpha_1, \ldots, \alpha_m \in [0,1): \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$

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$\mathscr{T}_{A_i}$: set of all witness trees with root-label A_i

(Lemma 1) $$\le \sum_{\tau \in \mathscr{T}_{A_i}} \prod_{u \in \tau} \Pr(A_{[u]})$$

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The Moser-Tardos Algorithm

Variable framework for LLL (**CSP with independent variables**):

- mutually independent random variables $\mathscr{X} = \{X_1, \ldots, X_n\}$
- bad events $\mathscr{A} = \{A_1, \ldots, A_m\}$, where $A_i \in \mathscr{A}$ is determined by vbl$(A_i) \subseteq \mathscr{X}$

Moser-Tardos Algorithm:

draw independent samples of $X_1, \ldots, X_n$;

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Theorem [Moser-Tardos 2010]**:**

$\exists \alpha_1, \ldots, \alpha_m \in [0,1)$:

$$\forall i, \quad \Pr[A_i] \le \alpha_i \prod_{A_j \in \Gamma(A_i)} (1 - \alpha_j)$$

$\Rightarrow$

The Moser-Tardos algorithm terminates within $\sum_{i=1}^{m} \frac{\alpha_i}{1-\alpha_i}$ resamples in expectation

Algorithmic Lovász Local Lemma:

Moser's Algorithm and Entropic Proof

k-SAT

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \overbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}^{k \text{ variables}}$$

- dependency degree d: each clause C_i intersects $\leq d$ other clauses
- uniform independent

  $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$

- bad event A_i: $p = 2^{-k}$

  C_i is violated

Moser-Tardos Algorithm:

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Theorem [Moser-Tardos 2010]**:**

$$d \leq 2^{k-2} \Leftrightarrow 4pd \leq 1 \quad \Longrightarrow$$

The Moser-Tardos algorithm terminates after $O(m/d)$ iterations in expectation

Moser's *Fix-It* Algorithm

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- Inclusive neighborhood:

$$\Gamma^+(C_i) \triangleq \Gamma(C_i) \cup \{C_i\} = \left\{ C_j \mid \mathrm{vbl}(C_j) \cap \mathrm{vbl}(C_i) \neq \varnothing \right\}$$

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- **Correctness**: any clause is fixed at most once in top-level

top-level **Fix**(C_i) returned $\Longrightarrow$ C_i remains satisfied

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$d < 2^{k-3} \implies$ total # of calls to Fix() is $O(m \log m + \log n)$ with high probability

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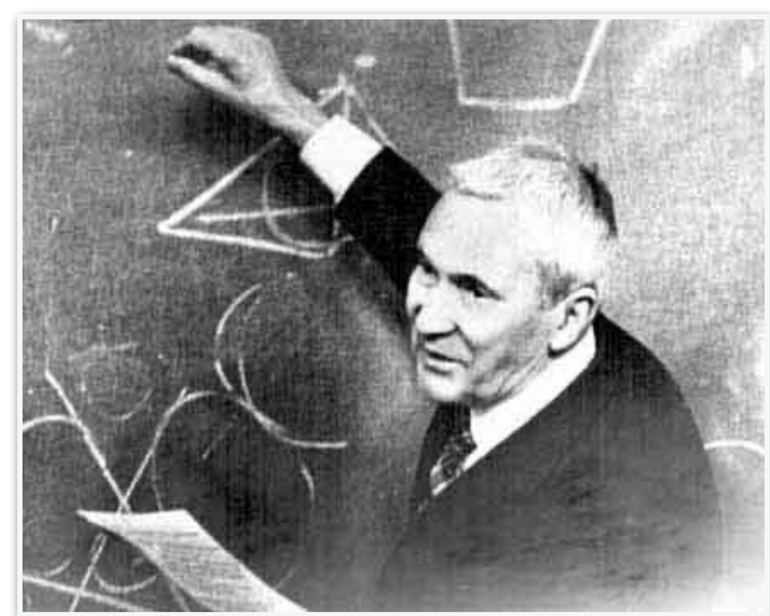
Incompressibility Principle

"**Lossless compression of random data is impossible.**"

Incompressibility Principle:

For any injective function $\mathtt{Enc} : \{0,1\}^N \xrightarrow{\text{1-1}} \{0,1\}^*$, for uniform random $s \in \{0,1\}^N$, for any integer $l > 0$,

$$\Pr\left[\text{length of } \mathtt{Enc}(s) \leq N - l\right] < 2^{1-l}$$



Andrey Kolmogorov
(1903–1987)

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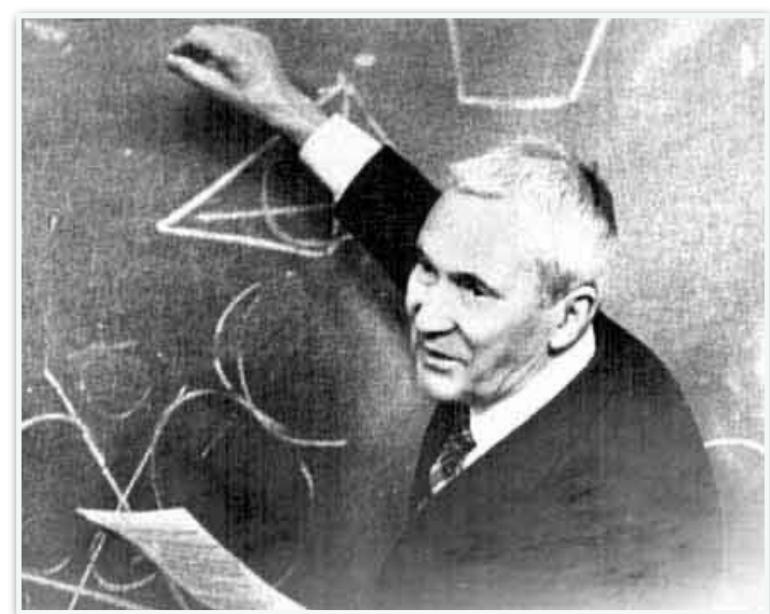
$$\left|\{0,1\}^N\right| = 2^N$$

# of strings of $\le N - l$ bits

$$= \sum_{i \le N-l} 2^i < 2^{N-l+1}$$



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- k-CNF $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ with dependency degree $d < 2^{k-3}$

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- t calls to **Fix()** $\Longrightarrow$ we can compress random bits:

$n + tk$ random bits $\xrightarrow{\mathtt{Enc}_\Phi}$ $\leq n + O(m \log m) + t(\lceil \log_2(d+1) \rceil + 2)$ bits

Simulate(Φ, t):

Run Moser's Algorithm on k-CNF Φ for up to t calls to Fix();

printf("$X_1 \ldots X_n$"); `//the current` $X_1, \ldots, X_n$ `in Moser's algorithm`

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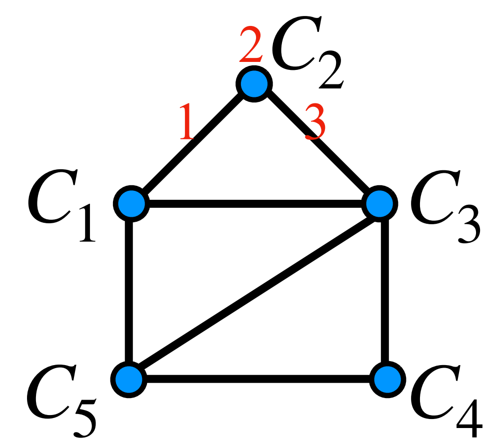

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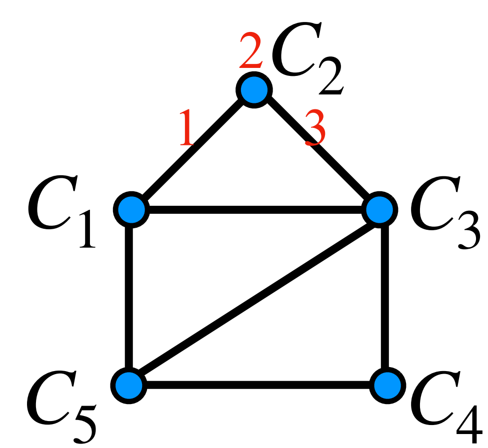

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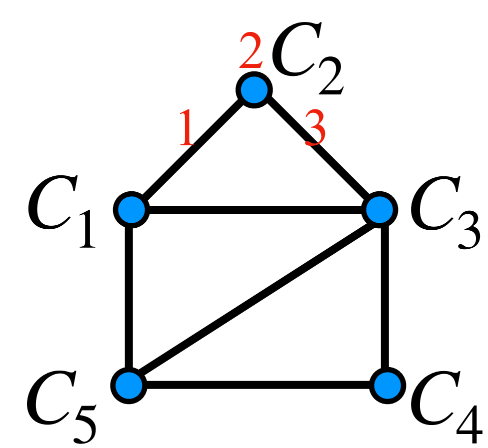

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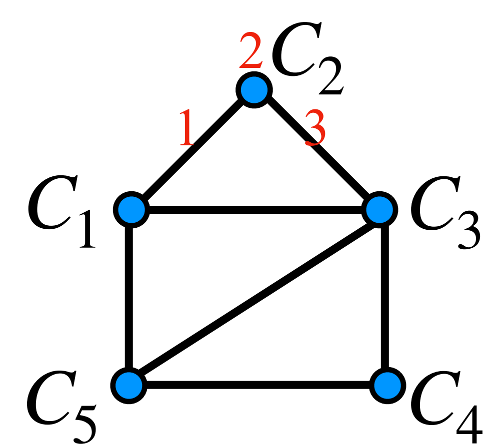

- **Recursion trees** + **final assignment**

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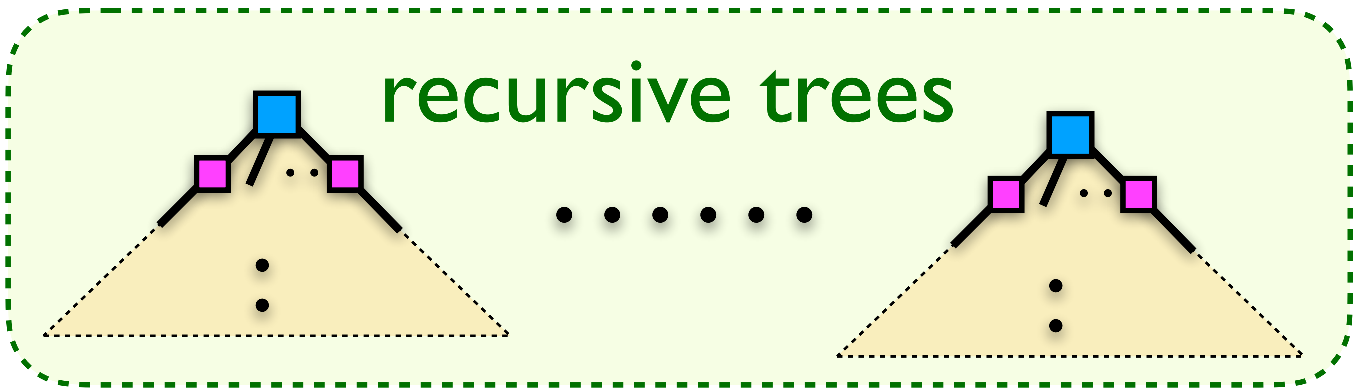

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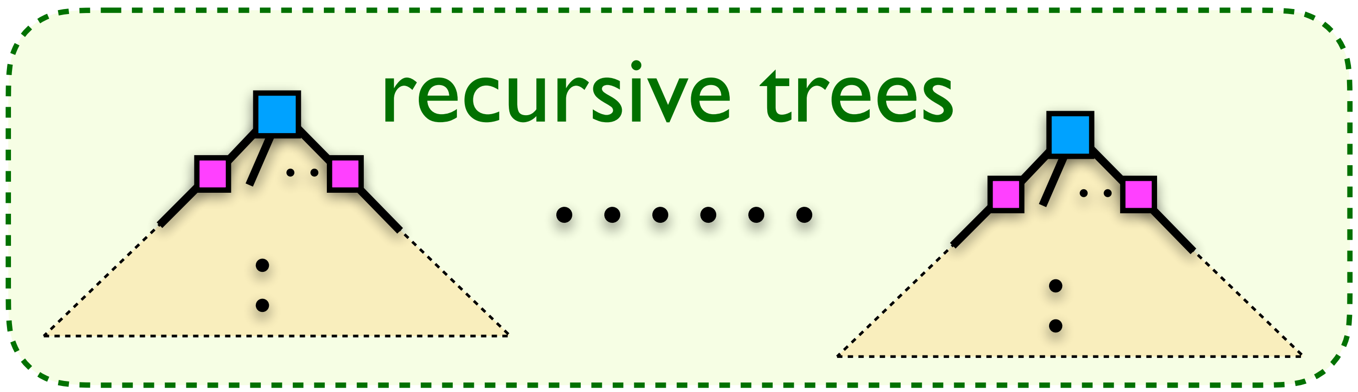

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$X_1, \ldots X_n$

Observation: Fix(C_i) is called $\Longrightarrow$ C_i is violated at the moment
$\Longrightarrow$ current values of all $X_j \in$ vbl(C_i) is uniquely determined

- **Output string**:

  (98(2(1(1)(3))(3))(4(2)))(126(3(2⋯

  ⋯⋯⋯(110100101010110111

  t calls to Fix() $X_1, \ldots, X_n$

- **Recursion trees** + **final assignment**

Moser's Algorithm:

draw uniform $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

while $\exists$ a violated clause C_i :

  Fix(C_i);

Fix(C_i)**:**

resample all variables in vbl(C_i);

while $\exists$ violated $C_j \in \Gamma^+(C_i)$:

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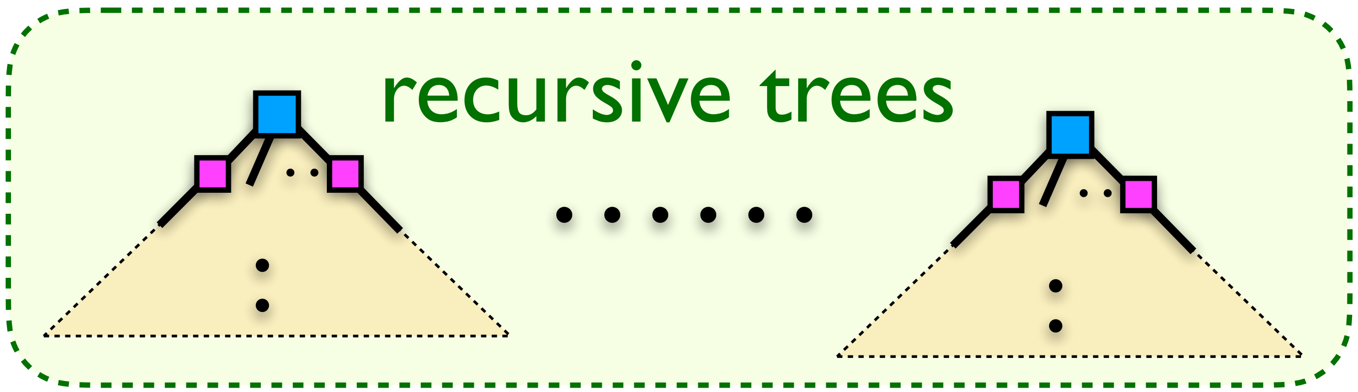

+

final assignment

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Recover

$n + tk$ random bits used by the algorithm

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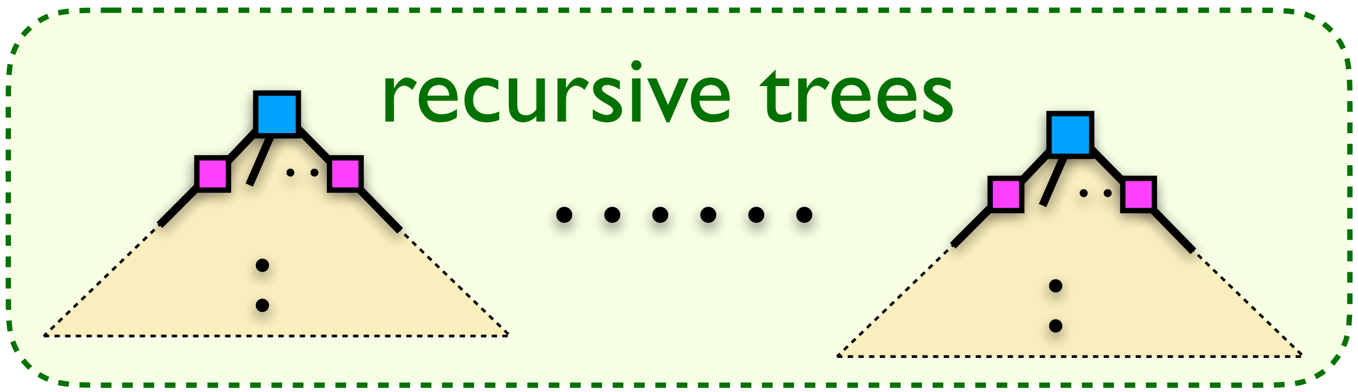

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- **Output string**:

  $\underbrace{\text{(98(2(1(1)(3))(3))(4(2)))(126(3(2}\cdots\cdots\cdots}_{t \text{ calls to Fix()}}\underbrace{\text{(11010010101 0110111}}_{X_1, \ldots, X_n}$

1-1 mapping

$\xleftarrow{\mathtt{Enc}_\Phi}$

$n + tk$ random bits used by the algorithm

- **Recursion trees** + **final assignment**

Simulate(Φ, t):

Run Moser's Algorithm on k-CNF Φ for up to t calls to Fix();

printf("$X_1 \ldots X_n$");

Moser's Algorithm:

draw uniform $X_1, \ldots, X_n \in \{\mathtt{T}, \mathtt{F}\}$;

while $\exists$ a violated clause C_i :

printf("$(i$");

Fix(C_i);

Fix(C_i):

resample all variables in vbl(C_i);

while $\exists$ violated $C_j \in \Gamma^+(C_i)$:

let $r =$ rank of C_j in $\Gamma^+(C_i)$;

printf("$(r$") and **Fix(C_j)**;

printf("$)$")

- **Output string**:
  - $\leq m \times$ "$(i$", where $i \in [m]$ at top-level
  - $\leq t \times$ "$(r$", where $r \in [d+1]$
  - $\leq t \times$ "$)$"
  - "(" ")" form prefix of legal parenthesization
  - n bits $X_1, \ldots, X_n$ in the end

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$$\left.\right\} \leq n + m(\lceil \log_2 m \rceil + 2) + t(\lceil \log_2 (d+1) \rceil + 2) \text{ bits}$$

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- Assume $\geq t$ calls made to **Fix**():

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$\mathtt{Enc}_\Phi$

$\leq n + m(\lceil \log_2 m \rceil + 2)$
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(LLL condition $d < 2^{k-3}$) $\leq n + m(\log_2 m + 3) + t(k-1)$

Incompressibility Principle:

For any **injective** function $\texttt{Enc} : \{0,1\}^N \xrightarrow{1\text{-}1} \{0,1\}^*$, for uniform random $\boldsymbol{s} \in \{0,1\}^N$, for any integer $l > 0$,

$$\Pr\left[\text{length of } \texttt{Enc}(\boldsymbol{s}) \leq N - l\right] < 2^{1-l}$$

- Assume $\geq t$ calls made to **Fix**():

$n + tk$ random bits used by the algorithm $\Rightarrow$ $\texttt{Enc}_\Phi$ $\Rightarrow$ $\leq n + m(\lceil \log_2 m \rceil + 2) + t(\lceil \log_2 (d+1) \rceil + 2)$ bits

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(LLL condition $d < 2^{k-3}$) $\quad \le n + m(\log_2 m + 3) + t(k-1)$

- For any $t \ge m(\log_2 m + 3) + \log_2 n + 1$:

this can only occur with probability $< \dfrac{1}{n}$

Moser's *Fix-It* Algorithm

- k-CNF: $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

$$\Phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge \overbrace{(x_3 \vee \neg x_4 \vee \neg x_5)}^{\text{k variables}}$$

Moser's Algorithm:

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Theorem [Moser 2009]**:**

$d < 2^{k-3} \implies$ total # of calls to Fix() is $O(m \log m + \log n)$
with high probability

with probability 1-O(1/n)

Lovász Local Lemma:

The Probabilistic Method

Algorithmic Lovász Local Lemma:

Moser-Tardos Algorithm

Moser's Algorithm and Entropic Proof

Summary

- The **Moser-Tardos** algorithm:

LLL condition $\Rightarrow$ Efficiently find $X_1, \ldots, X_n$ that avoids all $A_1, \ldots, A_m$

$A_1, \ldots, A_m$ are defined on independent random variables $X_1, \ldots, X_n$

- The proof based on ***resampling table*** and ***witness trees***

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- The proof based on ***resampling table*** and ***witness trees***

- What's next:
    - tighter LLL condition for algorithms
    - beyond independent variables: **non-variable framework**
    - more than construction: **sampling LLL**