Advanced Algorithms (Fall 2024)

Linear Programming

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Outline

Outline

Typical Combinatorial Optimizatiion Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

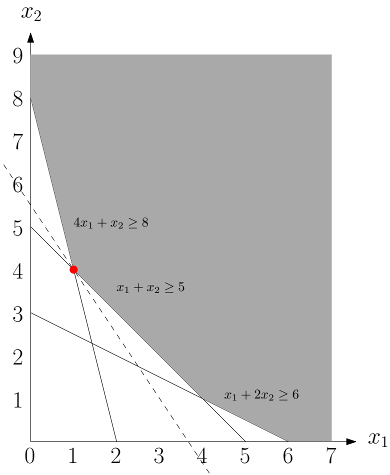
Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

Example:

- Shortest Path, Minimum Spanning Tree
- Maximum Independent Set, Maximum Matching, Knapsack Packing

- CO problem $\Longleftrightarrow$ Integer Program (IP) $\xrightarrow{\text{relax?}}$ Linear Program (LP)
- In general: Integer programming is NP-hard; linear programming is in P

$$\begin{aligned} \min \quad & 7x_1 + 4x_2 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 2x_2 \geq 6 \\ & 4x_1 + x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$



- optimum solution: $x_1 = 1, x_2 = 4$
- optimum value $= 7 \times 1 + 4 \times 4 = 23$

Standard Form of Linear Programs

$$
\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\geq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\geq b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots & \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\geq b_m \\
x_1, x_2, \cdots, x_n &\geq 0
\end{aligned}
$$

- n: number of variables $\qquad m$: number of constraints

- Other considerations: $\leq$ constraints? equlities?
- variables can be negative? maximization problem?

Standard Form of Linear Programs

$$x := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \qquad c := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n,$$

$$A := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n\times m}, \quad b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m.$$

$$
\begin{aligned}
\min \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\geq b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &\geq b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots & \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\geq b_m \\
x_1, x_2, \cdots, x_n &\geq 0
\end{aligned}
$$

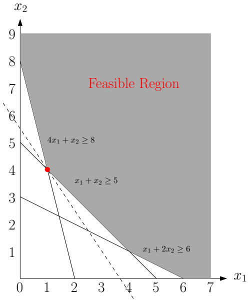
Standard Form of Linear Program

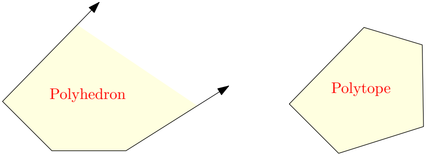
$$
\begin{aligned}
\min \quad & c^{\mathrm{T}}x \\
Ax &\geq b \\
x &\geq 0
\end{aligned}
$$

- $\geq$: coordinate-wise less than or equal to

Preliminaries

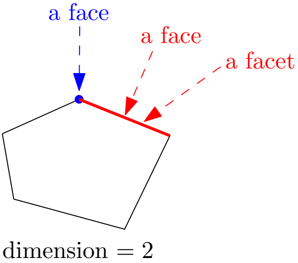
- feasible region: the set of x's satisfying $Ax \geq b, x \geq 0$
- a polyhedron is the intersection of finite number of closed half-spaces
- so, feasible region is a polyhedron
- if every coordinate has an upper and lower bound in the polyhedron, then the polyhedron is a polytope

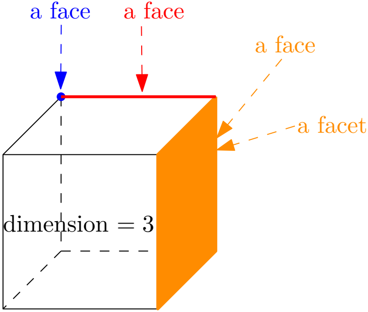




Given a polytope $\mathcal{P} \subseteq \mathbb{R}^n$:

- The dimension of $\mathcal{P}$ is n minus the maximum number of linearly-independent equalities satisfied by all points in $\mathcal{P}$.
- Assume the linear inequality $a^{\mathrm{T}}x \leq b$ holds for every $x \in \mathcal{P}$, and some $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x = b$. Then $\{x \in \mathcal{P} : a^{\mathrm{T}}x = b\}$ is said to be a face of $\mathcal{P}$.
- A face of $\mathcal{P}$ is also a polytope.
- Assume the dimension of $\mathcal{P}$ is d. Then a face of $\mathcal{P}$ of dimension $d-1$ is said to be a facet of $\mathcal{P}$.



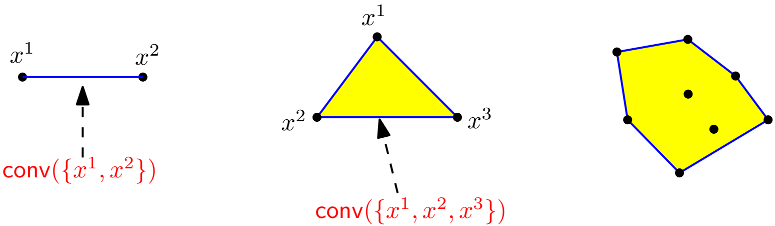


Preliminaries

- x is a convex combination of $\{x^{(1)}, x^{(2)}, \cdots, x^{(t)}\}$ if the following condition holds: there exist $\lambda_1, \lambda_2, \cdots, \lambda_t \in [0, 1]$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_t = 1, \qquad \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_t x^{(t)} = x$$

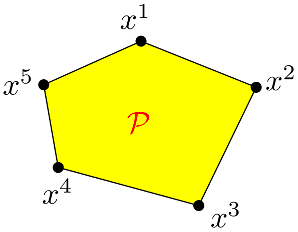
- the convex hull of a set of S of points in $\mathbb{R}^n$, denoted as conv(S), is the set of convex combinations of S



Terminology and Preliminaries

- let $\mathcal{P}$ be polytope, $x \in \mathcal{P}$. If there are no other points $x', x'' \in \mathcal{P}$ such that x is a convex combination of x' and x'', then x is called a vertex/extreme point of $\mathcal{P}$

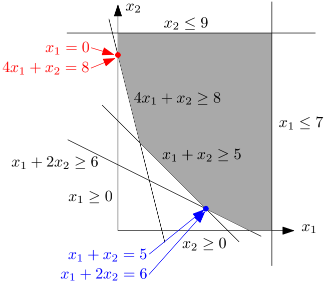
Lemma A polytope has finite number of vertices, and it is the convex hull of the vertices.



$\mathcal{P} = \text{conv}(\{x^1, x^2, x^3, x^4, x^5\})$

Terminology and Preliminaries

Lemma Let $x \in \mathbb{R}^n$ be a vertex of a polytope. Then, there are n constraints in the definition of the polytope, such that x is the unique solution to the linear system obtained from the n constraints by replacing inequalities to equalities.



Lemma If the feasible region of a linear program is a polytope, then the opimum value can be attained at some vertex of the polytope.

Special cases (for minimization linear programs):

- if feasible region is empty, then its value is ∞
- if the feasible region is unbounded, then its value can be $-\infty$

Outline

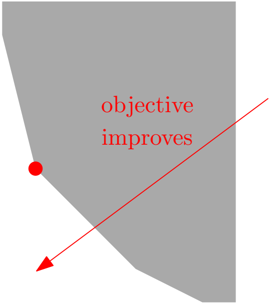
Algorithms for Linear Programming

algorithm	running time	practice
Simplex Method	exponential time	fast
Ellipsoid Method	polynomial time	slow
Interior Point Method	polynomial time	fast

Simplex Method

- [Dantzig, 1946]

- move from one vertex to another, so as to improve the objective
- repeat until we reach an optimum vertex



- the number of iterations might be expoentially large; but algorithm runs fast in practice

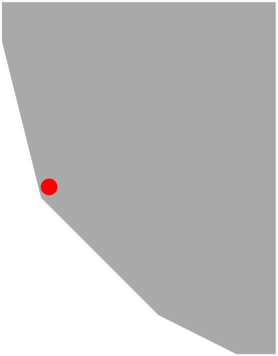
- [Spielman-Teng,2002]: smoothed analysis

Interior Point Method

- [Karmarkar, 1984]

- keep the solution inside the polytope
- design penalty function so that the solution is not too close to the boundary
- the final solution will be arbitrarily close to the optimum solution

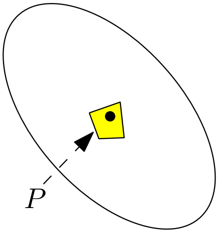
- polynomial time



Ellipsoid Method

- [Khachiyan, 1979]
- used to decide if the feasible region is empty or not

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the elliposid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



- polynomial time, but impractical

Q: The exact running time of these algorithms?

- it depends on many parameters: #variables, #constraints, #(non-zero coefficients), magnitude of integers
- precision issue

Open Problem

Can linear programming be solved in strongly polynomial time algorithm?

Applications of Linear Programming

- domain: computer science, mathematics, operations research, economics
- types of problems: transportation, scheduling, clustering, network routing, resource allocation, facility location

Research Directions

- polynomial time exact algorithm
- polynomial time approximation algorithm
- sub-routines for the branch-and-bound metheod for integer programming
- other algorithmic models: online algorithm, distributed algorithms, dynamic algorithms, fast algorithms

Typical Combinatorial Optimizatiion Problem

Input: $[n]$: ground set

$\mathcal{S}$: feasible sets: a family of subsets of U, often implicitly given

$w_i, i \in [n]$: values/costs of elements

Output: the set $S \in \mathcal{S}$ with the minimum/maximum $w(S) := \sum_{i \in S} w_i$

Def. For any $S \subseteq [n]$, we use $\chi^S \in \{0,1\}^{[n]}$ to denote the indicator vector for S:

$$\chi_i^S = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Examples

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\text{BM}} := \text{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\text{GM}} := \text{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\text{ST}} := \text{conv}\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Examples

Travelling Salesman Problem (TSP) Polytope

- Given the complete graph $G = (V, \binom{V}{2})$
- $\mathcal{P}_{\mathrm{TSP}} :=$ conv$(\{\chi^S, S \subseteq \binom{V}{2}$ is a TSP tour of V$\})$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

- Mechanic description of $\mathcal{P}$:

$$\sum_{i \in S} w_i x_i \leq \max_{S \in \mathcal{S}} \sum_{i \in S} w_i \qquad \forall w \in \mathbb{R}^{[n]}$$

- However, the description is often useless; many constraints are redundant
- It is often interesting and important to find the facet-defining constraints; those are the constraints that can not be removed

1. In some cases, $\mathcal{P}$ has polynomial number of facets
2. In some cases, $\mathcal{P}$ has exponential number of facets, but has an efficient separation oracle.
3. In some cases, $\mathcal{P}$ does not have an efficient separation oracle, unless P = NP.

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Def. A polytope $\mathcal{P} \subseteq [0,1]^n$ is said to be integral, if all vertices of $\mathcal{P}$ are in $\{0,1\}^n$.

Lemma For a $\mathcal{Q} \subseteq [0,1]^n$, if $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$ and $\mathcal{Q}$ is integral, then $\mathcal{Q} = \mathcal{P}$.

Proof.

- $\mathcal{P} \subseteq \mathcal{Q}$, as every vertex of $\mathcal{P}$ is χ^S for some $S \in \mathcal{S}$, and $\chi^S \in \mathcal{Q}$.
- $\mathcal{Q} \subseteq \mathcal{P}$: take some vertex x of $\mathcal{Q}$
- $\mathcal{Q}$ is integral $\implies$ x is integral $\implies$ $x = \chi^S$ for some $S \subseteq [n]$
- As $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$, $x = \chi^S$ for some $S \in \mathcal{S}$
- $x \in \mathcal{P}$ $\square$

polytope of interest: $\mathcal{P} = \text{conv}\big(\{\chi^S : S \in \mathcal{S}\}\big)$

Lemma For a $\mathcal{Q} \subseteq [0,1]^n$, if $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$ and $\mathcal{Q}$ is integral, then $\mathcal{Q} = \mathcal{P}$.

- Often, it is easy to guarantee $\mathcal{Q} \cap \{0,1\}^n = \{\chi^S : S \in \mathcal{S}\}$
- The linear program that defines such a $\mathcal{Q}$ is often called a LP relaxation for the problem.
- The harder part is often to prove that $\mathcal{Q}$ is integral.

Outline

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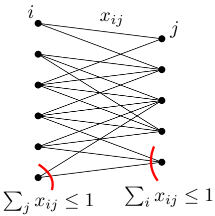
Bipartite Matching Polytope

Maximum Weight Bipartite Matching

Input: bipartite graph $G = (L \uplus R, E)$
edge weights $w \in \mathbb{Z}^E_{>0}$

Output: a matching $M \subseteq E$ so as to maximize $\sum_{e \in M} w_e$



Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

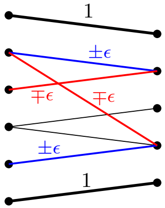
$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \le 1, \forall v \in L \cup R; \qquad x_e \ge 0, \forall e \in E.$$

Proof.

- take any x that satisfies the constraints
- prove: x non integral $\implies$ x non-vertex
- find $x', x'' \in \mathcal{P}$: $x' \neq x'', x = \frac{1}{2}(x' + x'')$
- case 1: fractional edges contain a cycle
  - color edges in cycle blue and red
  - x': $+\epsilon$ for blue edges, $-\epsilon$ for red edges
  - x'': $-\epsilon$ for blue edges, $+\epsilon$ for red edges
- case 2: fractional edges form a forest
  - color edges in leaf-leaf path blue and red
  - x': $+\epsilon$ for blue edges, $-\epsilon$ for red edges
  - x'': $-\epsilon$ for blue edges, $+\epsilon$ for red edges □



Outline

Def. A matrix $A \in \mathbb{R}^{m \times n}$ is said to be totally unimodular (TUM), if every sub-square of A has determinant in $\{-1, 0, 1\}$.

Theorem If a polytope $\mathcal{P}$ is defined by $Ax \geq b, x \geq 0$ with a totally unimodular matrix A and integral b, then $\mathcal{P}$ is integral.

Proof.

- Every vertex $x \in \mathcal{P}$ is the unique solution to the linear system (after permuting coordinates): $\begin{pmatrix} A' & 0 \\ 0 & I \end{pmatrix} x = \begin{pmatrix} b' \\ 0 \end{pmatrix}$, where
  - A' is a square submatrix of A with $\det(A') = \pm 1$, b' is a sub-vector of b,
  - and the rows for b' are the same as the rows for A'.
- Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, so that $A'x^1 = b'$ and $x^2 = 0$.
- Cramer's rule: $x_i^1 = \frac{\det(A_i'|b)}{\det(A')}$ for every $i \implies x_i^1$ is integer

  $A_i'|b$: the matrix of A' with the i-th column replaced by b □

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The following equation system may give a vertex:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Example for the Proof

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Equivalently, the vertex satisfies

$$\begin{pmatrix} a_{1,2} & a_{1,3} & 0 & 0 & 0 \\ a_{3,2} & a_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Lemma Let $A' \in \{0, \pm 1\}^{n \times n}$ such that every row of A' contains at most one 1 and one -1. Then $\det(A') \in \{0, \pm 1\}$.

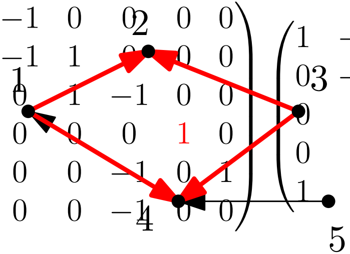
Proof.

- wlog assume every row of A' contains one 1 and one -1
  - otherwise, we can reduce the matrix
- treat A' as a directed graph: columns $\equiv$ vertices, rows $\equiv$ arcs
- #edges $=$ #vertices $\implies$ underlying undirected graph contains a cycle $\implies$ det$(A') = 0$ □

Lemma Let $A \in \{0, \pm 1\}^{m \times n}$ such that every row of A contains at most one 1 and one -1. Then A is TUM.

Coro. In the LP for s-t network flow problem with integer capacities, every vertex solution to the LP is integral.

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \color{red}{1} & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\begin{array}{cccccc} + & (1 & -1 & 0 & 0 & 0) \\ - & (0 & -1 & 1 & 0 & 0) \\ + & (0 & 0 & 1 & -1 & 0) \\ - & (1 & 0 & 0 & -1 & 0) \\ \\ = & (0 & 0 & 0 & 0 & 0) \end{array}$$

Lemma A matrix $A \in \{0,1\}^{m \times n}$ where the 1's on every row form an interval is TUM.

Proof.

- take any square submatrix A' of A,
- the 1's on every row of A' form an interval.
- $A'M$ is a matrix satisfying condition of first lemma, where

$$M = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \ \det(M) = 1.$$

- $\det(A'M) \in \{0, \pm 1\} \implies \det(A') \in \{0, \pm 1\}$. □

Example for the Proof

$$\begin{pmatrix} \color{red}0 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}0 & 0 \\ \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}0 & 0 \\ \color{red}0 & \color{red}0 & \color{red}1 & \color{red}1 & \color{red}1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \color{red}0 & \color{red}0 & \color{red}0 & \color{red}1 & \color{red}1 & 0 \\ \color{red}0 & \color{red}1 & \color{red}1 & \color{red}1 & \color{red}1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \implies \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

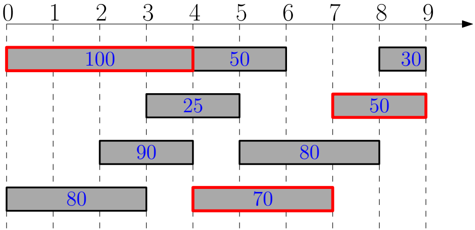
- (col 1, col 2 − col 1, col 3 − col 2, col 4 − col 3, col 5 − col 4)
- every row has at most one 1, at most one -1

Weighted Interval Scheduling Problem

Input: n activities, activity i starts at time s_i, finishes at time f_i, and has weight $w_i > 0$

i and j can be scheduled together iff $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: maximum weight subset of jobs that can be scheduled



- optimum value= 220
- Classic Problem for Dynamic Programming

Weighted Interval Scheduling Problem

Linear Program

$$\max \sum_{j \in [n]} x_j w_j$$
$$\sum_{j \in [n]: t \in [s_j, f_j)} x_j \leq 1 \qquad \forall t \in [T]$$
$$x_j \geq 0 \qquad \forall j \in [n]$$

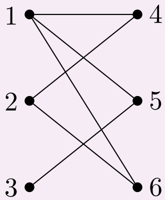
- The polytope is integral as the 1's in every column are consecutive.

Lemma The edge-vertex incidence matrix A of a bipartite graph is totally-unimodular.

Proof.

- $G = (L \uplus R, E)$: the bipartite graph
- A': obtained from A by negating columns correspondent to R
- each row of A' has exactly one $+1$, and exactly one -1
- $\implies A'$ is TUM $\iff A$ is TUM ☐

Example



$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}$$

A different proof for the theorem we proved:

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e\in\delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

Proof.

The coefficient matrix for the constraints $\sum_{e\in\delta(v)} x_e \leq 1, \forall v \in L \cup R$ is the vertex-edge incidence matrix of the graph G. Therefore, the polytope is integral. □

- remark: bipartiteness is needed. The edge-vertex incidence matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ of a triangle has determinent 2.

Outline

Def. A separation oracle for a polytope $\mathcal{P} \subseteq \mathbb{R}^n$ is an algorithm that, given some $x^* \in \mathbb{R}^n$,

- either correctly claims that $x \in \mathcal{P}$,
- or outputs a linear constraint $a^{\mathrm{T}}x \leq b$ that separating x^* from $\mathcal{P}$: every $x \in \mathcal{P}$ satisfies $a^{\mathrm{T}}x \leq b$, but $a^{\mathrm{T}}x^* > b$. We say $a^{\mathrm{T}}x \leq b$ is a separation plane for x^*.

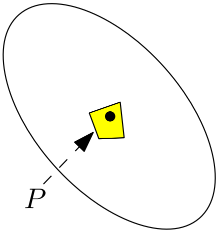
The separation oracle is efficient if its running time is polynomial in the size of the instance plus the size of x

- Clearly, if $\mathcal{P} \subseteq \mathbb{R}^n$ can be described using a polynomial-size LP, then it has an efficient separation oracle.
- However, there are cases where $\mathcal{P} \subseteq \mathbb{R}^n$ has exponential number of facets, but still admits an efficient separation oracle.

- We can use ellipsoid method to solve the LP $\min / \max w^{\mathrm{T}} x, x \in \mathcal{P}$, when $\mathcal{P}$ has an efficient separation oracle, using the ellipsoid method.

Ellipsoid Method

- maintain an ellipsoid that contains the feasible region
- query a separation oracle if the center of ellipsid is in the feasible region:
  - yes: then the feasible region is not empty
  - no: cut the elliposid in half, find smaller ellipsoid to enclose the half-ellipsoid, and repeat



Outline

s-t Cut Polytope

Def. Given a digraph $G = (V, E)$, C is a s-t cut in G, if s and t are disconnected in $(V, E \setminus C)$.

- $\mathcal{P}_{\mathrm{min-cut}} := \mathrm{conv}\big(\{\chi^C : C \text{ is a } s\text{-}t \text{ cut in } G\}\big)$

Theorem $\mathcal{P}_{\mathrm{min-cut}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in P} x_e \geq 1 \qquad \forall \text{ simple } s\text{-}t \text{ path } P \qquad (*)$$

$$x_e \in [0, 1] \qquad \forall e \in E$$

Q: Given $x \in [0, 1]^E$, how can we check if x satisfies all constraints in (*)?

A: Use shortest path algorithm with weights $(x_e)_{e \in E}$.

Theorem $\mathcal{P}_{\mathrm{min-cut}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in P} x_e \geq 1 \qquad \forall \text{ simple } s\text{-}t \text{ path } P \qquad (*)$$
$$x_e \in [0,1] \qquad \forall e \in E$$

Proof of Lemma.

- Given $x \in [0,1]^E$ satisfying (*)
- $d_x(v), v \in V$: length of shortest path from s to v, with x being the weights; so $d_x(s) = 0$ and $d_x(t) \geq 1$
- randomly choose a real $\theta \in (0,1)$
- $S := \{v \in V : d_x(v) \leq \theta\}, T := V \setminus S = \{v \in V : d_x(v) > \theta\}$
- $C := E(S,T)$

Claim For an edge $(u, v) \in E$, we have

$$\Pr[(u, v) \in C] \leq \max\{d_x(v) - d_x(u), 0\}.$$

Proof.

- $(u, v) \in C$ happens only if $d_x(u) < \theta \leq d_x(v)$.
- This happens with probability at most $\max\{d_x(v) - d_x(u), 0\} \leq x_{(u,v)}$. □

Proof of Lemma, Continued

- $\mathbb{E}_\theta[\chi^C] \leq x$
- We can define a random set C' so that $C' \supseteq C$ happens with probability 1, and $\mathbb{E}_\theta[\chi^{C'}] = x$.
- So $x \in$ conv$(\{\chi^{C'} : C'$ is a s-t cut in $G\})$ □

Outline

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{ST}} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in E} x_e &= n - 1 & & \\ \sum_{e \in E[S]} x_e &\leq |S| - 1 & \forall S \subseteq V, 2 \leq |S| \leq n - 1 & \quad (*) \\ x_e &\geq 0 & \forall e \in E & \end{aligned}$$

- Spanning trees correspond to bases of graphic matroid for G
- Later we prove a more general theorem on matroid polytopes

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\mathrm{ST}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E[S]} x_e \le |S| - 1 \qquad \forall S \subseteq V, 2 \le |S| \le n-1 \qquad (*)$$

$$x_e \ge 0 \qquad \forall e \in E$$

Q: How can we check if all constraints in (*) are satisfied?

A: $\xrightarrow{\text{reduce}}$ densest sub-graph $\xrightarrow{\text{reduce}}$ maximum flow

Checking if $\sum_{e \in E[S]} x_e \leq |S| - 1, \forall S \subseteq V$

- We need to check if $\exists S \subseteq V, \frac{\sum_{e \in E[S]} x_e}{|S|-1} > 1$:

- Guess a vertex $v \in S$; set $w_v = 0$ and $w_u = 1$ for every $u \in V \setminus \{v\}$
- The problem becomes to check if $\exists S \subseteq V, \frac{\sum_{e \in E[S]} x_e}{\sum_{u \in S} w_u} > 1$
- This is a (weighted) densest subgraph problem
- Exercise: It can be solved using maximum flow

Outline

General Graph Perfect Matching Polytope

General Perfect Matching Polytope

- Given a graph $G = (V, E)$, where $|V|$ is even
- $\mathcal{P}_{\text{GPM}} := \text{conv}\left(\{\chi^M : M \subseteq E \text{ is a perfect matching in } G\}\right)$

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\text{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S, V \setminus S)} x_e \geq 1 \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\text{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S, V \setminus S)} x_e \geq 1 \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

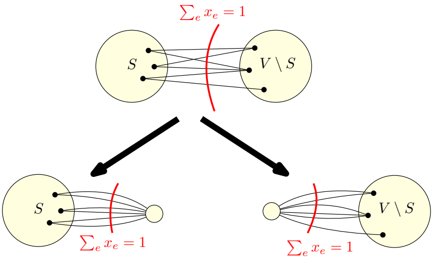
Proof of General Perfect Matching Polytope Theorem

- Clearly, every $x \in \mathcal{P}_{\text{GPM}}$ satisfies all the LP constraints
- We prove the LP polytope is integral; this implies lemma
- We choose the counter-example G with the smallest $|V| + |E|$, and focus on a non-integral vertex x of the LP polytope

Proof of General Perfect Matching Polytope Theorem

- $x_e = 0$ for some $e \in E$: e could be removed.
- $x_e = 1$ for some $e \in E$: e and its 2 end vertices could be removed.
- So $x_e \in (0, 1)$ for every $e \in E$.
- Every $v \in V$ has degree at least 2.
- Every $v \in V$ has degree exactly 2: G is union of disjoint cycles, x would not be a vertex of LP polytope.
- Assume some $v \in V$ has degree at least 3; $|E| \geq |V| + 1$.
- x is the unique solution to a system ot n linear equations from the LP.
- So, some linear equation is

$$\sum_{e \in E(S, V \setminus S)} x_e = 1 \text{ for some } S \subseteq V \text{ with } |S| \geq 3, |V \setminus S| \geq 3$$



Proof of General Perfect Matching Polytope Theorem

- Consider two instances: $(G/V, x'), (G/(V \setminus S), x'')$
- Both x' and x'' satisfy the LP constraints for their respective graphs.

- By the minimality assumption:

$$x' \in \text{conv}(\{\chi^M : M \text{ is a perfect matching in } G/S\})$$
$$x'' \in \text{conv}(\{\chi^M : M \text{ is a perfect matching in } G/(V \setminus S)\})$$

- Decompose x' and x'' into a convex combinations of matchings
- Each $e \in E(S, V \setminus S)$ has the same fraction in combinations
- "Concatenate" two convex combinations into one convex combinations of matching in G. So x can not be a vertex. □

Theorem (General Perfect Matching Polytope Theorem)
$\mathcal{P}_{\text{GPM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in V$$

$$\sum_{e \in E(S, V \setminus S)} x_e \geq 1 \qquad \forall S \subseteq V, |S| \text{ is odd} \qquad (*)$$

$$x_e \geq 0 \qquad \forall e \in E$$

Q: How can we check if all constraints in (*) are satisfied?

A: Use the Gomory-Hu Tree structure.

- inequality in (*) can be replaced by $\sum_{e \in E[S]} x_e \leq \frac{|S|-1}{2}$
- more convenient to obtain general matching polytope

General Matching Polytope

- Given a graph $G = (V, E)$
- $\mathcal{P}_{\mathrm{GM}} := \text{conv}\left(\left\{\chi^M : M \subseteq E \text{ is a matching in } G\right\}\right)$

Theorem (General Matching Polytope Theorem) $\mathcal{P}_{\mathrm{GM}}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leq 1 && \forall v \in V \\ \sum_{e \in E(S)} x_e &\leq \frac{|S| - 1}{2} && \forall S \subseteq V, |S| \text{ is odd} \qquad (1) \\ x_e &\geq 0 && \forall e \in E \end{aligned}$$

Remark

- For all the polytopes, we identified a set of linear inequalities that are sufficient to define the polytope.
- However, not all the constraints are facet-defining.
- Only facet-defining constraints are necessarily; other constraints could be removed. (We keep all the constraints.for convenience of description.)
- Example: in spanning tree polytope, $\sum_{e \in E[S]} x_e \leq |S| - 1$ is not needed if $(S, E[S])$ is disconnected, or contains a bridge. In this case, the constraint does not define a facet.

Outline

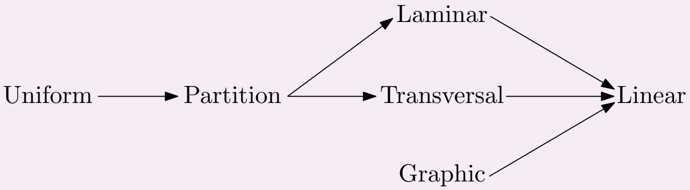
Outline

Recall Definition and Examples of Matroid

Def. A (finite) matroid $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite set (called the ground set) and $\mathcal{I}$ is a family of subsets of E (called independent sets) with the following properties:

1. $\emptyset \in \mathcal{I}$.
2. (downward-closed property) If $B \subsetneq A \in \mathcal{I}$, then $B \in \mathcal{I}$.
3. (augmentation/exchange property) If $A, B \in \mathcal{I}$ and $|B| < |A|$, then there exists $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$.

Relationship between matroids



Other Terminologies Related To a Matroid $\mathcal{M} = (E, \mathcal{I})$

- A subset of E that is not independent is dependent.
- A maximal independent set is called a basis (plural: bases)
- A minimal dependent set is called a circuit

- Graphic matroid for a connected graph $G = (V, E)$:
  basis $\Longleftrightarrow$ spanning tree   circuit $\Longleftrightarrow$ cycle

Lemma All bases of a matroid have the same size.

Proof.

- Assume two A and A' are both bases of $\mathcal{M}$ and $|A| > |A'|$
- By exchange property: $\exists i \in A \setminus A', A' \cup \{i\} \in \mathcal{I}$
- contradiction with that A' is a basis $\square$

- Recall: Matroid Rank Function:

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the rank of any $A \subseteq E$ is defined as

$$r_{\mathcal{M}}(A) = \max\left\{|A'| : A' \subseteq A, A' \in \mathcal{I}\right\}.$$

The function $r_{\mathcal{M}} : 2^E \to \mathbb{Z}_{\geq 0}$ is called the rank function of $\mathcal{M}$.

- $r_{\mathcal{M}}(A)$ is size of maximum independent subset of A

Trivial properties of $r_{\mathcal{M}}$

- $r_{\mathcal{M}}(\emptyset) = 0$
- $r_{\mathcal{M}}(A \cup \{i\}) - r_{\mathcal{M}}(A) \in \{0, 1\}$ for every $A \subseteq E, i \in E \setminus A$

Theorem The rank function $r_{\mathcal{M}}$ of a matroid $\mathcal{M} = (E, \mathcal{I})$ is submodular.

Greedy algorithm finds max ind. subset of any given $X \subseteq E$:

1: $S \leftarrow \emptyset$
2: **while** $\exists e \in X \setminus S$ s.t. $S \cup \{e\} \in \mathcal{I}$ **do**
3: let e be an arbitrary element satisfying the condition
4: $S \leftarrow S \cup \{e\}$
5: **return** S

Proof of Submodularity of $r_{\mathcal{M}}$.

- Take $A \subsetneq E, i, j \in E \setminus A, i \neq j$, need to prove: $r_{\mathcal{M}}(A \cup \{i, j\}) - r_{\mathcal{M}}(A \cup \{i\}) \leq r_{\mathcal{M}}(A \cup \{j\}) - r_{\mathcal{M}}(A)$
- if not, then LHS $= 1$, RHS $= 0$
- S: max ind. subset of A, S': max ind. subset of $A \cup \{i\}$
- $|S| = r_{\mathcal{M}}(A), |S'| = r_{\mathcal{M}}(A \cup \{i\})$, $S' = S$ or $S' = S \cup \{i\}$
- RHS $= 0 \implies S \cup \{j\} \notin \mathcal{I}$, LHS $= 1 \implies S' \cup \{j\} \in \mathcal{I}$
- contradiction □

Lemma A function $r : 2^E \to \mathbb{R}$ is the rank function of a matroid if and only if

1. $r(\emptyset) = 0$
2. $r(A \cup \{i\}) - r(A) \in \{0, 1\}$ for all $A \subseteq E, i \notin E \setminus A$
3. r is submodular.

Proof.

- Define $\mathcal{I} = \{A \subseteq E : r(A) = |A|\}$.
- Claim: $(E, \mathcal{I})$ is a matroid and r is its rank function.
- ①, ② $\implies \mathcal{I}$ is closed under taking subsets
- $A, A' : r(A) = |A|, r(A') = |A'|, |A| < |A'|$
- $U := A \cup A' : r(U) \geq r(A') > r(A), \qquad A \subsetneq U$
- ③ $\implies \exists i \in U \setminus A = A' \setminus A : r(A \cup \{i\}) > r(A)$
- $i \in A' \setminus A$ and $r(A \cup \{i\}) = r(A) + 1 = |A \cup \{i\}|$
- so, $A \cup \{i\} \in \mathcal{I} \implies$ exchange property □

Derivatives of Matroids

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and an element $e \in E$, the matroid obtained from $\mathcal{M}$ by removing e, denoted as $\mathcal{M} \setminus e$, is defined as follows:

$$\mathcal{M} \setminus e = (E \setminus e, \{A \subseteq E \setminus e : A \in \mathcal{I}\}).$$

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and an element $e \in E$, the matroid obtained from $\mathcal{M}$ by contracting e, denoted as $\mathcal{M}/e$, is defined as follows:

$$\mathcal{M}/e = (E \setminus e, \{A \subseteq E \setminus e : A \cup \{e\} \in \mathcal{I}\}).$$

Derivatives of Matroids

Def. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and a subset $E' \subseteq E$, the matroid of $\mathcal{M}$ restricted to E', denoted as $\mathcal{M}[E']$, is defined as follows:

$$\mathcal{M}[E'] = (E', \{A \subseteq E' : A \in \mathcal{I}\}).$$

Def. For a matroid $\mathcal{M} = (E, \mathcal{I})$, the dual matroid $\mathcal{M}^* = (E, \mathcal{I}^*)$ is defined so that the bases in $\mathcal{M}^*$ are exactly the complements of the bases in $\mathcal{I}$.

Theorem $\mathcal{M}^*$ is a matroid.

Outline

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

- Recall: $\chi^A \in \{0,1\}^E, \quad \chi_i^A = \begin{cases} 1 & i \in A \\ 0 & i \notin A \end{cases}$

Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \Big\{ x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E \Big\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof of Matroid Polytope Theorem

- $\mathcal{Q} := \left\{x \in [0,1]^E : \sum_{i\in A} x_i \leq r_{\mathcal{M}}(A), \forall A \subseteq E\right\}$
- $\mathcal{Q} \cap \{0,1\}^E = \{\chi^A : A \in \mathcal{I}\}$; it suffices to prove $\mathcal{Q}$ is integral
- Focus on the counter example with the smallest $|E|$
- assume some vertex x of $\mathcal{Q}$ is non-integral
- If $x_e = 0$ for some $e \in E$, removing e gives a smaller counterexample
- If $x_e = 1$ for some $e \in E$, contracting e gives a smaller counterexample
- So, $x_e \in (0,1)$ for every $e \in E$.

Proof of Matroid Polytope Theorem

Def. We say a set $A \subseteq E$ is tight if $x(A) = r_{\mathcal{M}}(A)$. Let $\mathcal{T}$ be the family of all tight subsets of E.

Lemma If $A, B \in \mathcal{T}$, then both $A \cup B$ and $A \cap B$ are in $\mathcal{T}$.

Proof.

$$x(A) + x(B) = r_{\mathcal{M}}(A) + r_{\mathcal{M}}(B)$$
$$\geq r_{\mathcal{M}}(A \cup B) + r_{\mathcal{M}}(A \cap B) \geq x(A \cup B) + x(A \cap B).$$

- equality: A and B are tight
- first inequality: $r_{\mathcal{M}}$ is submodular
- second inequality: $x(S) \leq r_{\mathcal{M}}(S)$ for every $S \subseteq E$

But $x(A) + x(B) = x(A \cup B) + x(A \cap B)$. So, both inequalities hold with equality. □

Proof of Matroid Polytope Theorem

Def. A chain is a sequence of subsets $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_t$ of E.

- We use span$(\mathcal{S})$ for span$(\{\chi^S : S \in \mathcal{S}\})$, for any $\mathcal{S} \subseteq \mathcal{T}$.

Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have span$(\mathcal{C}) =$ span$(\mathcal{T})$.

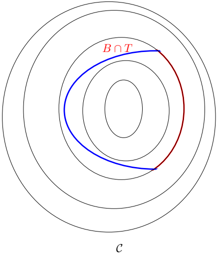
Proof of Key Lemma

- We say two sets B and T conflict with each other, if $B \not\subseteq T$ and $T \not\subseteq B$.
- Define $\tau(B) := \{T \in \mathcal{C} : B \text{ conflicts with } T\}, \forall B$
- Assume span$(\mathcal{C}) \subsetneq$ span$(\mathcal{T})$
- Let $B = \arg\min_{B \in \mathcal{T}, \chi^B \notin \text{span}(\mathcal{C})} |\tau(B)|$

Proof of Matroid Polytope Theorem

Proof of Key Lemma

- Let $T \in \mathcal{C}$ be a set contradicting with B;
- We prove $\tau(B \cup T), \tau(B \cap T) \subsetneq \tau(B)$.



- For $\tau(B \cup T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S does not conflict with $B \cup T$, and may conflict with B.
  - $S \supsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cup T$.
- For $\tau(B \cap T) \subseteq \tau(B)$:
  - $S \subsetneq T$: S not conflict with $B \implies S$ not conflict with $B \cap T$.
  - $S \supsetneq T$: S does not conflict with $B \cap T$, and may conflict with B.
- "$\neq$" : B conflicts with T, but $B \cup T$ and $B \cap T$ do not.

Proof of Matroid Polytope Theorem

Proof of Key Lemma

- By our choice of B, we have $\chi^{B\cup T}, \chi^{B\cap T} \in \text{span}(\mathcal{C})$.
- However, as $\chi^B = \chi^{B\cup T} + \chi^{B\cap T} - \chi^T$ and all the three vectors are in span($\mathcal{T}$), contradiction with $\chi^B \notin \text{span}(\mathcal{C})$. □

Recall the key lemma:

Lemma (Key Lemma) Let $\mathcal{C}$ be a longest chain of tight subsets of E (i.e., subsets in $\mathcal{T}$). Then, we have span($\mathcal{C}$) = span($\mathcal{T}$).

- Therefore, $x \in [0,1]^E$ is defined by the system of linear equations correspondent to $\mathcal{C}$.
- $|\mathcal{C}| = |E|$, the chain $\mathcal{C}$ is of full length.
- The system gives an integer solution x. Contradiction. □

What we proved:

Matroid Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid polytope for $\mathcal{M}$ is defined as

$$\mathcal{P}_{\mathcal{M}} := \text{conv}(\{\chi^A : A \in \mathcal{I}\}).$$

Theorem (Matroid Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}} = \Big\{x \in [0,1]^E : x(S) \leq r_{\mathcal{M}}(S), \forall S \subseteq E\Big\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Outline

Matroid Basis Polytope

- Given a matroid $\mathcal{M} = (E, \mathcal{I})$
- The matroid basis polytope for $\mathcal{M}$ is defined as
$$\mathcal{P}_{\mathcal{M}}^{\text{basis}} := \text{conv}(\{\chi^A : A \in \mathcal{I}, \text{rank}_{\mathcal{M}}(A) = \text{rank}_{\mathcal{M}}(E)\}).$$

Theorem (Matroid Basis Polytope Theorem) For a matroid $\mathcal{M} = (E, \mathcal{I})$, we have

$$\mathcal{P}_{\mathcal{M}}^{\text{basis}} = \left\{ x \in [0,1]^E : x(S) \le r_{\mathcal{M}}(S), \forall S \subseteq E; x(E) = r_{\mathcal{M}}(E) \right\},$$

where $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq E$.

Proof.

- $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is a face (not necessarily a facet) of $\mathcal{P}_{\mathcal{M}}$.
- $\mathcal{P}_{\mathcal{M}}$ is integral $\implies$ $\mathcal{P}_{\mathcal{M}}^{\text{basis}}$ is integral □

Recall: Spanning Tree Polytope

Spanning Tree Polytope

- Given a connected graph $G = (V, E)$
- $\mathcal{P}_{\rm ST} :=$ conv $\left(\left\{\chi^T : T \subseteq E \text{ is a spanning tree of } G\right\}\right)$

Theorem (Spanning Tree Polytope Theorem) $\mathcal{P}_{\rm ST}$ is the set of vectors $x \in \mathbb{R}^E$ satisfying the following inequalities:

$$\begin{aligned} \sum_{e \in E} x_e &= n - 1 & & \\ \sum_{e \in E[S]} x_e &\leq |S| - 1 & \forall S \subseteq V, 2 \leq |S| \leq n-1 \quad & (*) \\ x_e &\geq 0 & \forall e \in E & \end{aligned}$$

- Graphic matroid:
  - independent sets $\leftrightarrow$ spanning forests
  - bases $\leftrightarrow$ spanning trees.
- So, $\mathcal{P}_{\mathrm{ST}}$ is the set of $x \in [0,1]^E$ satisfying

$$x(E') \leq n - \mathsf{CC}(E'), \forall E' \subseteq E; \quad x(E) = n - 1,$$

  where $\mathsf{CC}(E')$ is the number of connected components in (V, E').
- It suffices to consider the case where $E' = E[S]$ for some connected set $S \subseteq V$, in which case $n - \mathrm{CC}(E') = |S| - 1$.
- $\implies$ Spanning Tree Polytope Theorem.

Theorem (Matroid Intersection Polytope Theorem) Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be two matroids with the common ground set E. Then

$$\begin{aligned}&\mathrm{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2\}\big) = \mathcal{P}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2}\\ &= \Big\{x \in [0,1]^E : x(S) \le r_{\mathcal{M}_1}(S), x(S) \le r_{\mathcal{M}_2}(S), \forall S \subseteq E\Big\}.\end{aligned}$$

- We will not prove the theorem.
- A similar theorem works if we require A to be a basis for the matroid $\mathcal{M}_1$ or $\mathcal{M}_2$:

$$\begin{aligned}&\mathrm{conv}\big(\{\chi^A : A \in \mathcal{I}_1 \cap \mathcal{I}_2, \mathrm{rank}_{\mathcal{M}_1}(A) = \mathrm{rank}_{\mathcal{M}_1}(E)\}\big)\\ &= \mathcal{P}^{\mathrm{basis}}_{\mathcal{M}_1} \cap \mathcal{P}_{\mathcal{M}_2}\end{aligned}$$

Applications

Bipartite Matching Polytope

- Given bipartite graph $G = (L \cup R, E)$
- $\mathcal{P}_{\mathrm{BM}} := \mathrm{conv}\big(\{\chi^M : M \text{ is a matching in } G\}\big)$

Theorem $\mathcal{P}_{\mathrm{BM}}$ is the set of $x \in \mathbb{R}^E$ satisfying the following constraints:

$$\sum_{e \in \delta(v)} x_e \leq 1, \forall v \in L \cup R; \qquad x_e \geq 0, \forall e \in E.$$

- A matching is an independent set of two partition matroids, one for each side of the bipartite graph.
- Matching polytope is intersection of two partition matroid polytopes.

Applications

Arborescence Polytope

- Given a directed graph $G = (V, E)$, a root $r \in V$
- $\mathcal{P}_{\mathrm{Arbo}} := \mathrm{conv}(\{\chi^{E'} : E' \text{ is an arborescence of } G \text{ rooted at } r\})$

- We define two matroids:
  - Graphic Matroid: we ignore the directions of G, and require E' to be a spanning forest
  - Partition Matroid: we require every vertex other than r has in-degree at most 1
- E' is an arborescence if it is a basis of both polytopes.

Summary

- linear programming, simplex method, interior point method, ellipsoid method
- Polytopes with totally-unimodular coefficient matrix:
  - integral LP polytopes: bipartite matching polytope, s-t flow polytope, weighted interval scheduling polytope
- Matroid Polytope