Advanced Algorithms (Fall 2024)

Primal-Dual Algorithms

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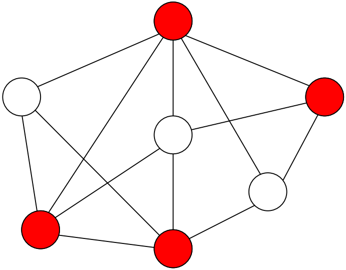
Nanjing University

Outline



Weighted Vertex Cover Problem

Input: graph $G = (V, E)$, vertex weights $w \in \mathbb{Z}_{>0}^V$

Output: vertex cover S of G, to minimize $\sum_{v \in S} w_v$

LP Relaxation

$$\min \quad \sum_{v \in V} w_v x_v$$

$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$

$$x_v \geq 0 \qquad \forall v \in V$$

Dual LP

$$\max \quad \sum_{e \in E} y_e$$

$$\sum_{e \in \delta(v)} y_e \leq w_v \qquad \forall v \in V$$

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- Algorithm constructs integral primal solution x and dual solution y simultaneously.

Primal-Dual Algorithm for Weighted Vertex Cover Problem

```
x ← 0, y ← 0, all edges said to be uncovered
while there exists at least one uncovered edge do
    take such an edge e arbitrarily
    increasing y_e until the dual constraint for one end-vertex v of e becomes tight
    x_v ← 1, claim all edges incident to v are covered
return x
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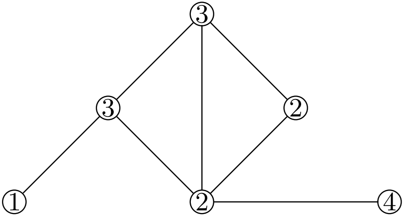

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2: **while** there exists at least one uncovered edge **do**
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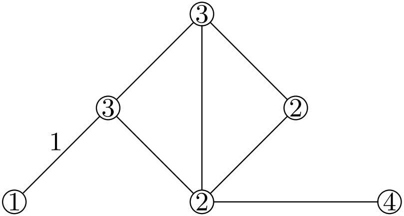

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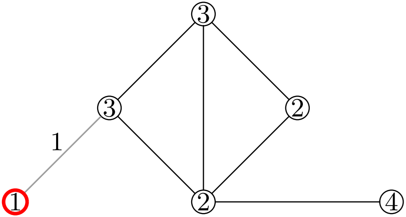

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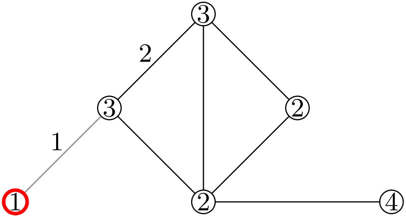

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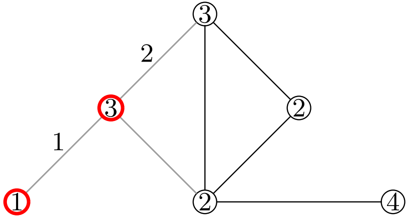

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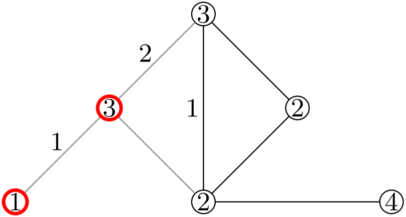

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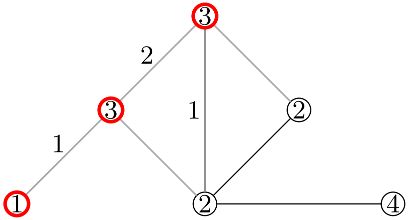

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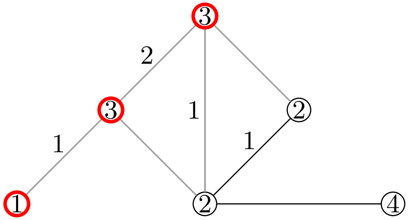

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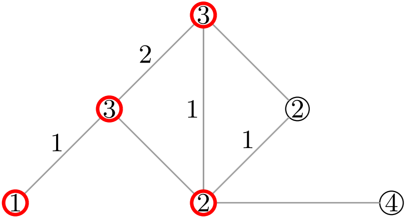

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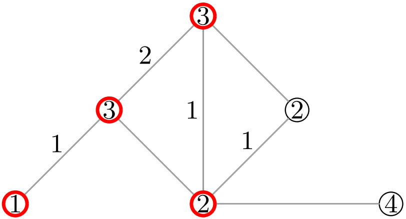

Lemma

1. x satisfies all primal constraints
2. y satisfies all dual constraints
3. $P \leq 2D \leq 2D^* \leq 2 \cdot \text{opt}$

   $P := \sum_{v \in V} x_v$: value of x

   $D := \sum_{e \in E} y_e$: value of y

   D^* : dual LP value

Proof of $P \le 2D$.

$$P = \sum_{v \in V} w_v x_v \le \sum_{v \in V} x_v \sum_{e \in \delta(v)} y_e = \sum_{(u,v) \in E} y_{(u,v)}(x_u + x_v)$$

$$\le 2 \sum_{e \in E} y_e = 2D. \qquad \square$$

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- a more general framework: construct an arbitrary maximal dual solution y; choose the vertices whose dual constraints are tight
- y is maximal: increasing any coordinate y_e makes y infeasible

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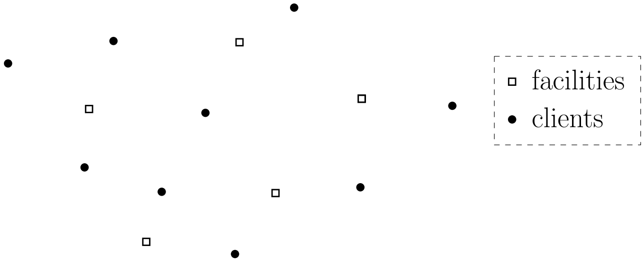
- primal-dual algorithms do not need to solve LPs
- LPs are used in analysis only
- faster than LP-rounding algorithm in general

Outline

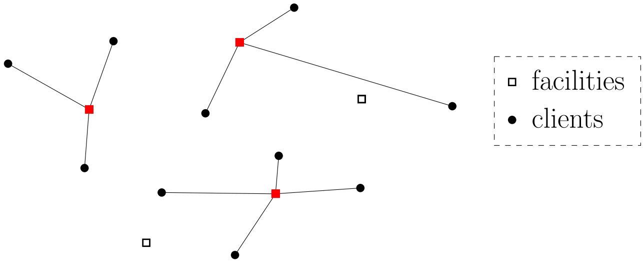
1. 2-Approximation Algorithm for Weighted Vertex Cover Using Primal-Dual
2. 3-Approximation Algorithm for Uncapacitated Facility Location Problem Using Primal Dual



Uncapacitated Facility Location Problem

Input: F: pontential facilities C: clients

d: (symmetric) metric over $F \cup C$ $(f_i)_{i \in F}$: facility opening costs



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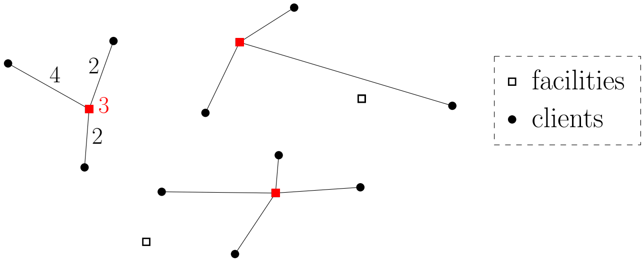

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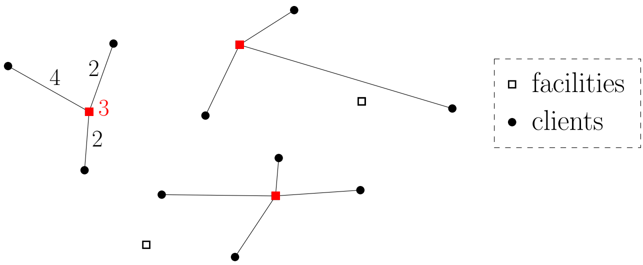
- 1.488-approximation [Li, 2011]



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- 1.488-approximation [Li, 2011]
- 1.463-hardness of approximation, 1.463 $\approx$ root of $x = 1 + 2e^{-x}$

- y_i: open facility i?
- $x_{i,j}$: connect client j to facility i?

Basic LP Relaxation

$$\min \quad \sum_{i \in F} f_i y_i + \sum_{i \in F, j \in C} d(i,j) x_{i,j}$$

$$\begin{aligned} \sum_{i \in F} x_{i,j} &\geq 1 & \forall j \in C \\ x_{i,j} &\leq y_i & \forall i \in F, j \in C \\ x_{i,j} &\geq 0 & \forall i \in F, j \in C \\ y_i &\geq 0 & \forall i \in F \end{aligned}$$

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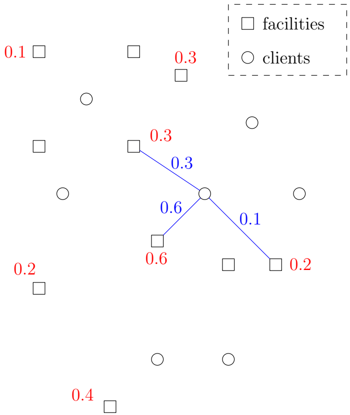

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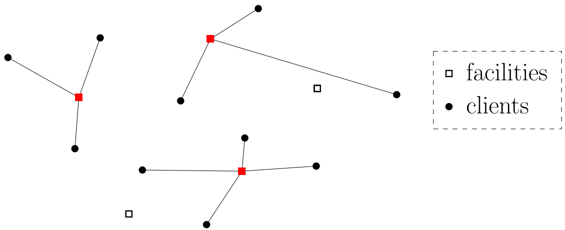

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- both LPs have exponential size, but the final algorithm can run in polynomial time

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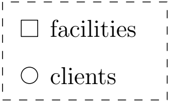
- α_j: budget of j
- dual constraints: total budget in any star is $\leq$ its cost
- $\implies$ opt $\geq$ total budget $=$ dual value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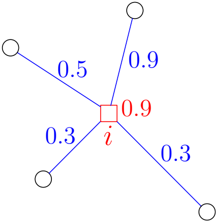
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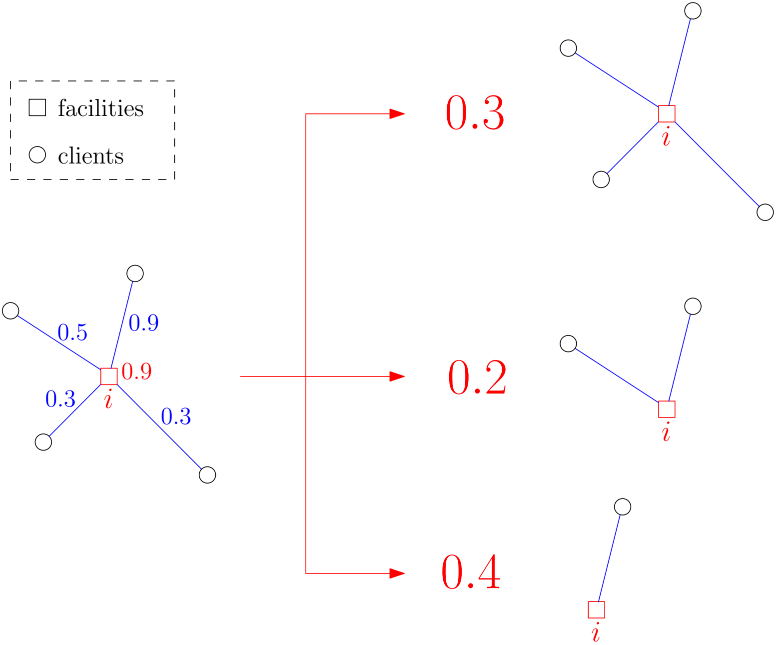
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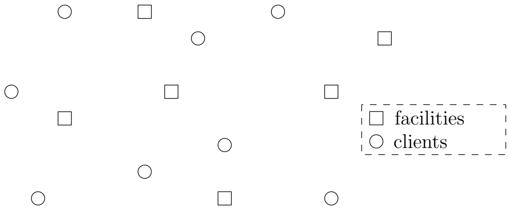

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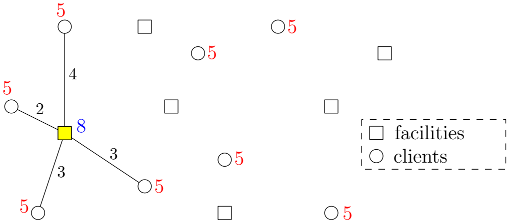

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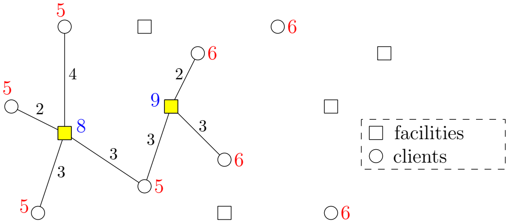

Construction of Dual Solution α

$\alpha_j \leftarrow 0, \forall j \in C$
while exists at least one active client **do**
    increase the budgets α_j for all active clients j at uniform rate, until (at least) one new client is frozen

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- α_j's can only increase
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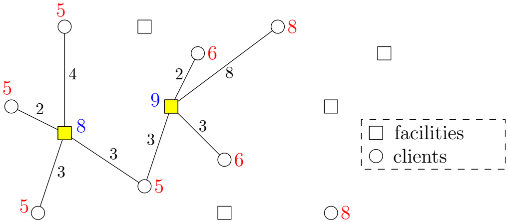

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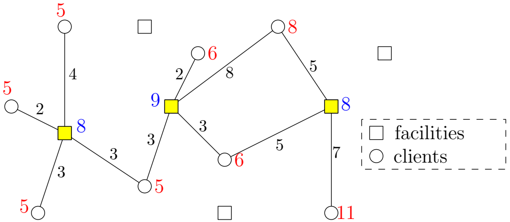

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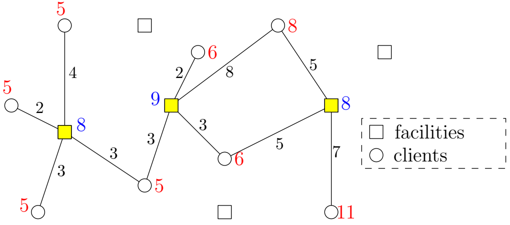

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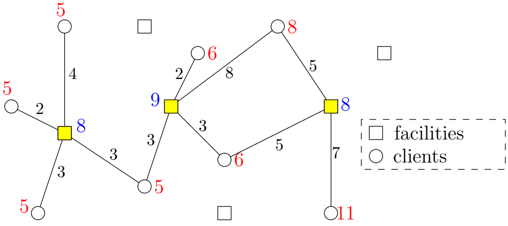

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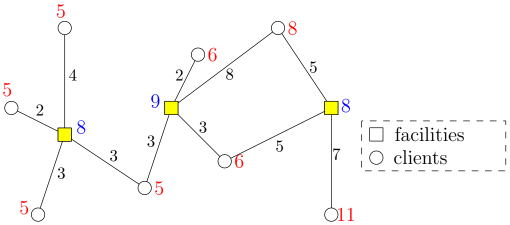

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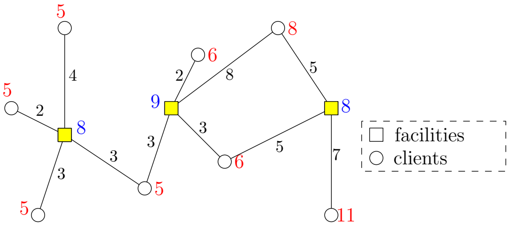

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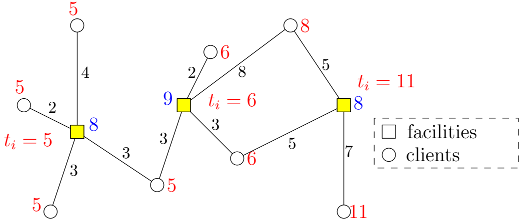

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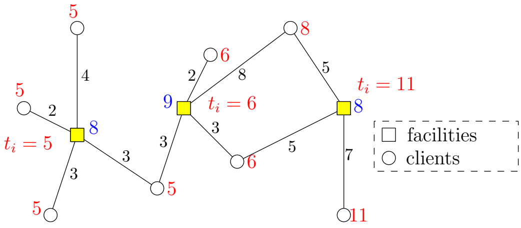

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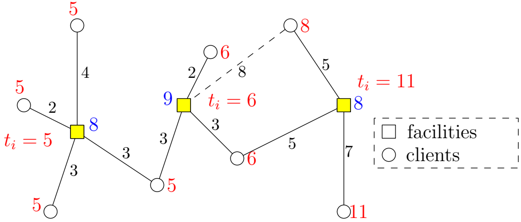

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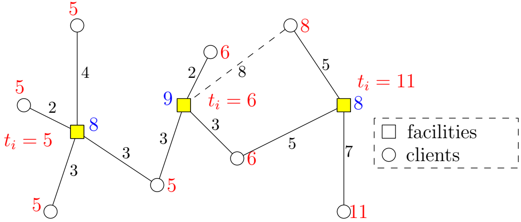

$\alpha_j > d(i, j)$: j contributes to i, (solid lines)

$\alpha_j = d(i, j)$: j does not contribute to i, but its budget is just enough for it to connect to i (dashed lines)

$\alpha_j < d(i, j)$: budget of j is not enough to connect to i

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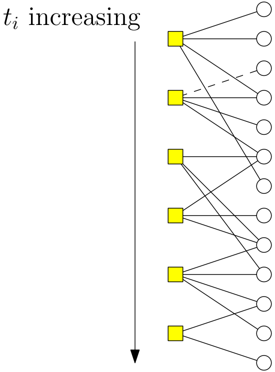

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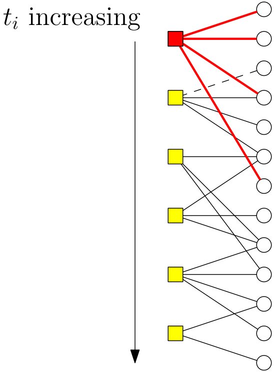

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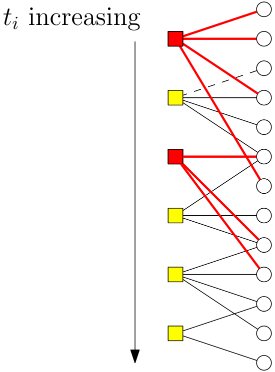

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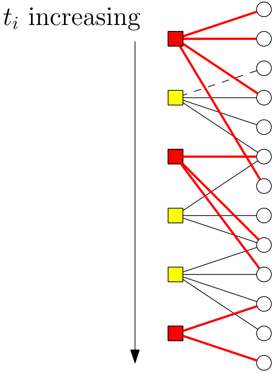

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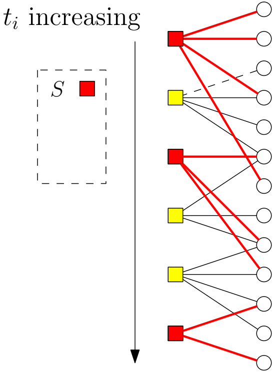

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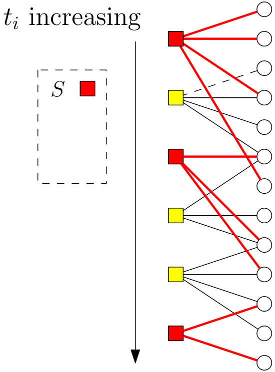
- C_1: clients that make contributions
- C_2: clients that do not make contributions



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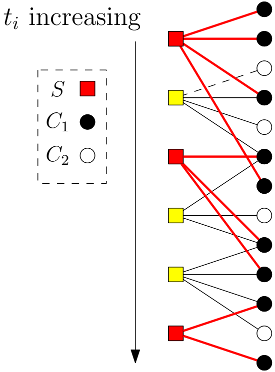

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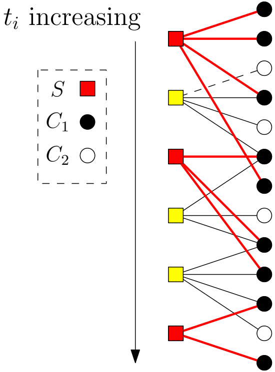

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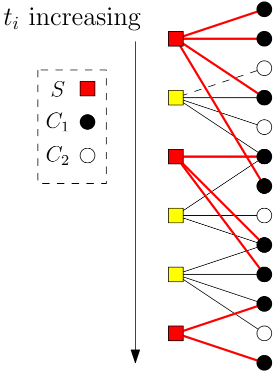

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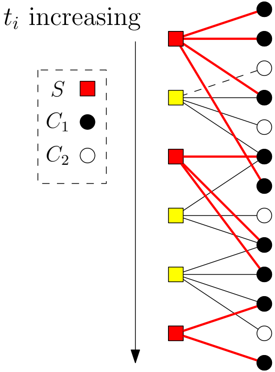

Lemma

- $f + \sum_{j \in C_1} c_j \le \sum_{j \in C_1} \alpha_j$
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- stronger statement:

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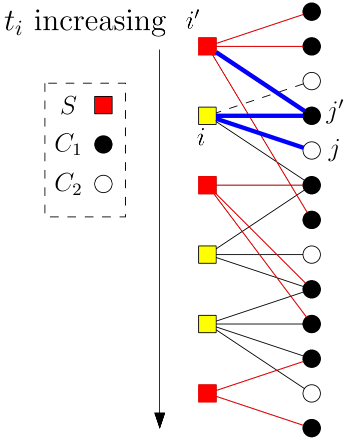
Proof of $\forall j \in C_2, c_j \leq 3\alpha_j$



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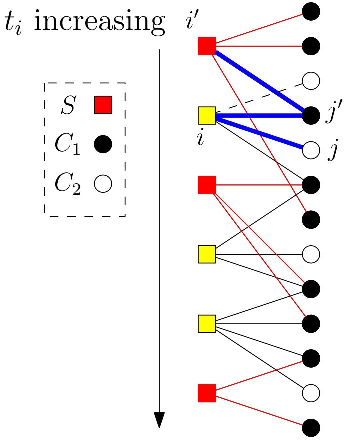

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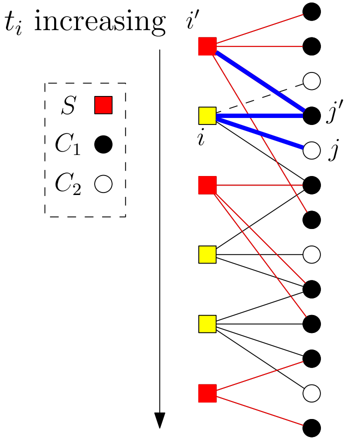

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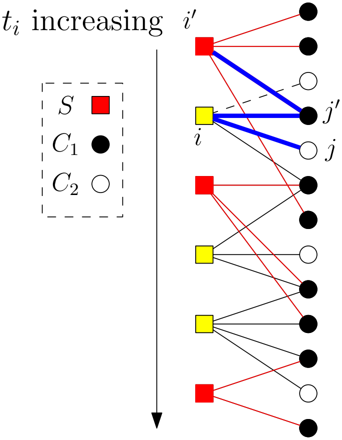

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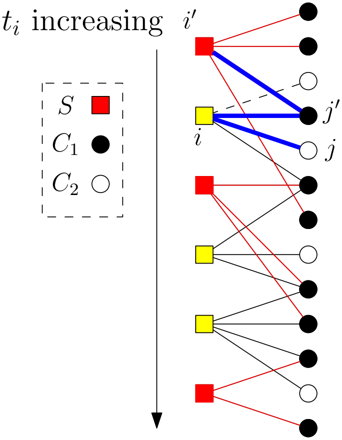

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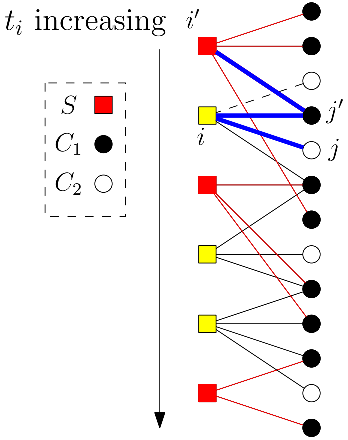

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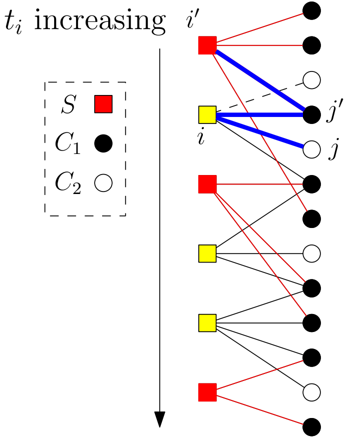

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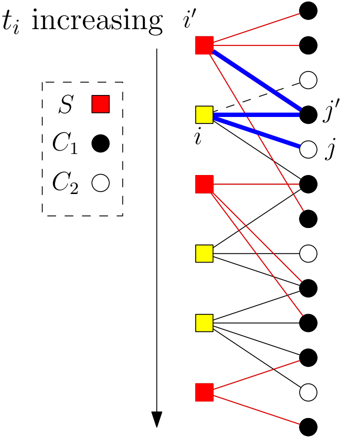

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- $d(j,i') \leq d(j,i) + d(i,j') + d(j',i') \leq \alpha_j + \alpha_j + \alpha_j = 3\alpha_j$