Combinatorics
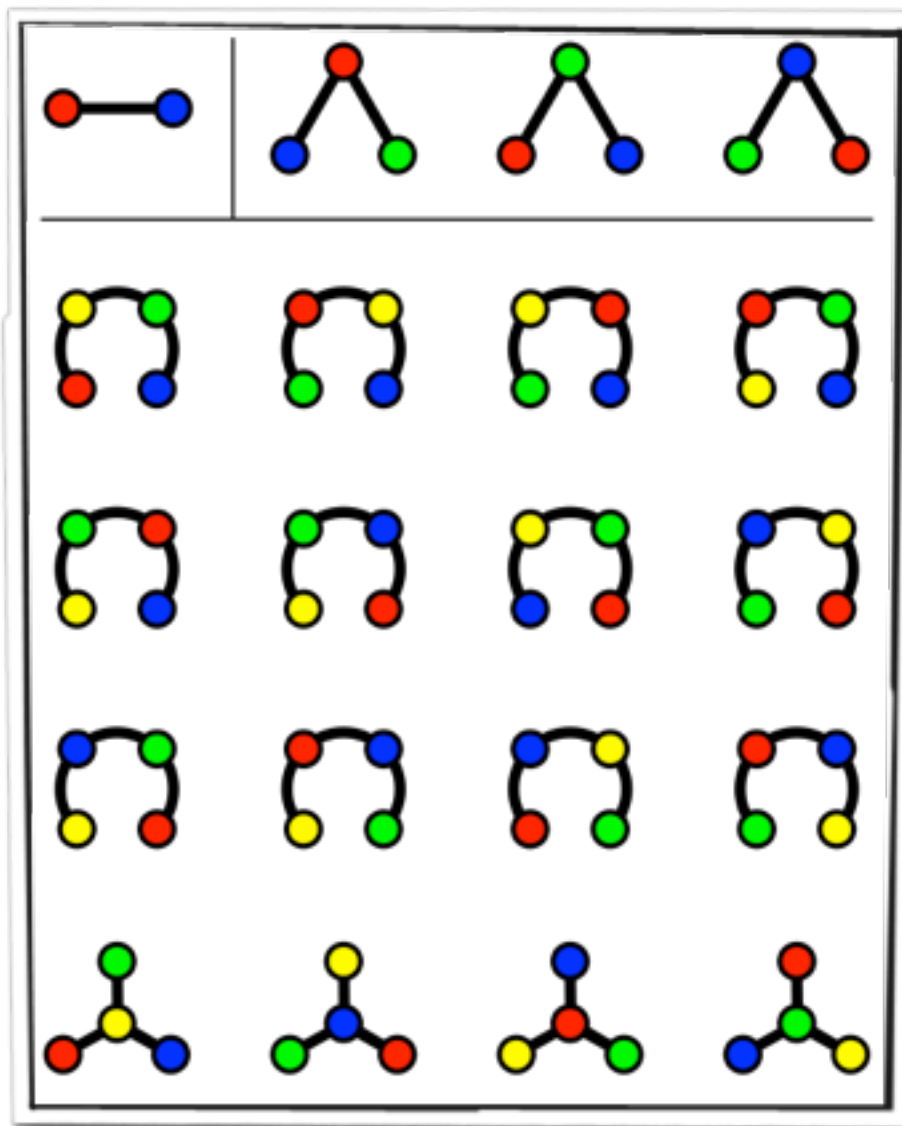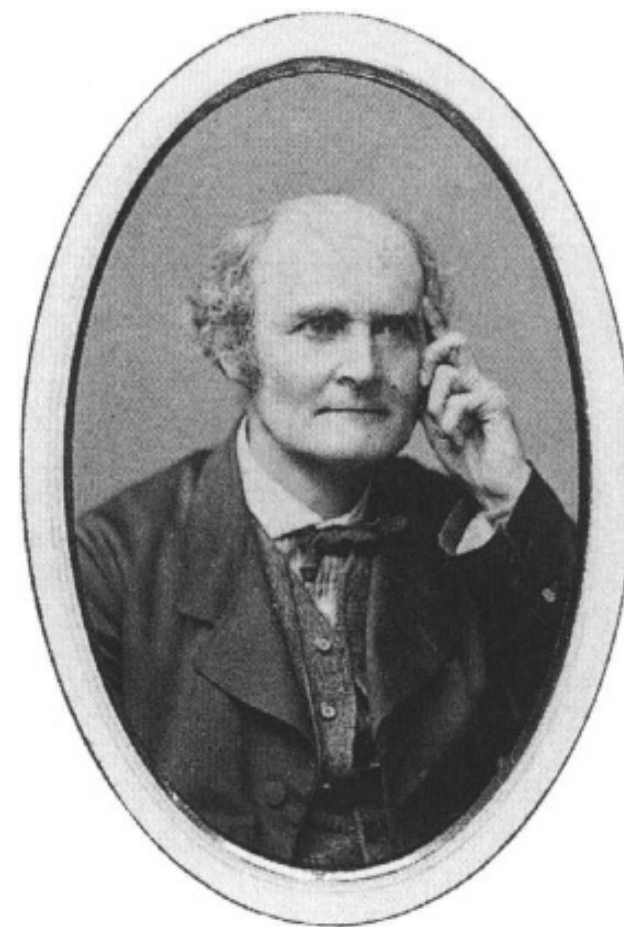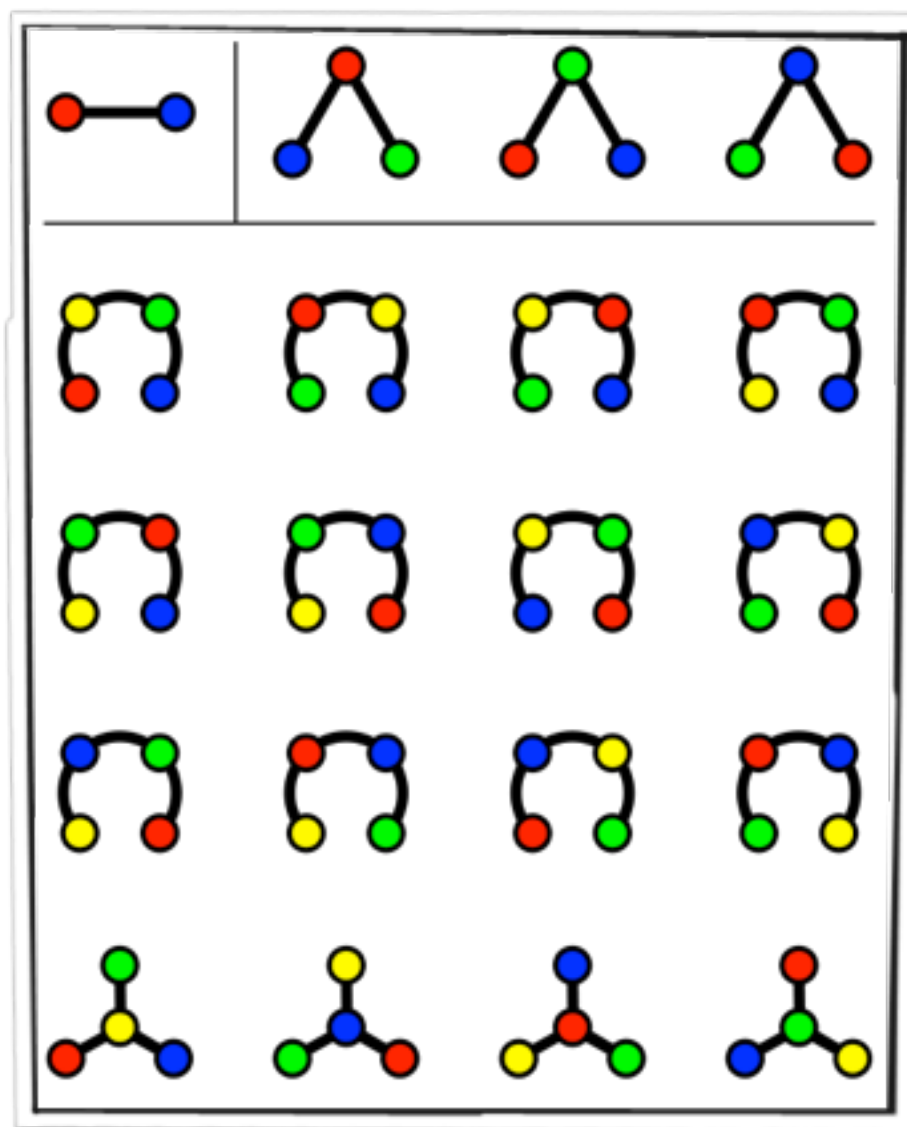
南京大学
尹一通

Counting (labeled) trees

*"How many different trees
can be formed from
n distinct vertices?"*

Cayley's formula:

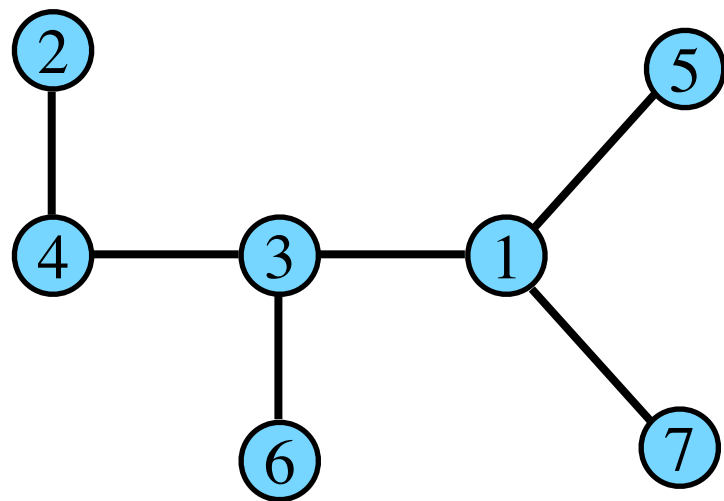There are n^{n-2} trees on n distinct vertices.

Arthur Cayley

Prüfer Code

leaf : vertex of degree 1

removing a leaf from T, still a tree

T_1 :

u_i : $2, 4, 5, 6, 3, 1$

v_i : $4, 3, 1, 3, 1, 7$

$T_1 = T$;

for $i = 1$ to n-1

  u_i : smallest leaf in T_i ;

  (u_i, v_i) : edge in T_i ;
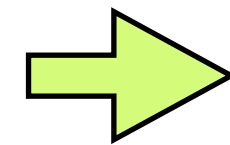
  T_{i+1} = delete u_i from T_i ;

Prüfer code:

$(v_1, v_2, \ldots, v_{n\text{-}2})$
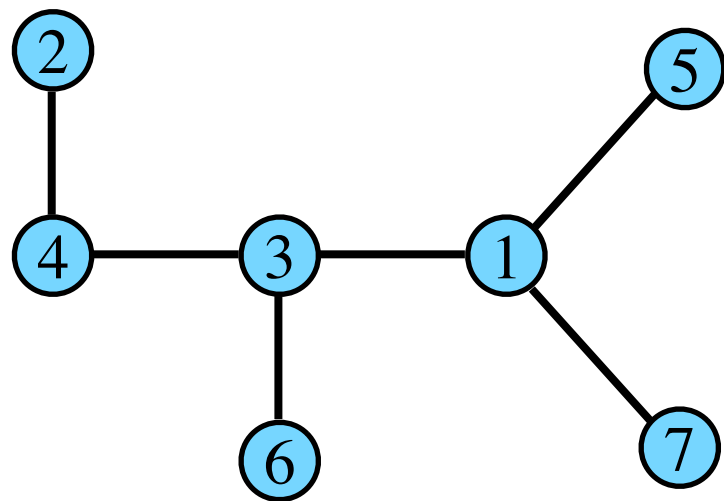
edges of T : (u_i, v_i), $1 \le i \le n-1$

$v_{n-1} = n$

u_i : smallest leaf in T_i
a tree has ≥ 2 leaves

$\Bigg\} \Rightarrow$

n is never deleted
$u_i \ne n$

T :

Only need to recover
every u_i from $(v_1, v_1, \ldots, v_{n-2})$.

u_i is the smallest number not in
$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

u_i : $2, 4, 5, 6, 3, 1$
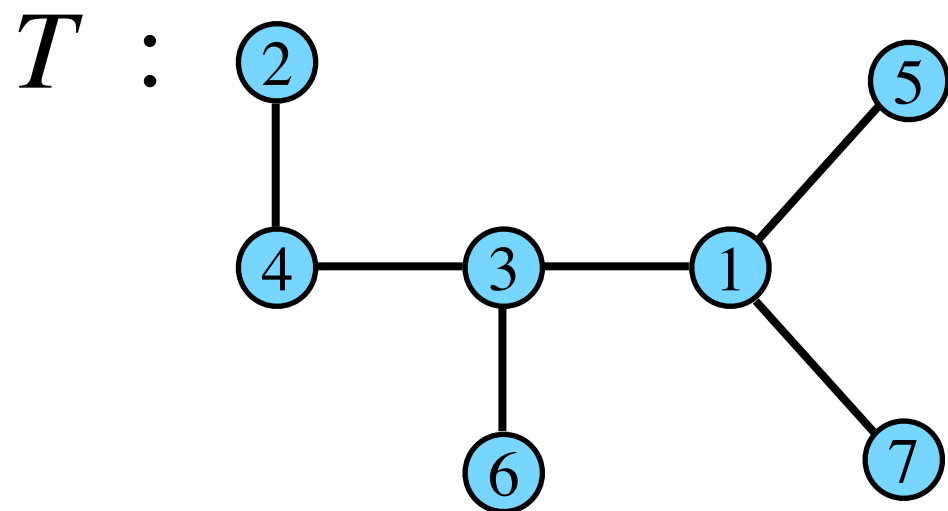
v_i : $4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$

u_i is the smallest number not in
$\{u_1, \dots, u_{i-1}\} \cup \{v_i, \dots, v_{n-1}\}$

$\forall$ vertex v in T,

      # occurrences of v in $u_1, u_2, \dots, u_{n-1}, v_{n-1}$:   1

      # occurrences of v in edges (u_i, v_i), $1 \le i \le n\text{-}1$: $\deg_T(v)$

T :

          # occurrences of v in
Prüfer code: $(v_1, v_2, \dots, v_{n\text{-}2})$

$\deg_T(v)\text{-}1$

u_i : $2, 4, 5, 6, 3, 1$
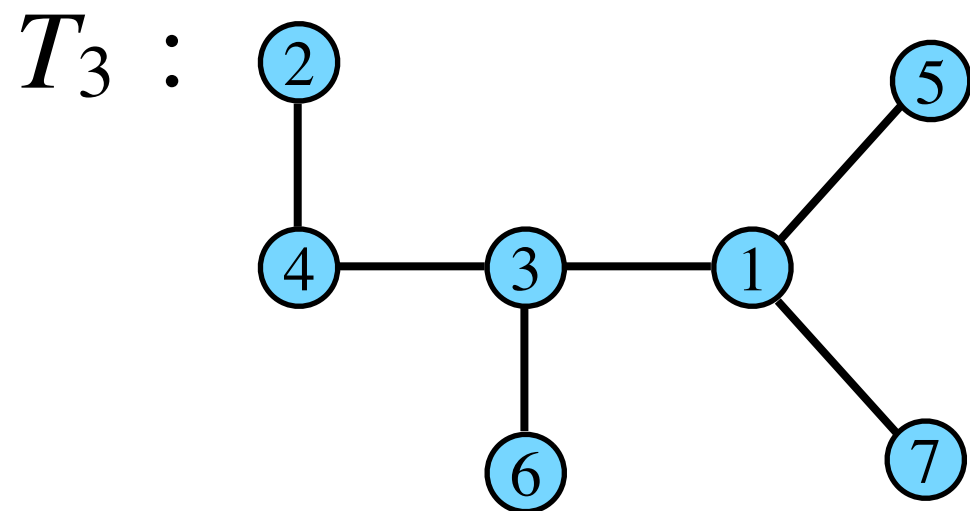
v_i : $4, 3, 1, 3, 1, 7$

$(v_1, v_2, \dots, v_{n\text{-}2})$

u_i is the smallest number not in
$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

$\forall$ vertex v in T_i,

    # occurrences of v in $u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}$:   1

    # occurrences of v in edges (u_j, v_j), $i \leq j \leq n-1$: $\deg_{T_i}(v)$

T_3 :

occurrences of v in $(v_i, \ldots, v_{n-2})$

$$\deg_{T_i}(v) - 1$$

leaf v of T_i :

    in $\{u_i, u_{i+1}, \ldots, u_{n-1}, v_{n-1}\}$
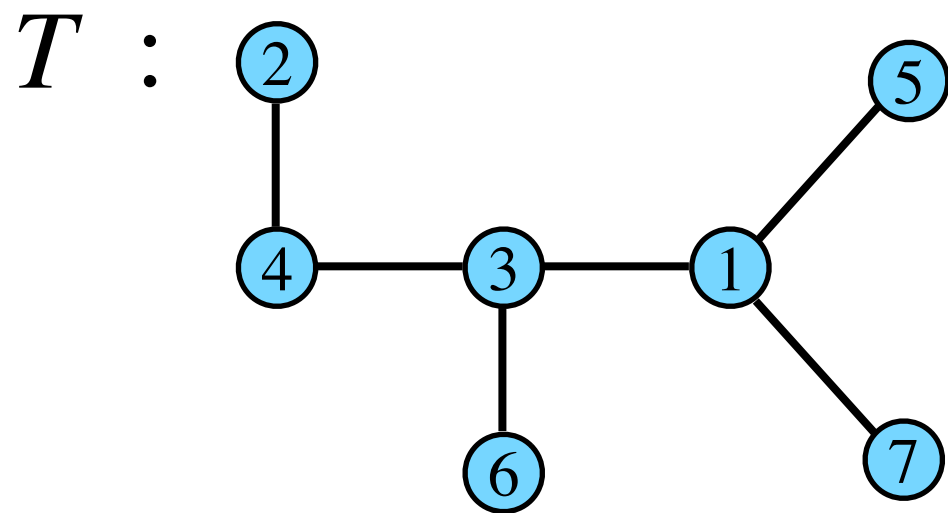
    not in $\{v_i, v_{i+1}, \ldots, v_{n-2}\}$

u_i : ~~2, 4~~, 5, 6, 3, 1

v_i : ~~4, 3~~, 1, 3, 1, 7

$(v_1, v_2, \ldots, v_{n-2})$

u_i : smallest leaf in T_i

u_i is the smallest number not in
$$\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$$

T :

u_i: $2, 4, 5, 6, 3, 1$

v_i: $4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$

T = empty graph;

$v_{n-1} = n$;

for $i = 1$ to n-1

   u_i: smallest number not in

      $\{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

  add edge (u_i, v_i) to T ;

Prüfer code is reversible ➡ 1-1

every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$

is decodable to a tree ➡ onto

T :

$u_i: 2, 4, 5, 6, 3, 1$

$v_i: 4, 3, 1, 3, 1, 7$

$(v_1, v_2, \ldots, v_{n-2})$

T = empty graph;
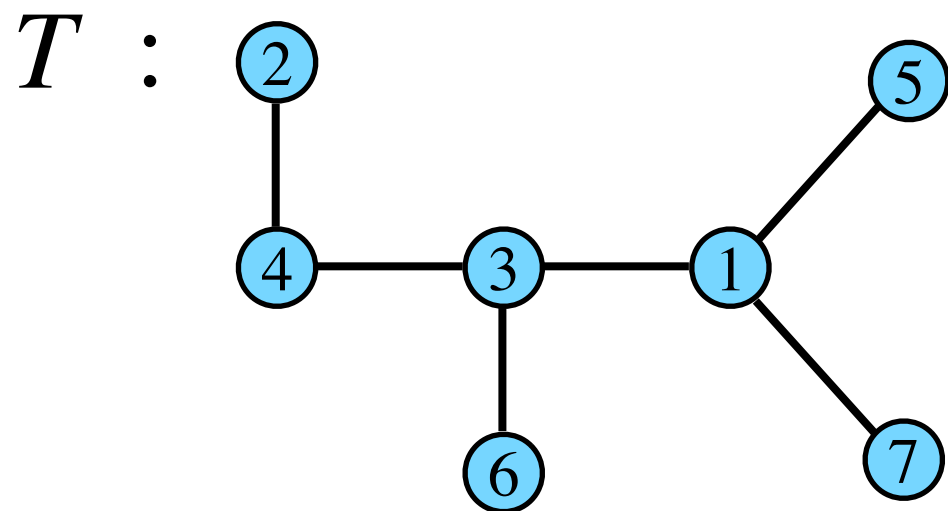
$v_{n-1} = n$;

for $i = 1$ to $n-1$

$\quad u_i$: smallest number not in

$\qquad \{u_1, \ldots, u_{i-1}\} \cup \{v_i, \ldots, v_{n-1}\}$

add edge (u_i, v_i) to T ;

Prüfer code is reversible $\implies$ 1-1

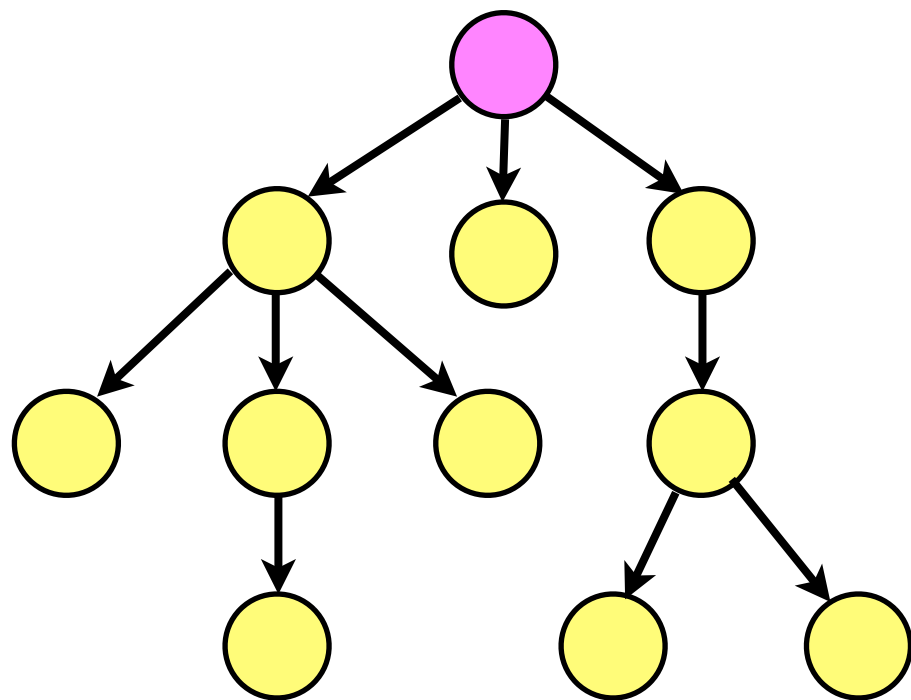every $(v_1, v_2, \ldots, v_{n-2}) \in \{1, 2, \ldots, n\}^{n-2}$

is decodable to a tree $\implies$ onto

Cayley's formula:

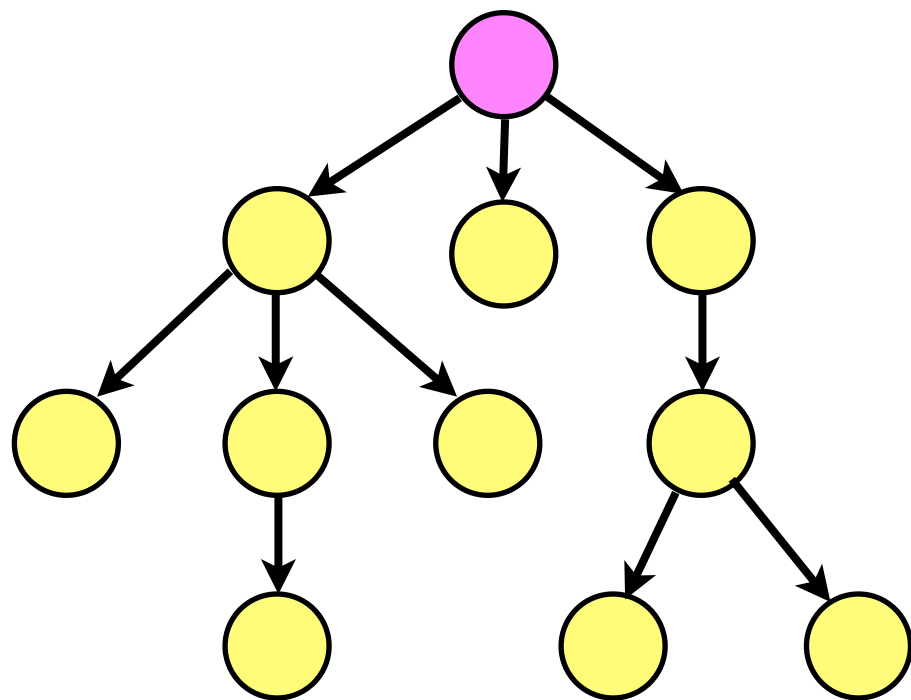There are n^{n-2} trees on n distinct vertices.

Double Counting

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*
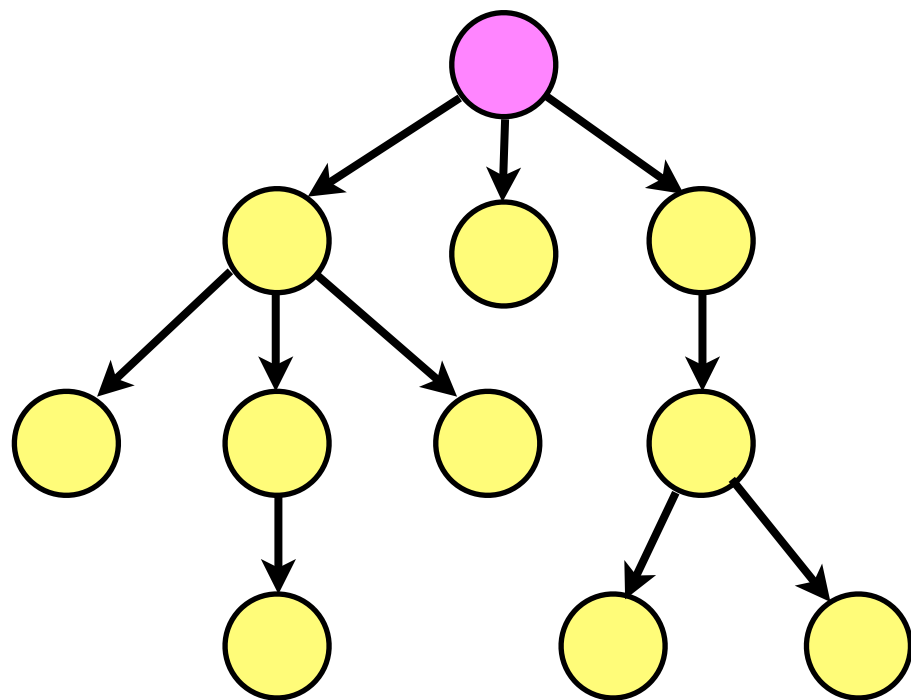
From a tree:
- pick a root;
- pick an order of edges.

$$T_n \, n(n-1)!$$

$$= n! T_n$$

T_n : # of trees on *n* distinct vertices.

of *sequences* of adding *directed edges*
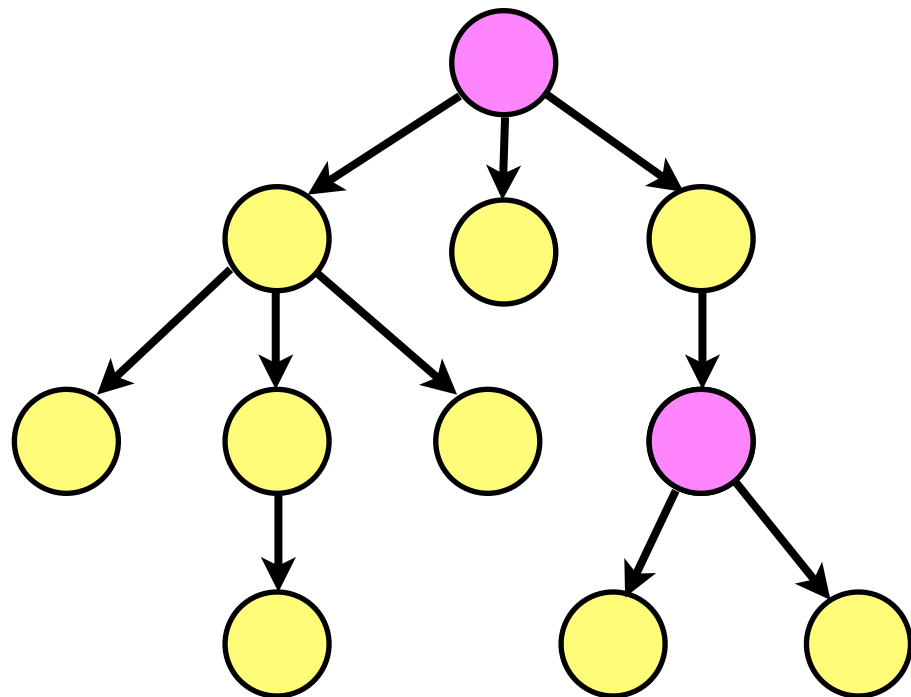to an empty graph to form a *rooted tree*

From an empty graph:

- add edges one by one

of *sequences* of adding *directed edges*
to an empty graph to form a *rooted tree*

From an empty graph: • add edges one by one

Start from n ~~isolated vertices~~
rooted trees

Each step joins 2 trees.

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph: • add edges one by one

8

4

7

3

5

2   1   9

6

10

Start from *n* rooted trees.

After adding *k* edges

n-k rooted trees

add an edge

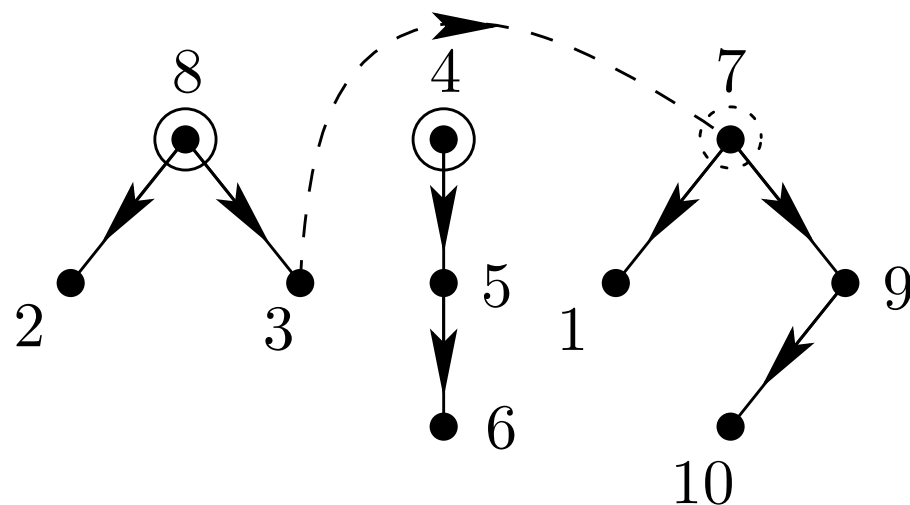| any vertex	→	root of another tree
n		*n-k-*1

of *sequences* of adding *directed edges* to an empty graph to form a *rooted tree*

From an empty graph: • add edges one by one

$$\prod_{k=0}^{n-2} n(n-k-1)$$

$$= n^{n-1} \prod_{k=1}^{n-1} k$$

$$= n^{n-2} n!$$

Start from *n* rooted trees.

After adding *k* edges
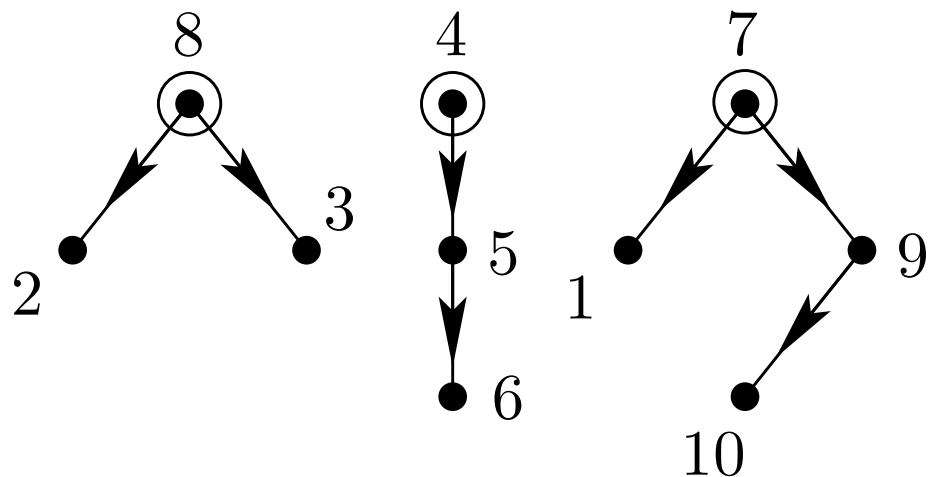
n-k rooted trees

add an edge

any vertex $\longrightarrow$ root of another tree

n n-k-1

of *sequences* of adding *directed edges*
to an empty graph to form a *rooted tree*

From a tree:
- pick a root;
- pick an order of edges.

$$T_n\, n(n-1)!$$
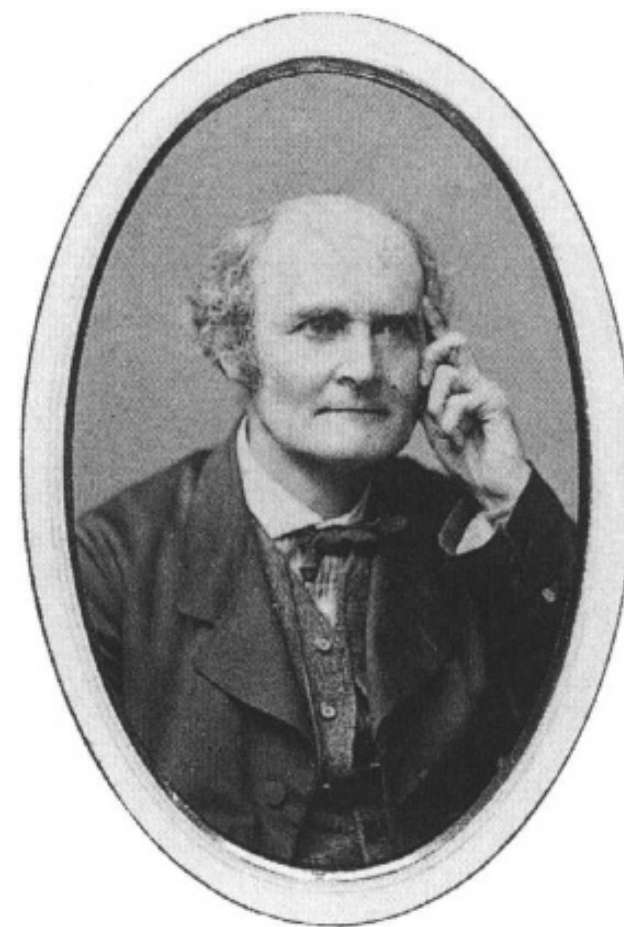
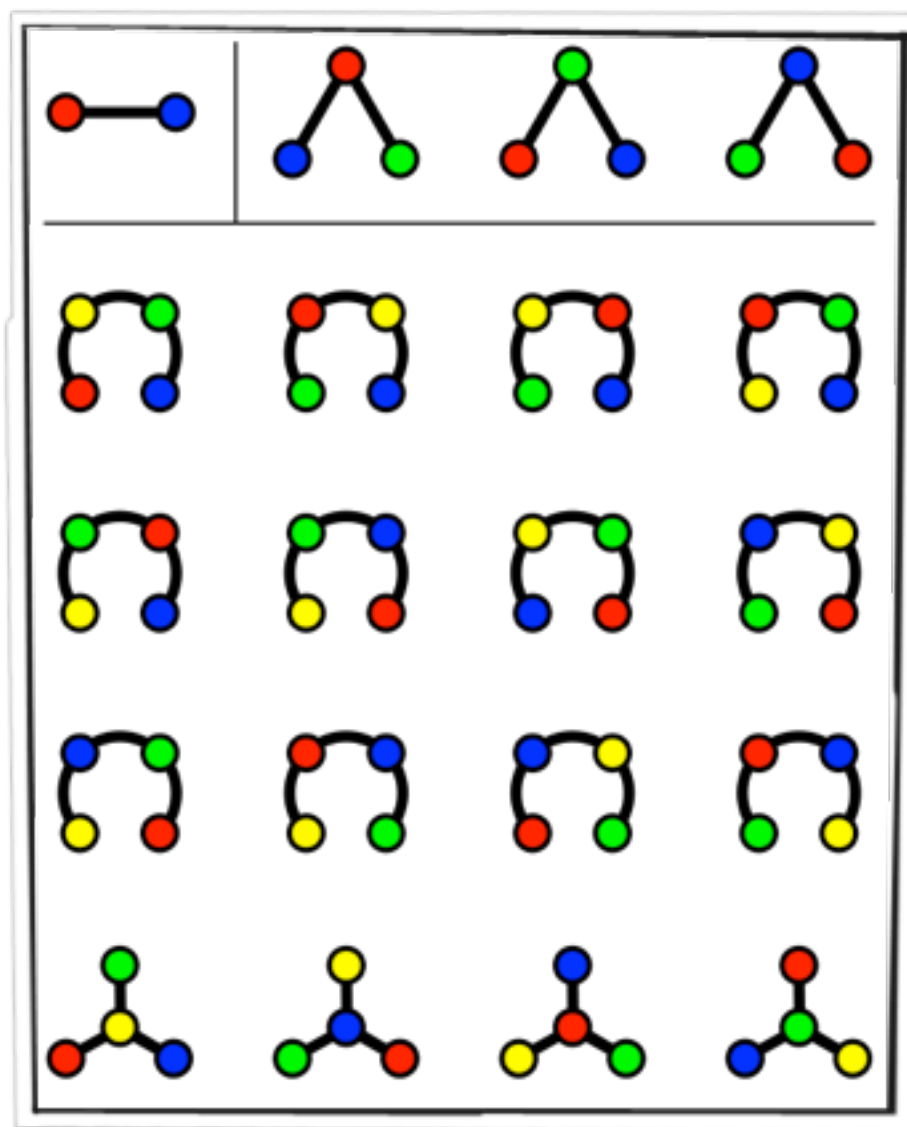$$= n!\,T_n$$

=

From an empty graph:
- add edges one by one

$$\prod_{k=2}^{n} n(k-1)$$

$$= n^{n-2} n!$$

$$T_n = n^{n-2}$$

Cayley's formula:

There are n^{n-2} trees on n distinct vertices.

Arthur Cayley

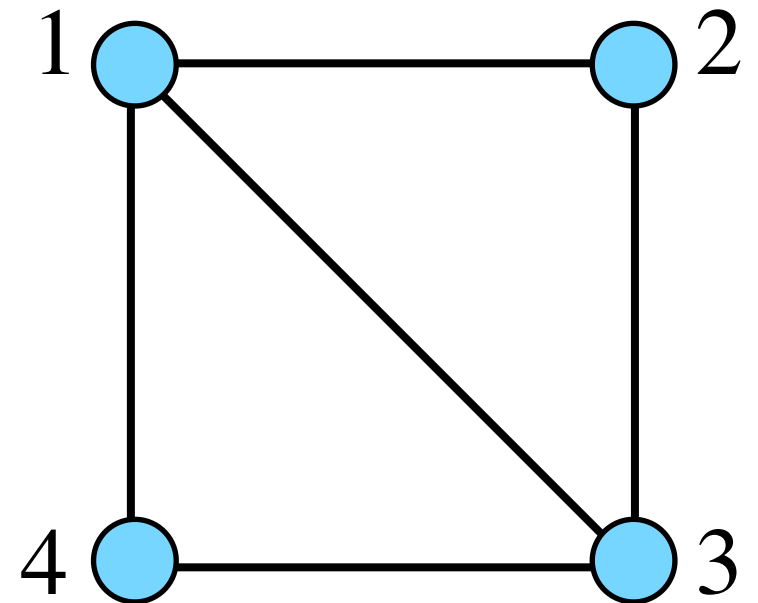# Graph *Laplacian*

Graph $G(V,E)$

adjacency matrix A

$$A(i,j) = \begin{cases} 1 & \{i,j\} \in E \\ 0 & \{i,j\} \notin E \end{cases}$$

diagonal matrix D

$$D(i,j) = \begin{cases} \deg(i) & i = j \\ 0 & i \neq j \end{cases}$$

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & d_n \end{bmatrix}$$

graph *Laplacian* L

$$L = D - A$$

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$
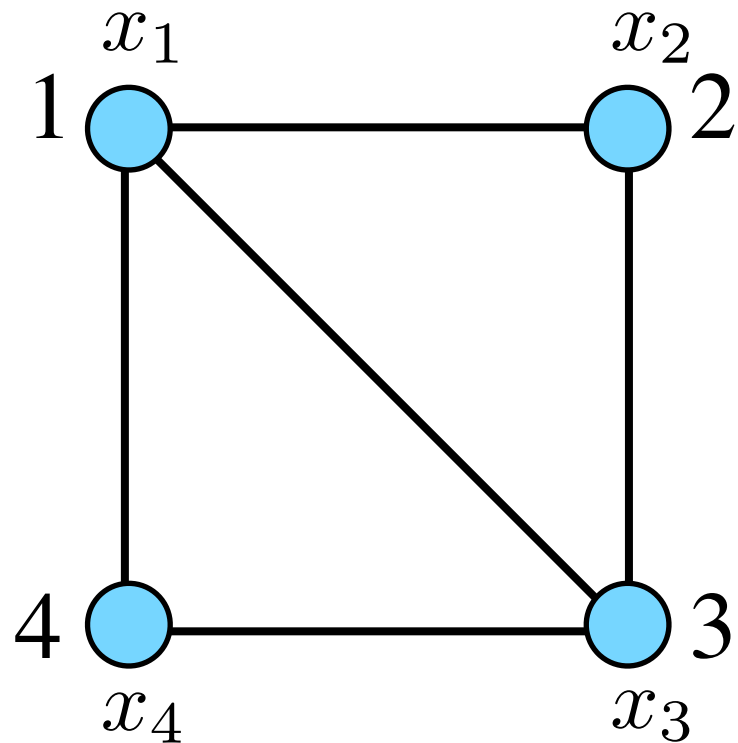
Graph *Laplacian*

graph *Laplacian* L

$$L(i,j) = \begin{cases} \deg(i) & i = j \\ -1 & i \neq j, \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

quadratic form:

$$xLx^T = \sum_i d_i x_i^2 - \sum_{ij \in E} x_i x_j = \frac{1}{2} \sum_{ij \in E} (x_i - x_j)^2$$
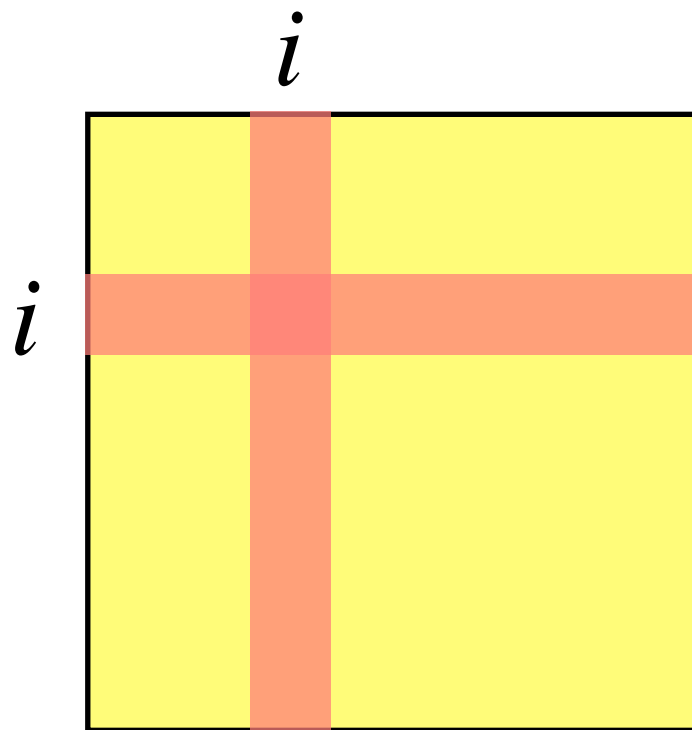
incidence matrix $B : \ n \times m$

$$i \in V, e \in E$$
$$B(i,e) = \begin{cases} 1 & e = \{i,j\}, i < j \\ -1 & e = \{i,j\}, i > j \\ 0 & \text{otherwise} \end{cases}$$

$$L = BB^T$$

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L by removing

ith row and ith collumn

$t(G)$: number of spanning trees in G

Kirchhoff's matrix-tree theorem

$L_{i,i}$: submatrix of L by removing
ith row and ith collumn

$t(G)$: number of spanning trees in G

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

$B_i: \ (n-1) \times m$
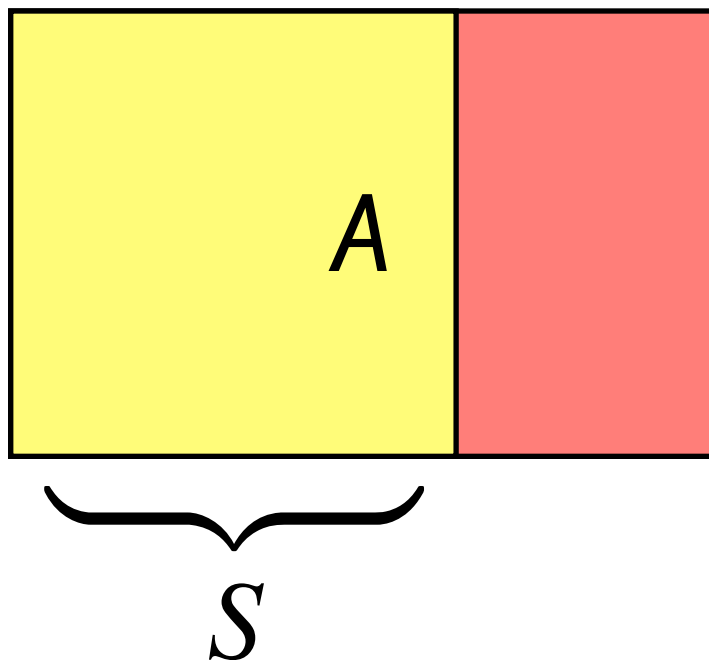
incidence matrix B removing ith row

$$L = BB^T$$

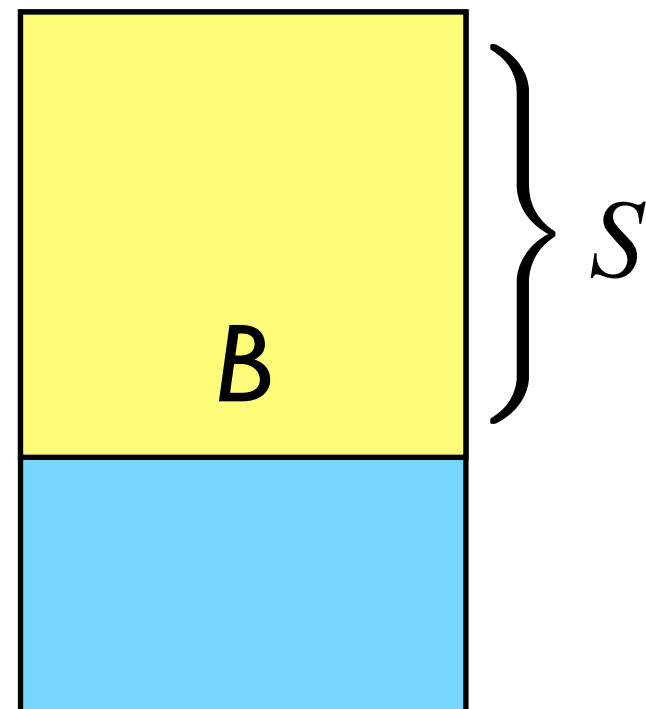$$L_{i,i} = B_i B_i^T \qquad \det(L_{i,i}) = \det(B_i B_i^T) = ?$$

Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$

$A : n \times m$

$B : m \times n$

A

B

S

S

Cauchy-Binet Theorem:

$$\det(AB) = \sum_{S \in \binom{[m]}{n}} \det(A_{[n],S}) \det(B_{S,[n]})$$

$$\det(L_{i,i}) = \det(B_i B_i^T)$$

$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S}) \det(B_{S,[n]\setminus\{i\}}^T)$$

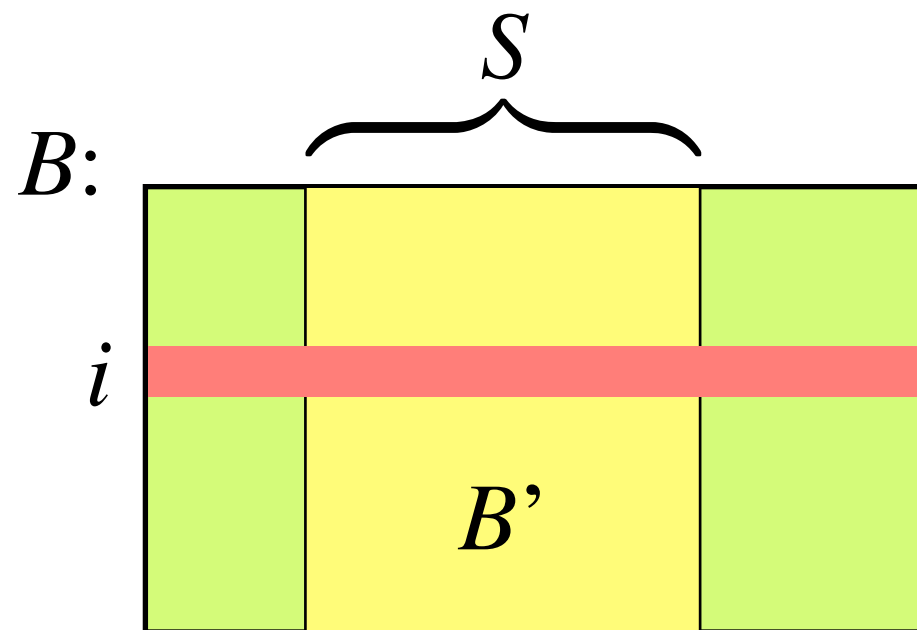$$= \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n]\setminus\{i\},S})^2$$

$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n] \setminus \{i\}, S})^2$$

$$j \in [n] \setminus \{i\}, e \in S$$

$$B_{[n] \setminus \{i\}, S}(j, e) = \begin{cases} 1 & e = \{j, k\}, j < k \\ -1 & e = \{j, k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\det(B_{[n] \setminus \{i\}, S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$
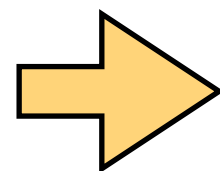
$$\det(B_{[n]\setminus\{i\},S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$

S

B:

i

B'

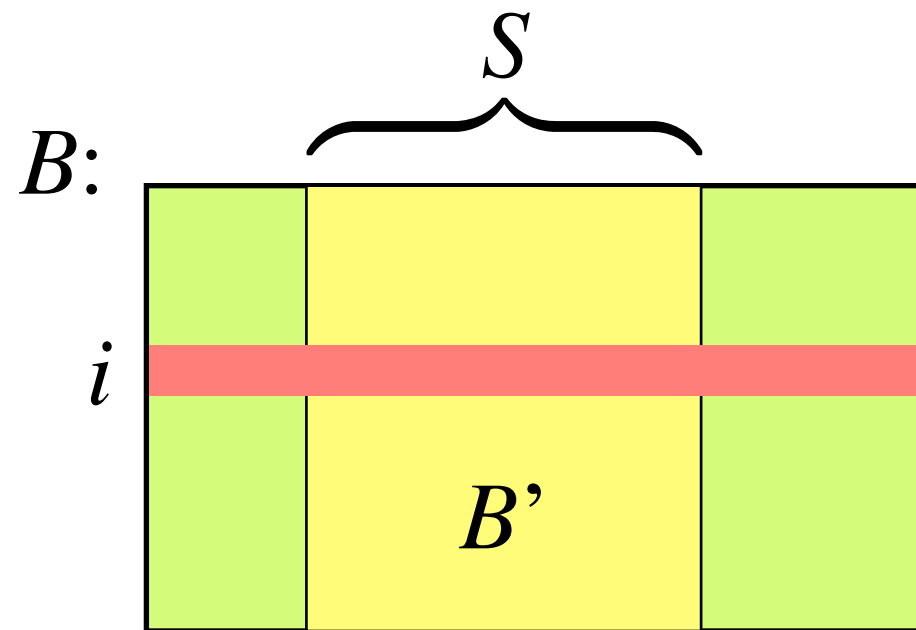$B' = B_{[n]\setminus\{i\},S}$

$(n-1) \times (n-1)$ **matrix:**

every column contains
at most one 1 and at most one -1
and all other entries are 0

$\implies$ $\det(B') \in \{-1, 0, 1\}$

$$\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}$$

$$\det(B') \neq 0 \text{ iff } S \text{ is a spanning tree}$$

S

B:

i

B'

S is not a spanning tree:

$\exists$ a connected component R

s.t. $i \notin R$

$\Rightarrow \quad \det(B') = 0$
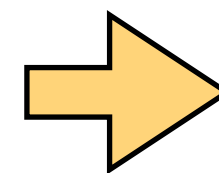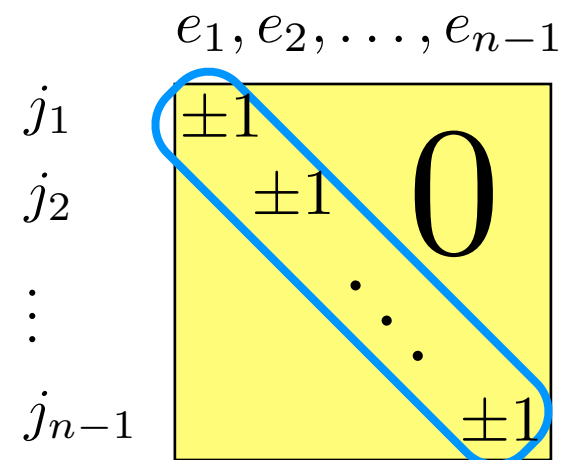
S is a spanning tree:

$\exists$ a leaf $j_1 \neq i$ with incident edge e_1, delete e_1
$\exists$ a leaf $j_2 \neq i$ with incident edge e_2, delete e_2
$\vdots$

vertices: $j_1, j_2, \ldots, j_{n-1}$
edges: $e_1, e_2, \ldots, e_{n-1}$

$e_1, e_2, \ldots, e_{n-1}$

j_1 ± 1
j_2 ± 1 0
$\vdots$ $\ddots$
j_{n-1} ± 1

$\Rightarrow \quad \det(B') = \pm 1$

Cauchy-Binet

$$\det(L_{i,i}) = \sum_{S \in \binom{[m]}{n-1}} \det(B_{[n] \setminus \{i\}, S})^2$$

$j \in [n] \setminus \{i\}, e \in S$

$$B_{[n] \setminus \{i\}, S}(j, e) = \begin{cases} 1 & e = \{j, k\}, j < k \\ -1 & e = \{j, k\}, j > k \\ 0 & \text{otherwise} \end{cases}$$

$$\det(B_{[n] \setminus \{i\}, S}) = \begin{cases} \pm 1 & S \text{ is a spanning tree of } G \\ 0 & \text{otherwise} \end{cases}$$

Kirchhoff's Matrix-Tree Theorem:

$$\forall i, \quad t(G) = \det(L_{i,i})$$

all n-vertex trees: spanning trees of K_n

$$L_{i,i} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

Cayley formula:

$$T_n = t(K_n) = \det(L_{i,i}) = n^{n-2}$$