Combinatorics

南京大学
尹一通

Matching Theory

System of Distinct Representatives
(Transversal)

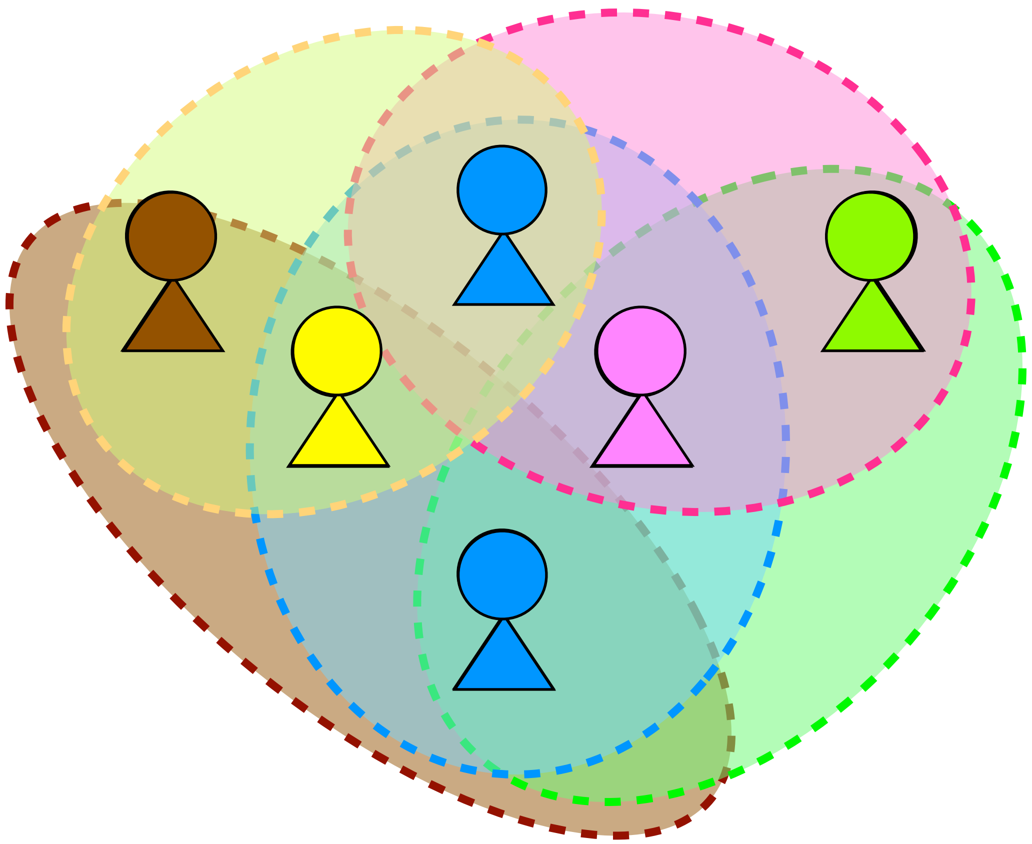
system of distinct representatives (SDR)

for sets $S_1, S_2, \ldots, S_m$

distinct $x_1, x_2, \ldots, x_m$

$$x_i \in S_i$$

for $i = 1, 2, \ldots, m$



Marriage Problem

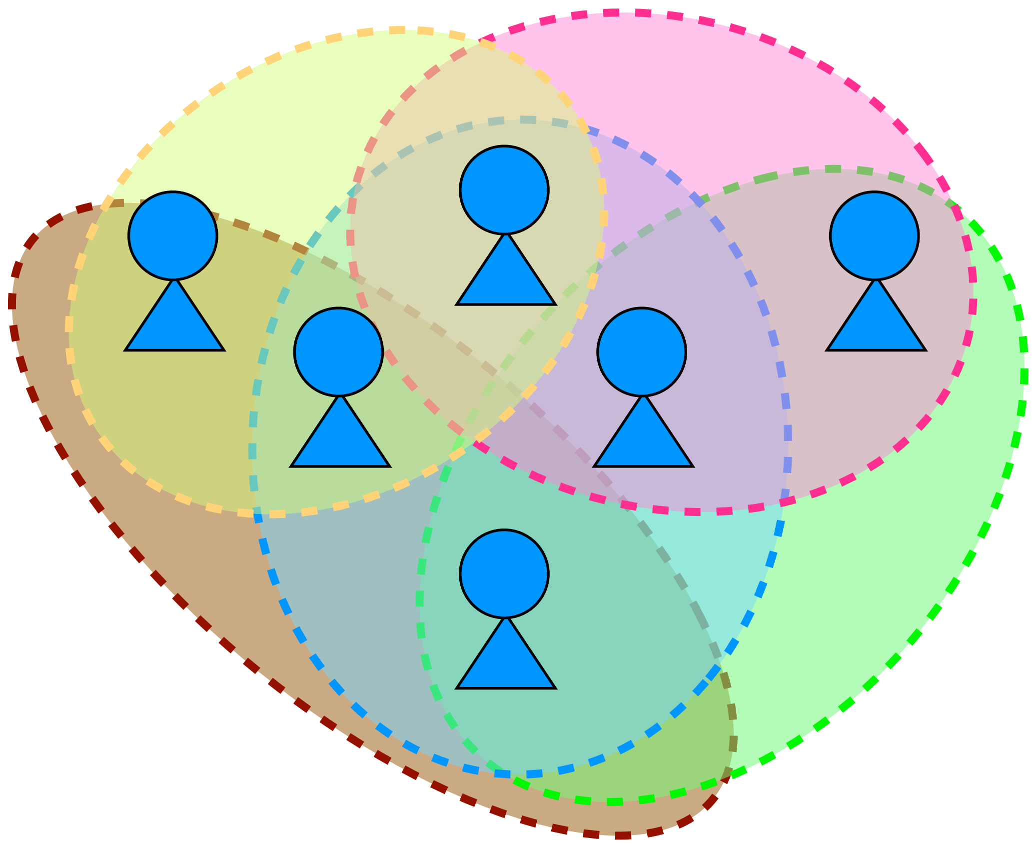
Does there exist an SDR for

$$S_1, S_2, \ldots, S_m \ ?$$

m girls

S_i : boys that girl i likes

"Is there a way of marrying these m girls?"



$S_1, S_2, \ldots, S_m$ have a SDR $\Rightarrow$

$\exists$ dinstinct $x_1 \in S_1, x_2 \in S_2, \ldots, x_m \in S_m$

$\Downarrow$

$\forall I \subseteq \{1, 2, \ldots, m\},$

$$\left|\bigcup_{i \in I} S_i\right| \ \geq |\{x_i \mid i \in I\}| \ \geq |I|.$$

distinct

$S_1, S_2, \ldots, S_m$ have a SDR $\Rightarrow$

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

Hall's Theorem (marriage theorem)

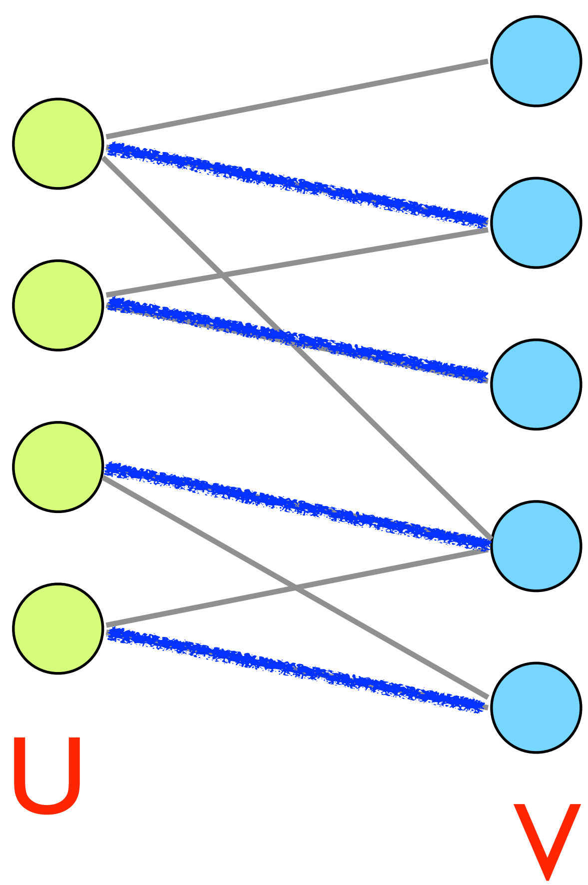
$S_1, S_2, \ldots, S_m$ have a SDR $\Longleftrightarrow$

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

Hall's Theorem (graph theory form)

A bipartite graph G(U,V,E) has a matching of U

$\Leftrightarrow$ $|N(S)| \geq |S|$ for all $S \subseteq U$



matching: edge independent set

$M \subseteq E$ with

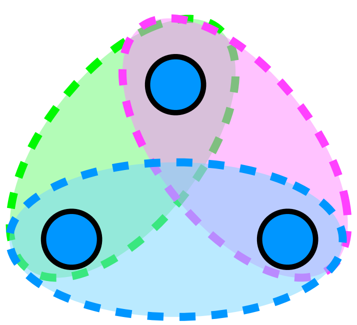
no $e_1, e_2 \in M$ share a vertex

$$N(S) = \{v \mid \exists u \in S, uv \in \mathrm{E}\}$$

Hall's Theorem (marriage theorem)

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR



critical family: $S_1, S_2, \ldots, S_k$ $\quad k < m$

$$\left|\bigcup_{i=1}^{k} S_i\right| = k$$

Induction on m : $\quad m = 1$, trivial

case.1: there is no critical family in $S_1, S_2, \ldots, S_m$

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

Hall's Theorem (marriage theorem)

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

case.1: there is no critical family in S_1, S_2, ..., S_m

$$\forall I \subseteq \{1, 2, \ldots, m\} \text{ that } |I| < m, \quad \left|\bigcup_{i\in I} S_i\right| > |I|$$

take an arbitrary $x \in S_m$ as representative of S_m

remove S_m and x $\quad S_i' = S_i \setminus \{x\} \quad i = 1, 2, ..., m\text{-}1$

$$\forall I \subseteq \{1, 2, \ldots, m-1\}, \quad \left|\bigcup_{i\in I} S_i'\right| \geq |I|$$

due to I.H. $S_1', \ldots, S_{m-1}'$ have a SDR $\{x_1, \ldots, x_{m-1}\}$

$x_1, \ldots, x_{m-1}$ and x form a SDR for S_1, S_2, ..., S_m

Hall's Theorem (marriage theorem)

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

say $|S_{m-k+1} \cup \cdots \cup S_m| = k \qquad k < m$

due to I.H. $S_{m-k+1}, \ldots, S_m$ have a SDR $X=\{x_1, \ldots, x_k\}$

$S_i' = S_i \setminus X \qquad i = 1, 2, \ldots, m-k$

$\forall I \subseteq \{1, 2, \ldots, m-k\},$

$$\left|\bigcup_{i=m-k+1}^{m} S_i \cup \bigcup_{i \in I} S_i\right| \geq k + |I|$$

$$\Rightarrow \quad \left|\bigcup_{i \in I} S_i'\right| \geq |I|$$

Hall's Theorem (marriage theorem)

$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

⇒ $S_1, S_2, \ldots, S_m$ have a SDR

case.2: there is a critical family in $S_1, S_2, \ldots, S_m$

say $|S_{m-k+1} \cup \cdots \cup S_m| = k$ $\quad k < m$

due to I.H. $S_{m-k+1}, \ldots, S_m$ have a SDR $X = \{x_1, \ldots, x_k\}$

$S_i' = S_i \setminus X \qquad i = 1, 2, \ldots, m-k$

$\forall I \subseteq \{1, 2, \ldots, m-k\}, \left|\bigcup_{i \in I} S_i'\right| \geq |I|$

due to I.H.

$S_1', \ldots, S_{m-k}'$ have a SDR $Y = \{y_1, \ldots, y_{m-k}\}$

X and Y form a SDR for $S_1, S_2, \ldots, S_m$

Hall's Theorem (marriage theorem)

$$S_1, S_2, \ldots, S_m \text{ have a SDR} \quad \Longleftrightarrow$$

$$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$$

Min-Max Theorems

- König-Egerváry theorem: in bipartite graph

  min vertex cover = max matching

- Dilworth's theorem: in poset

  min chain-decomposition = max antichain

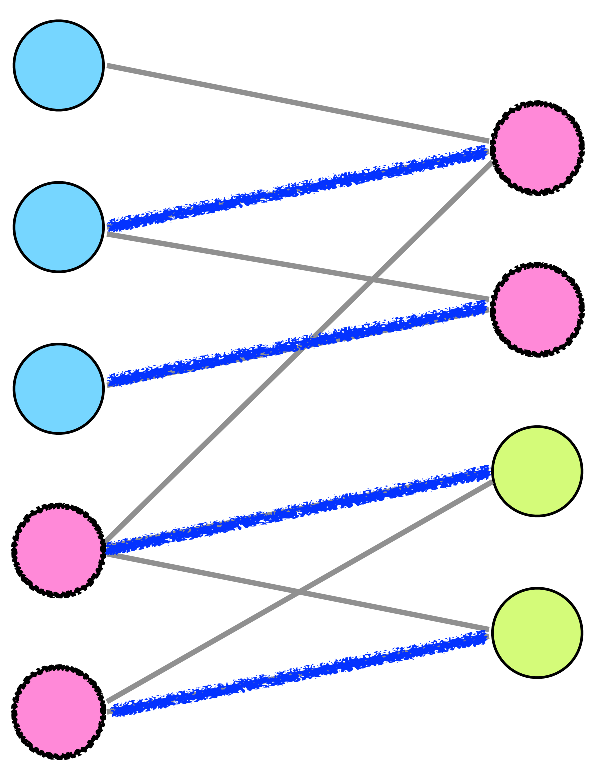
- Menger's theorem: in graph

  min vertex-cut = max vertex-disjoint paths

König-Egerváry theorem

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



matching: $M \subseteq E$

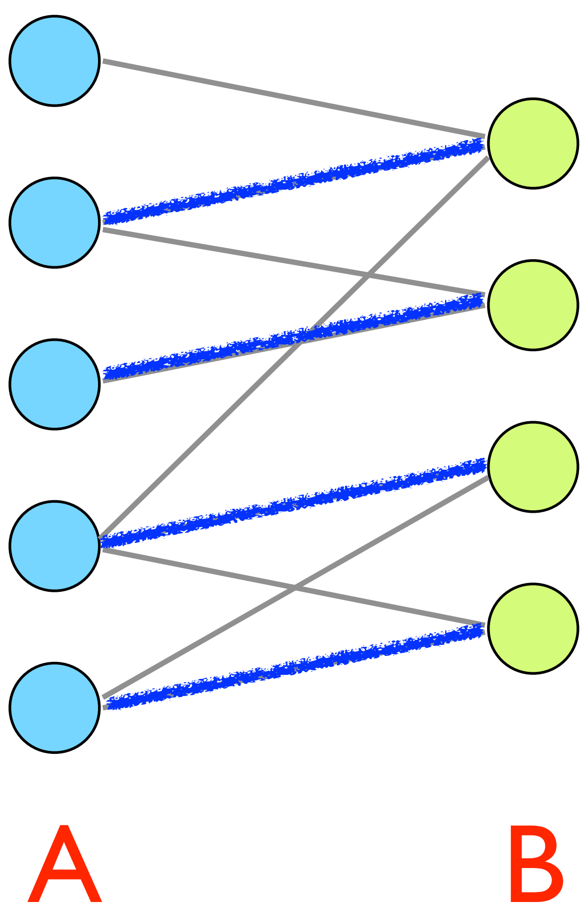
no $e_1, e_2 \in M$ share a vertex

vertex cover: $C \subseteq V$

all $e \in E$ adjacent to some $v \in C$

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



matching:

independent 1s

do not share row/column

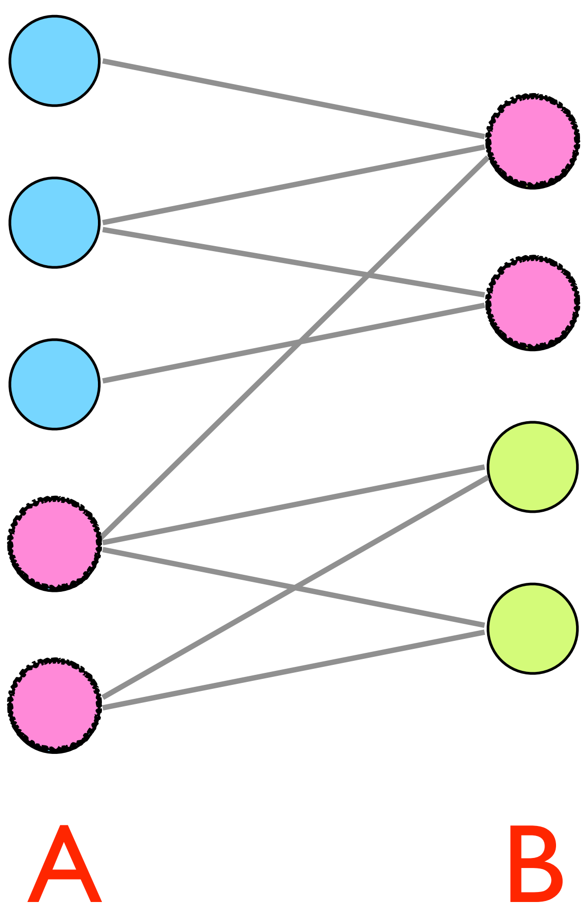
B

1	0	0	0
1	1	0	0
0	**1**	0	0
1	0	**1**	1
0	0	1	**1**

A

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.



vertex cover:

rows/columns covering all 1s

B

1	0	0	0
1	1	0	0
0	1	0	0
1	0	1	1
0	0	1	1

A

Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

König-Egerváry Theorem (matrix form)

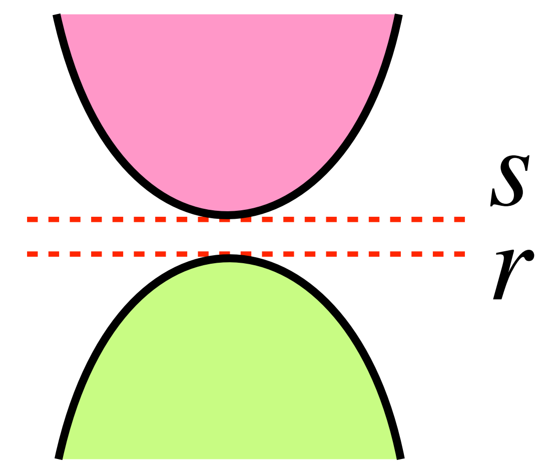
For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.

A : $m \times n$ 0-1 matrix

s : min # of rows/columns covering all 1's

r : max # of independent 1's

$$r \leq s$$



any r independent 1's

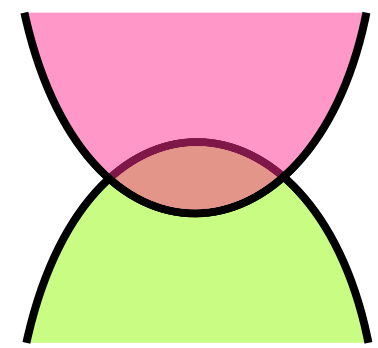
requires r rows/columns to cover

A : $m \times n$ 0-1 matrix

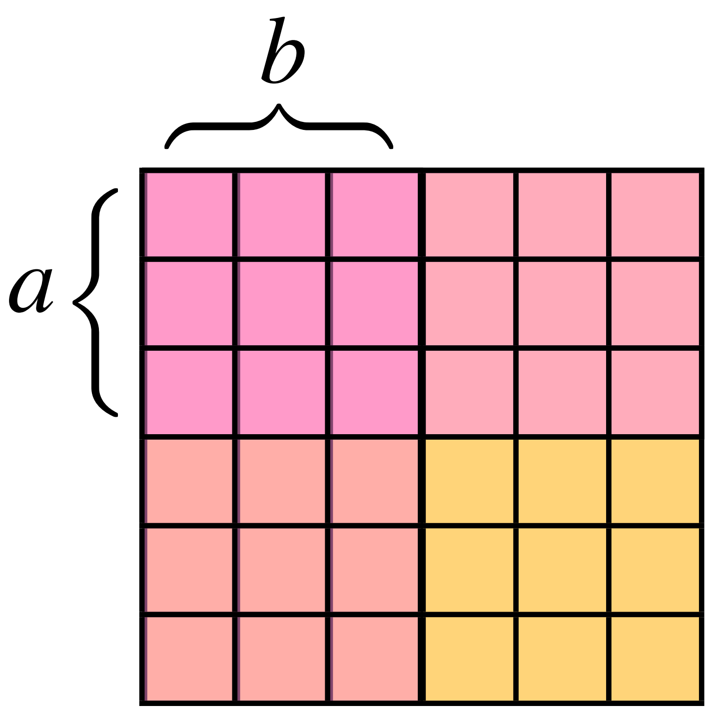
s : min # of rows/columns covering all 1's

r : max # of independent 1's

$r \geq s$



min covering: s = a rows + b columns



$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

D has b independent 1's

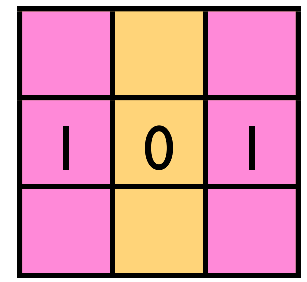
A has min covering: s = a rows + b columns

$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

S_2



S_1, S_2, ..., S_a have a SDR

otherwise $\exists 1 \le |I| \le a, \quad \left|\bigcup_{i\in I} S_i\right| < |I|$ (Hall)

C can be covered by $(a - |I|)$ rows $+ \left|\bigcup_{i\in I} S_i\right|$ columns

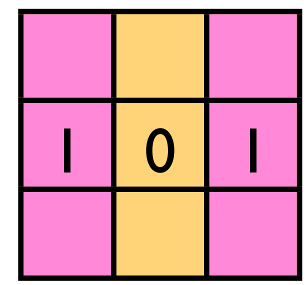
A has min covering: s = a rows + b columns

$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

S_2



S_1, S_2, ..., S_a have a SDR

otherwise $\exists 1 \le |I| \le a, \quad \left|\bigcup_{i\in I} S_i\right| < |I|$ (Hall)

C can be covered by $< a$ rows&columns

A can be covered by $< a+b$ rows&columns

contradiction!

A has min covering: s = a rows + b columns

$$A = \begin{bmatrix} B_{a \times b} & C_{a \times (n-b)} \\ D_{(m-a) \times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

$$S_i = \{j \mid C_{ij} = 1\}$$

$S_1, S_2, \ldots, S_a$ have a SDR

SDR: distinct $j_1, j_2, \ldots, j_a$

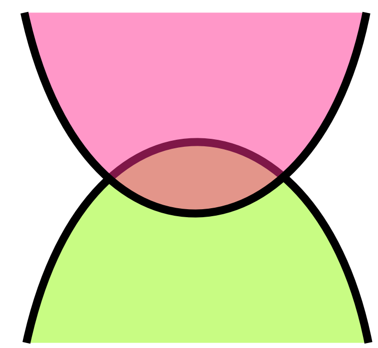
$$C(i, j_i) = 1$$

A : $m \times n$ 0-1 matrix

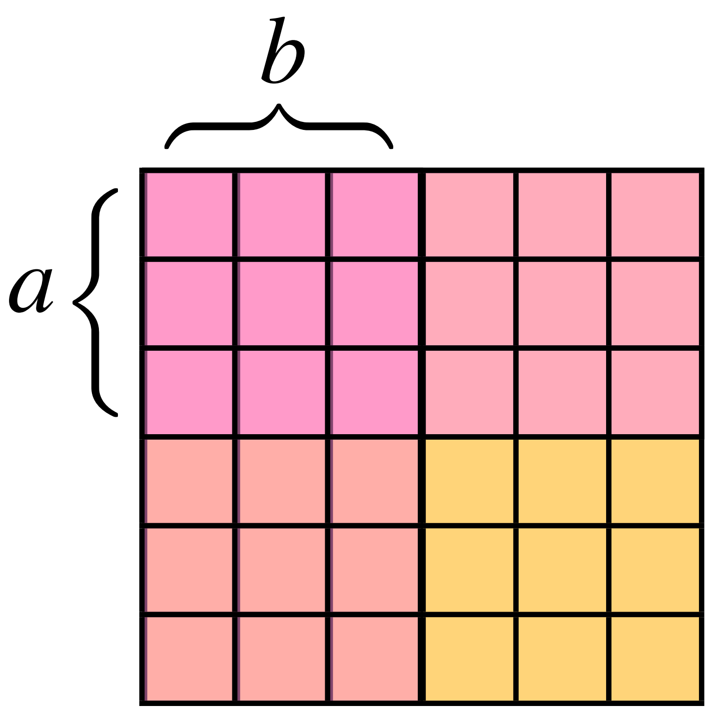
r : max # of independent 1's

s : min # of rows/columns covering all 1's

$$r \geq s$$



A has min covering: s = a rows + b columns



$$A = \begin{bmatrix} B_{a\times b} & C_{a\times(n-b)} \\ D_{(m-a)\times b} & \mathbf{0} \end{bmatrix}$$

C has a independent 1's

D has b independent 1's

König-Egerváry Theorem (matrix form)

For any 0-1 matrix, the maximum number of independent 1's equals the minimum number of rows and columns required to cover all the 1's.

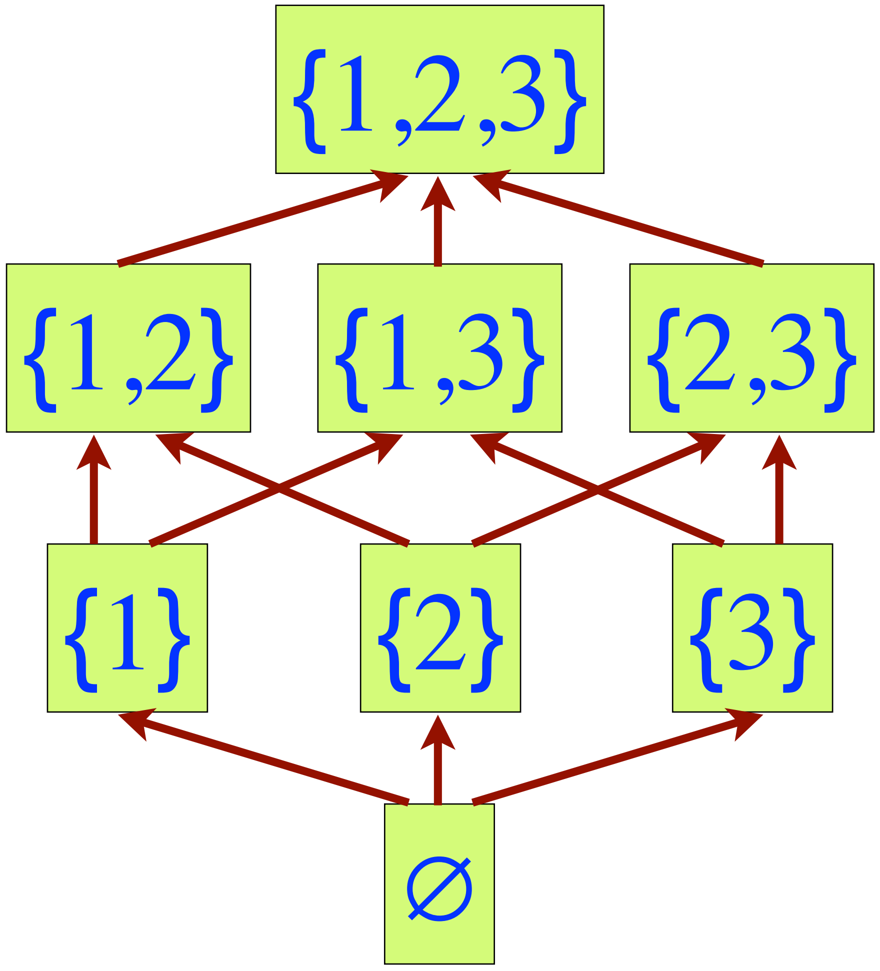
Theorem (König 1931, Egerváry 1931)

In a bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

Poset

$\mathcal{F} \subseteq 2^{[n]}$ with ⊆ define a

partially ordered set (poset)



reflexivity: A⊆A

antisymmetry:

A⊆B and B⊆A ⇒ A=B

transitivity:

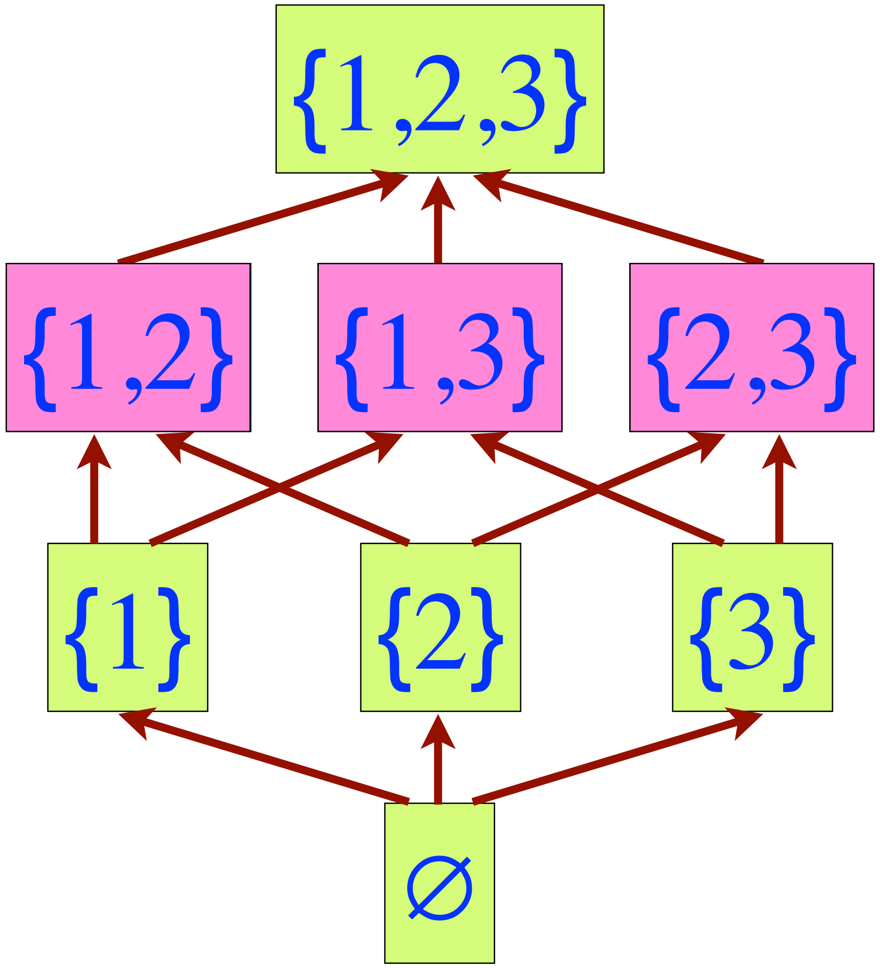
A⊆B and B⊆C ⇒ A⊆C

chain: $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$

antichain: $A_1, A_2, \ldots, A_r$ that $\forall\, A_i, A_j,\quad A_i \not\subseteq A_j$

Dilworth's Theorem

Size of the largest antichain in the poset P = size of the smallest partition of P into chains.



Dilworth's Theorem

Size of the largest antichain in the poset P = size of the smallest partition of P into chains.

Suppose: P has an antichain of size r.

P can be partitioned to s chains.

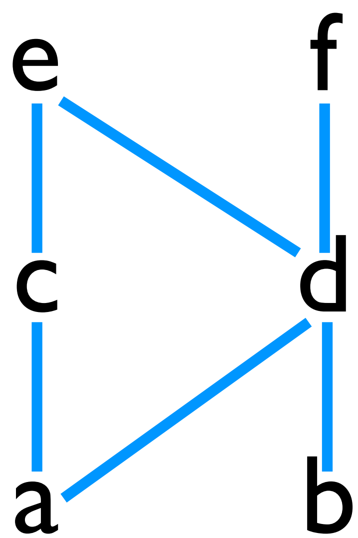
$r \leq s$

antichain A, chain C $|A \cap C| \leq 1$

We only need to prove:

There exist an antichain $A \subseteq P$ of size r and a partition of P into r chains.

poset P

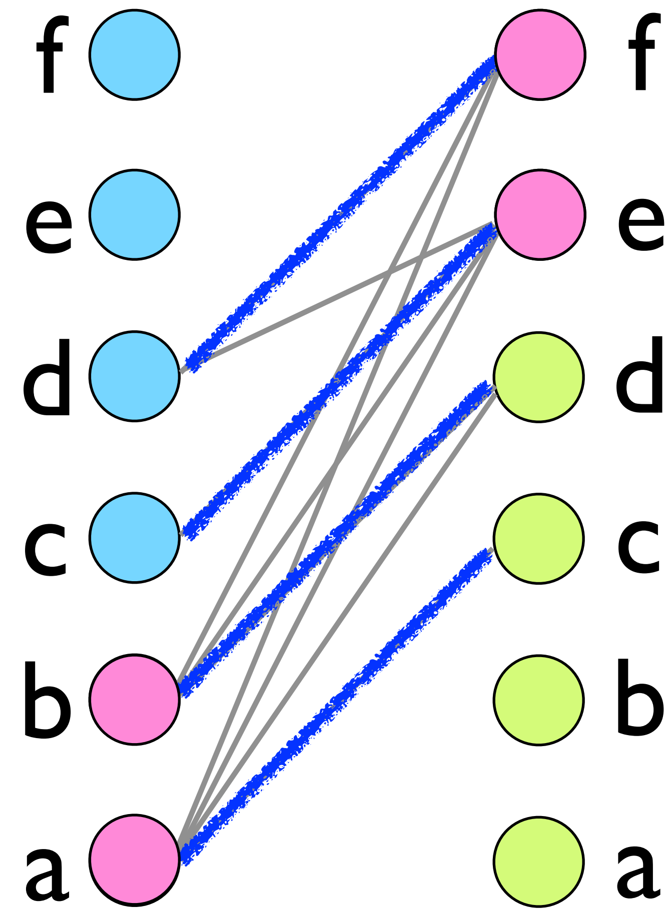


$|P| = n$

$G(U,V,E)$

$U=V=P$

$uv \in E$ if $u<v$

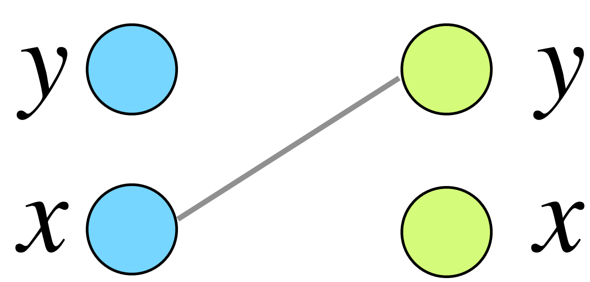


König-Egerváry Theorem:

$\exists$ matching M and vertex cover C, $|M| = |C| = k$

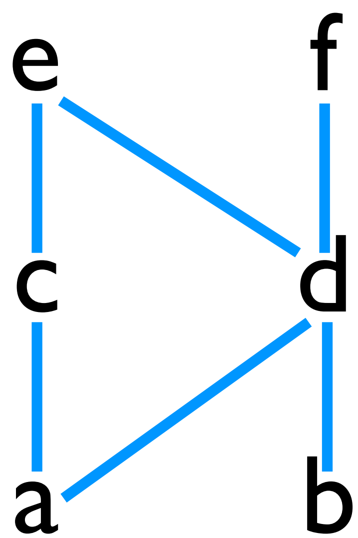
$x \in P$ uncovered by C ⇒ antichain $\geq n-k$

otherwise



C is not a vertex cover

poset P

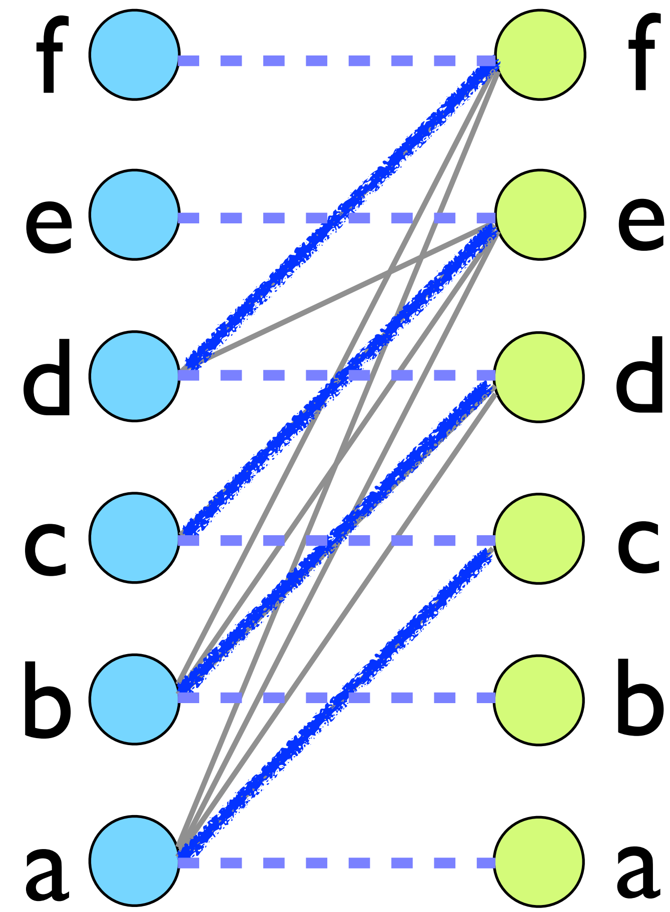


$|P| = n$

$G(U,V,E)$

$U=V=P$

$uv \in E$ if $u<v$



$\exists$ matching M and vertex cover C, $|M| = |C| = k$

$\exists$ antichain of size $\geq n\text{-}k$

decompose P into chains:

u,v in the same chain if $uv \in M$

chains = # unmatched vertices in U $= n\text{-}k$

Dilworth's Theorem

Suppose that the largest antichain in the poset P has size r. Then P can be partitioned into r chains.

$\exists$ antichain of size $\geq n\text{-}k$ = # chains

There exists an antichain $A \subseteq P$ and a partition of P into r chains such that $|A| = r$.

Hall's Theorem (marriage theorem)

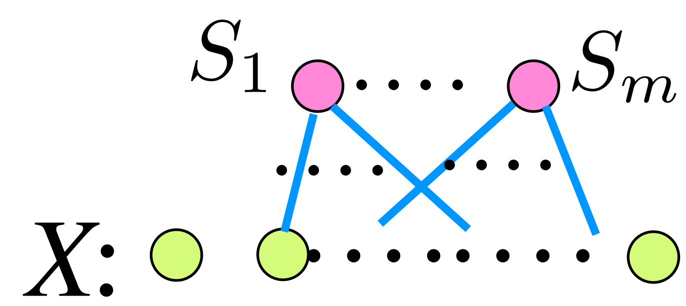
$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i\in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

let $X = S_1 \cup \cdots \cup S_m$

poset P: $X \cup \{S_1, \ldots, S_m\}$

$x < S_i$ if $x \in S_i$



X is the largest antichain in P.

$A \subseteq P$ is an antichain $\quad I = \{\, i \mid S_i \in A\}$ $\quad S_I = \bigcup_{i\in I} S_i$

$A \cap S_I = \emptyset \Rightarrow |A| \leq |I| + |X| - |S_I| \quad \leq |X|$

Hall condition

Hall's Theorem (marriage theorem)

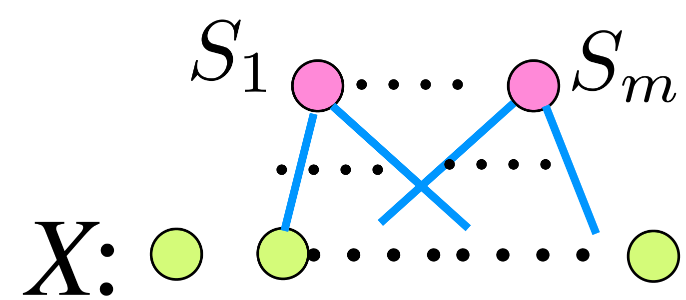
$\forall I \subseteq \{1, 2, \ldots, m\}, \quad \left|\bigcup_{i \in I} S_i\right| \geq |I|.$

$\Rightarrow$ $S_1, S_2, \ldots, S_m$ have a SDR

let $X = S_1 \cup \cdots \cup S_m$

poset P: $X \cup \{S_1, \ldots, S_m\}$

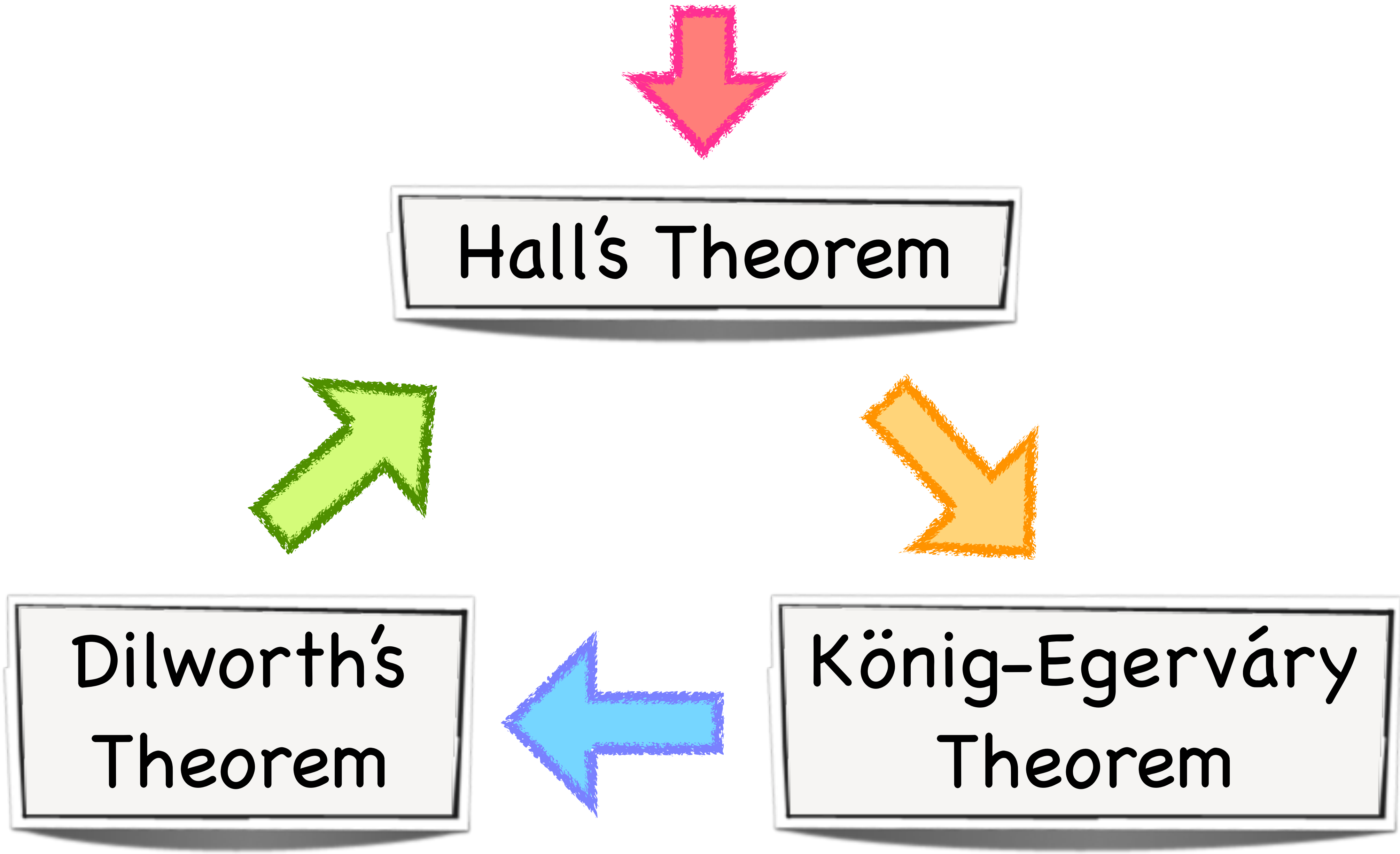
$x < S_i$ if $x \in S_i$



X is the largest antichain in P.

Dilworth: P is partitioned into $n = |X|$ chains

$\{S_1, x_1\}, \{S_2, x_2\}, \ldots, \{S_m, x_m\}, \{x_{m+1}\}, \ldots, \{x_n\}$


Hall's Theorem
König-Egerváry Theorem
Dilworth's Theorem

Erdős-Szekeres Theorem

A sequence of $> mn$ different numbers must contain either an increasing subsequence of length $m+1$, or a decreasing subsequence of length $n+1$.

$(a_1, \ldots, a_N)$ of N different numbers $\qquad N > mn$

poset P: $\{(i, a_i) \mid i = 1, 2, \ldots, N\}$

$$(i, a_i) \leq (j, a_j) \text{ if } a_i \leq a_j \text{ and } i \leq j$$

chain: increasing subseq

antichain: decreasing subseq

Use Dilworth!

Birkhoff - von Neumann Theorem

Every doubly stochastic matrix is a convex combination of permutation matrix.

doubly stochastic matrix A: $n{\times}n \quad A_{ij} \geq 0$

$$\forall j, \ \sum_i A_{ij} = 1 \quad \text{and} \quad \forall i, \ \sum_j A_{ij} = 1$$

permutation matrix P: $P_{ij} \in \{0, 1\}$

every row/column has precisely one 1

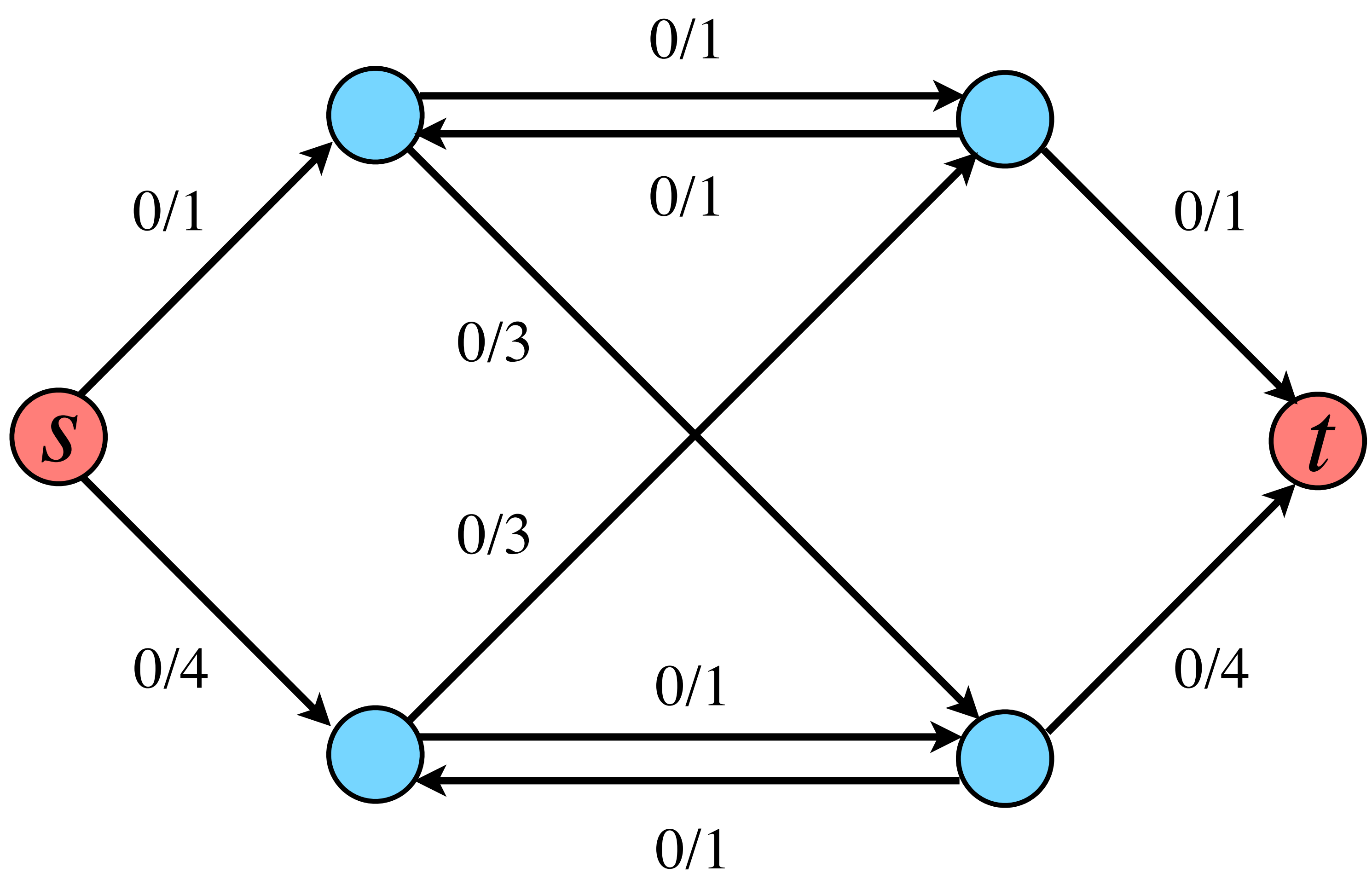
convex combination:

$$A = \sum_{i=1}^{m} \lambda_i P_i \qquad \lambda_i \geq 0 \quad \sum_{i=1}^{m} \lambda_i = 1$$

Flow

digraph: $D(V,E)$ source: s sink: t

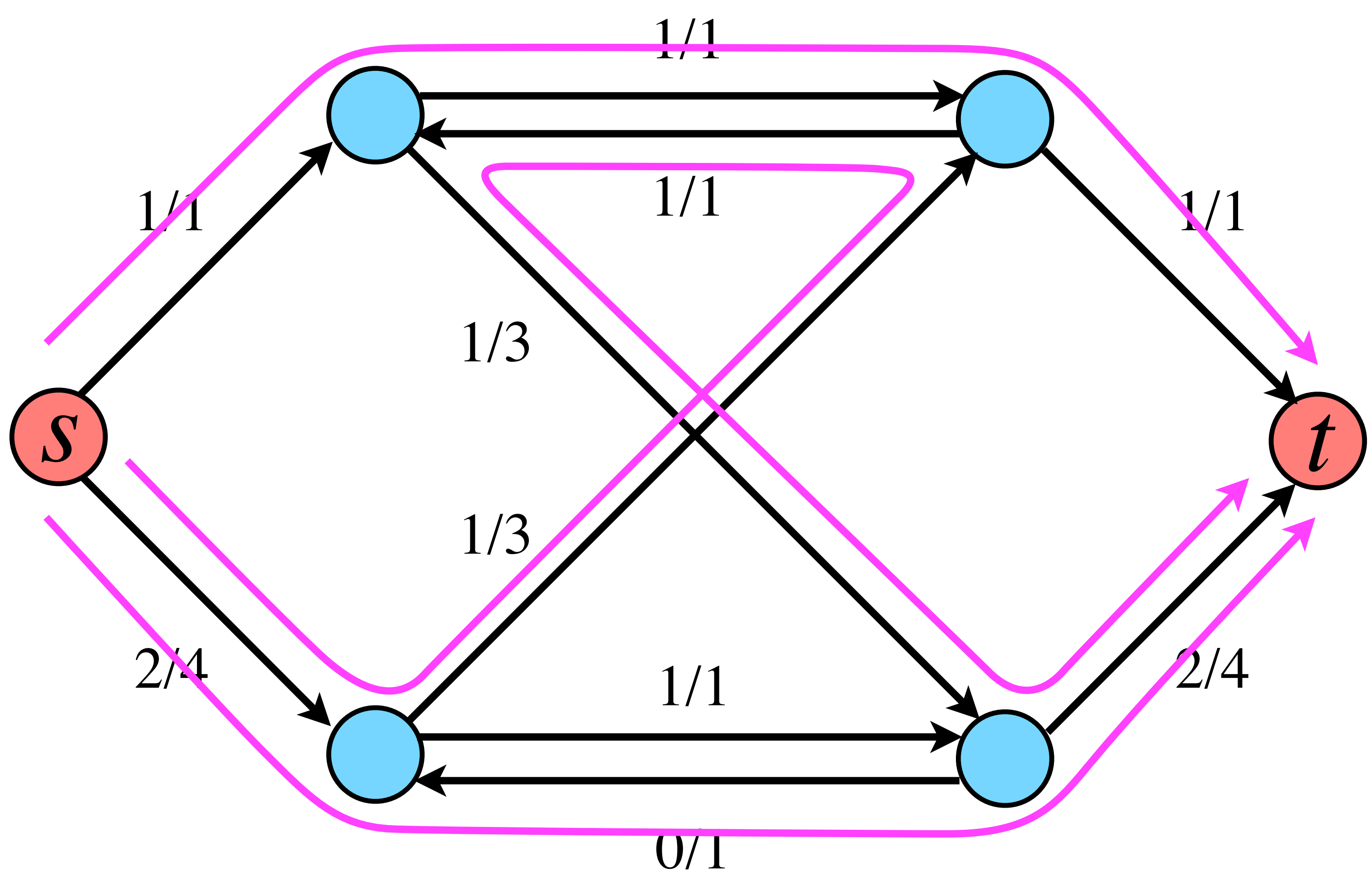
capacity: $c : E \to \mathbb{R}^+$



Flow

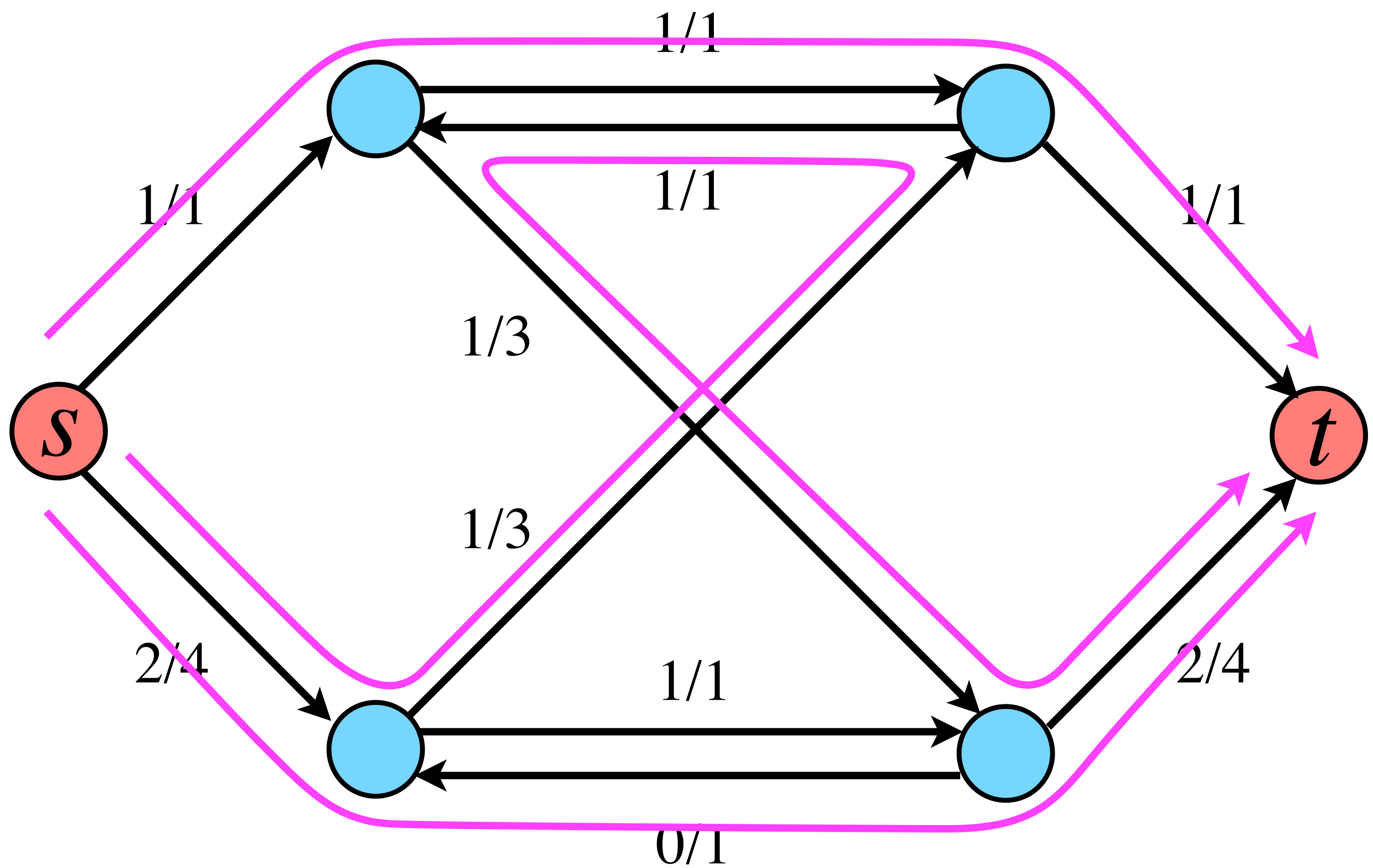
digraph: $D(V,E)$ source: s sink: t

capacity: $c : E \to \mathbb{R}^+$ flow: $f : E \to \mathbb{R}^+$



capacity: $\forall (u,v) \in E,\ f_{uv} \le c_{uv}$

conservation: $\forall u \in V \setminus \{s,t\},\ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

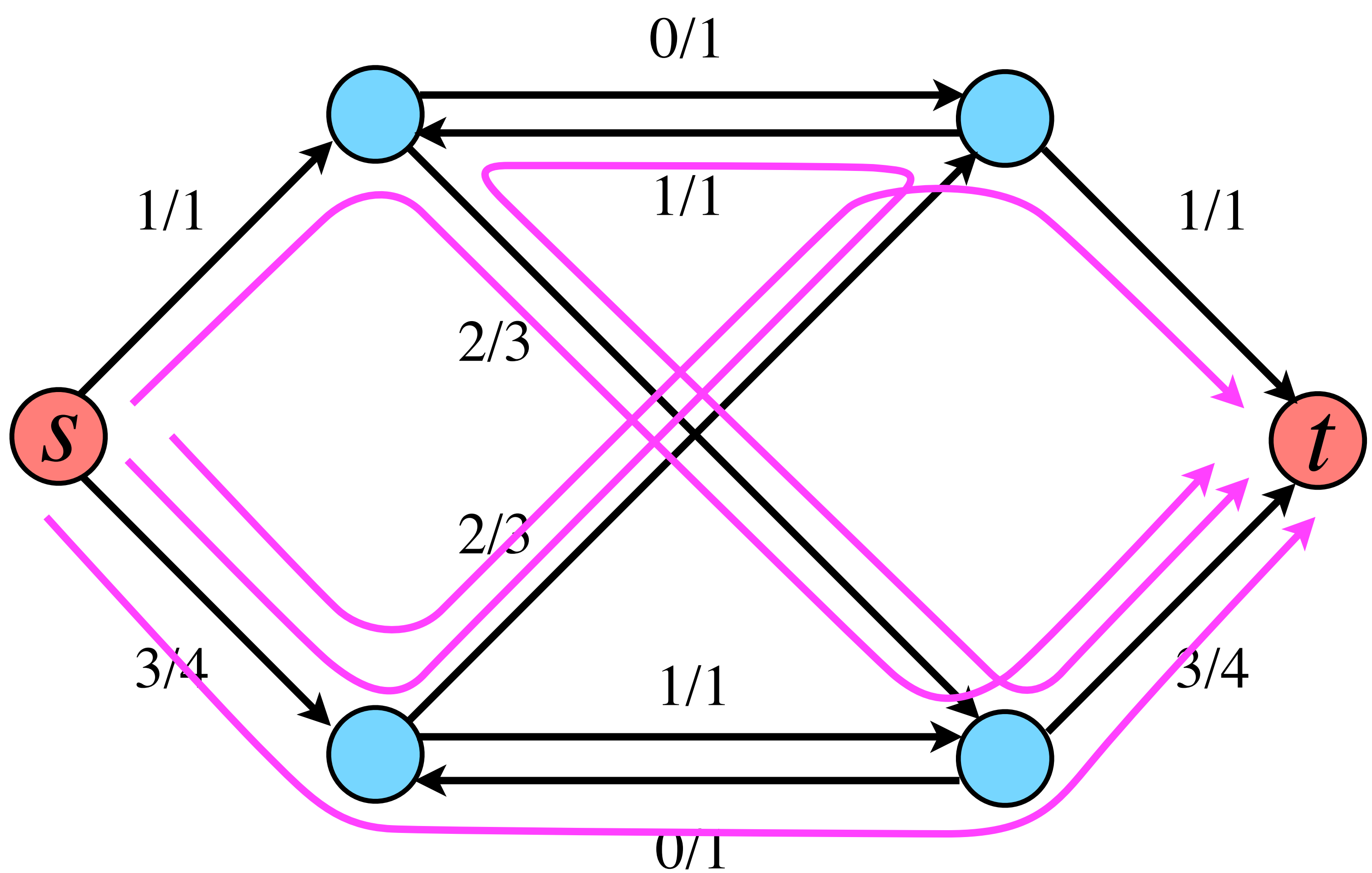


capacity: $\forall (u, v) \in E,\ f_{uv} \leq c_{uv}$

conservation: $\forall u \in V \setminus \{s, t\},\ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

value of flow: $$\sum_{(s,u)\in E} f_{su} = \sum_{(v,t)\in E} f_{vt}$$

maximum flow



capacity: $\forall (u,v) \in E,\ f_{uv} \le c_{uv}$

conservation: $\forall u \in V \setminus \{s,t\},\ \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv}$

value of flow: $$\sum_{(s,u)\in E} f_{su} = \sum_{(v,t)\in E} f_{vt}$$

maximum flow

Maximum Flow

digraph: $D(V,E)$ source: s sink: t

capacity: $c : E \to \mathbb{R}^+$

max $\sum_{u:(s,u)\in E} f_{su}$

s.t. $0 \le f_{uv} \le c_{uv}$ $\forall (u,v) \in E$

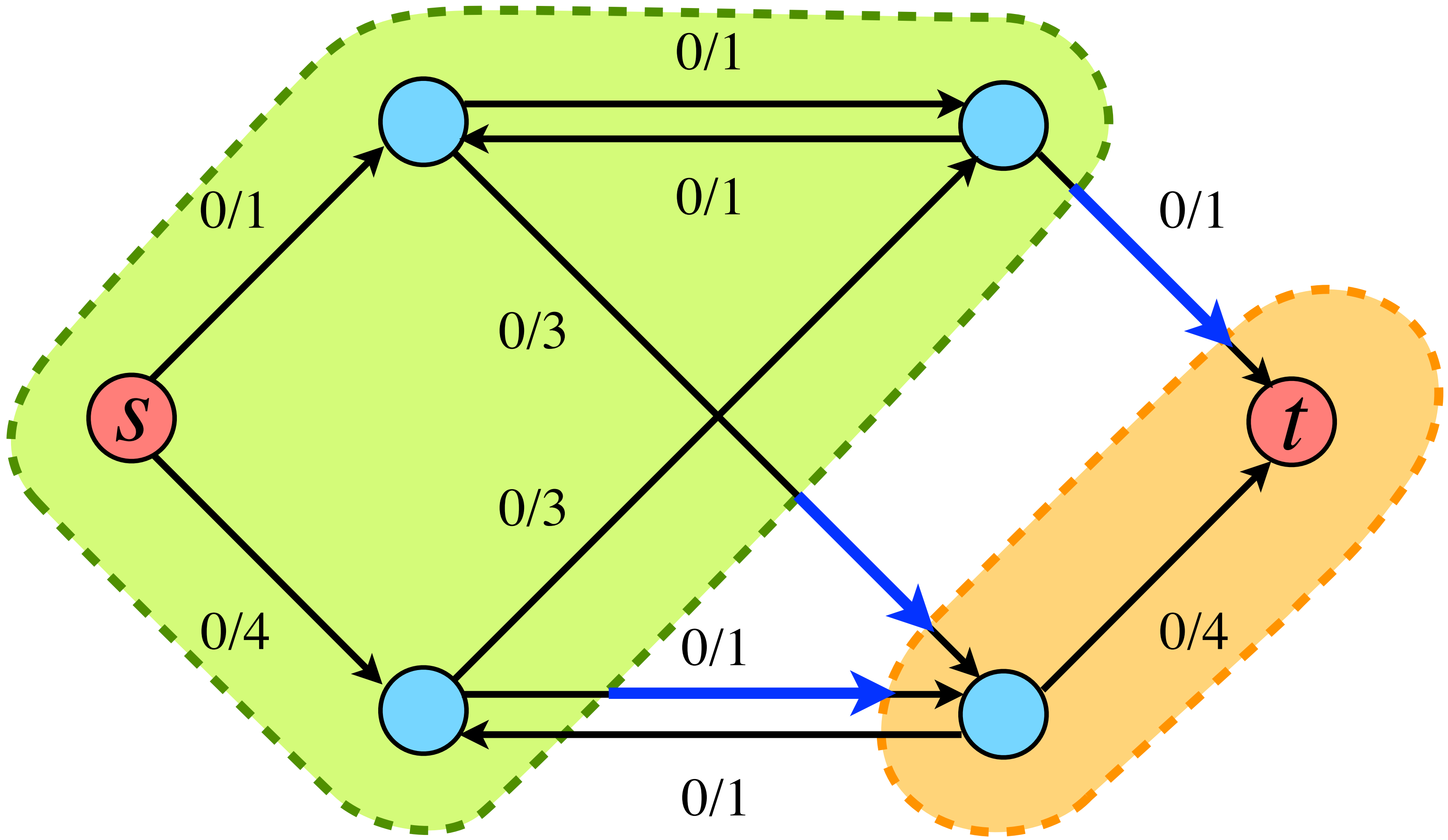
$\sum_{w:(w,u)\in E} f_{wu} - \sum_{v:(u,v)\in E} f_{uv} = 0$ $\forall u \in V \setminus \{s,t\}$

integral flow: $f_{uv} \in \mathbb{Z}$ $\forall (u,v) \in E$

Cut

digraph: $D(V,E)$ source: s sink: t

capacity: $c : E \rightarrow \mathbb{R}^{+}$



s-t cut: $S \subset V$

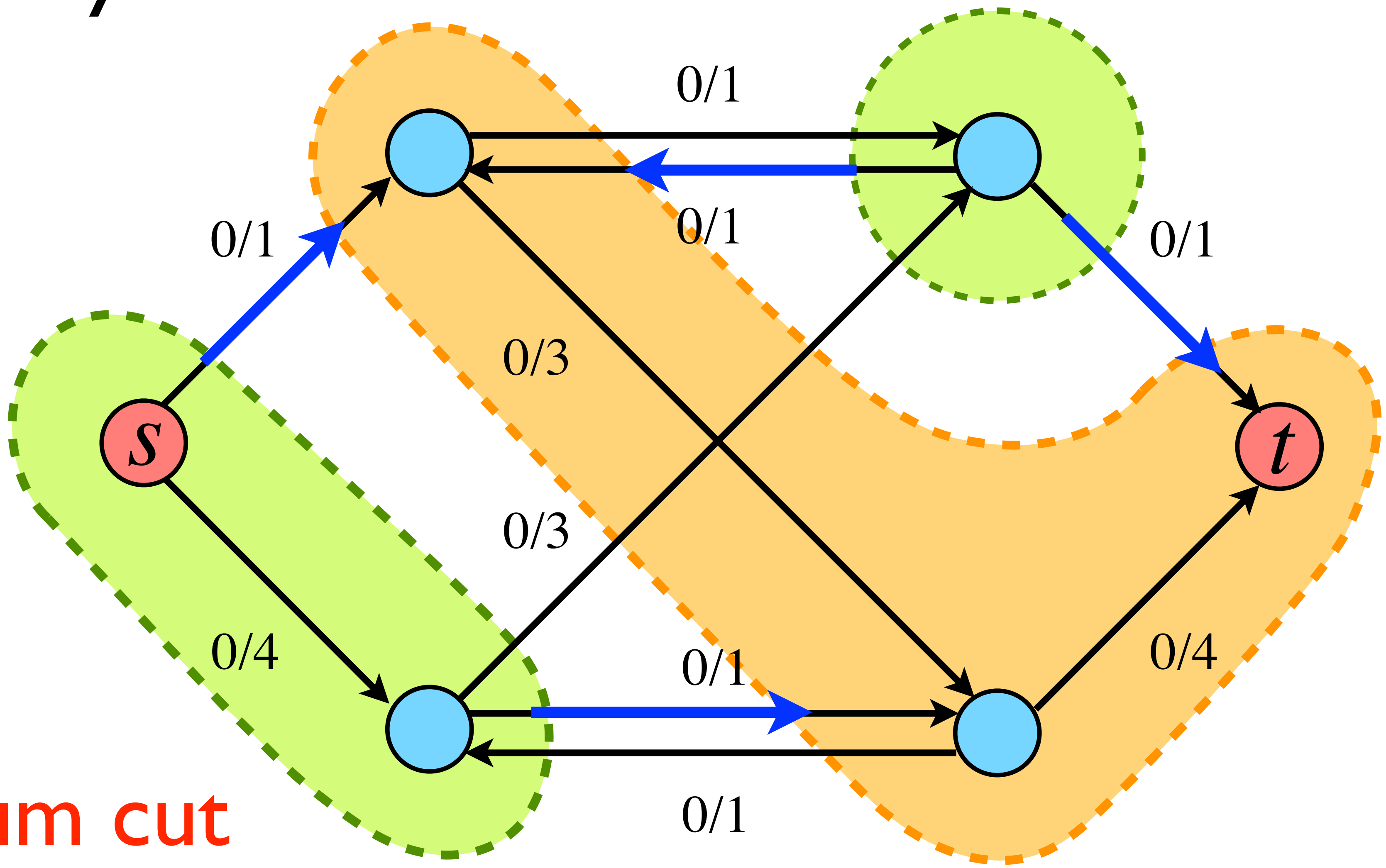
$s \in S, t \notin S$

$$\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$$

Cut

digraph: $D(V,E)$ source: s sink: t

capacity: $c: E \rightarrow \mathbb{R}^+$



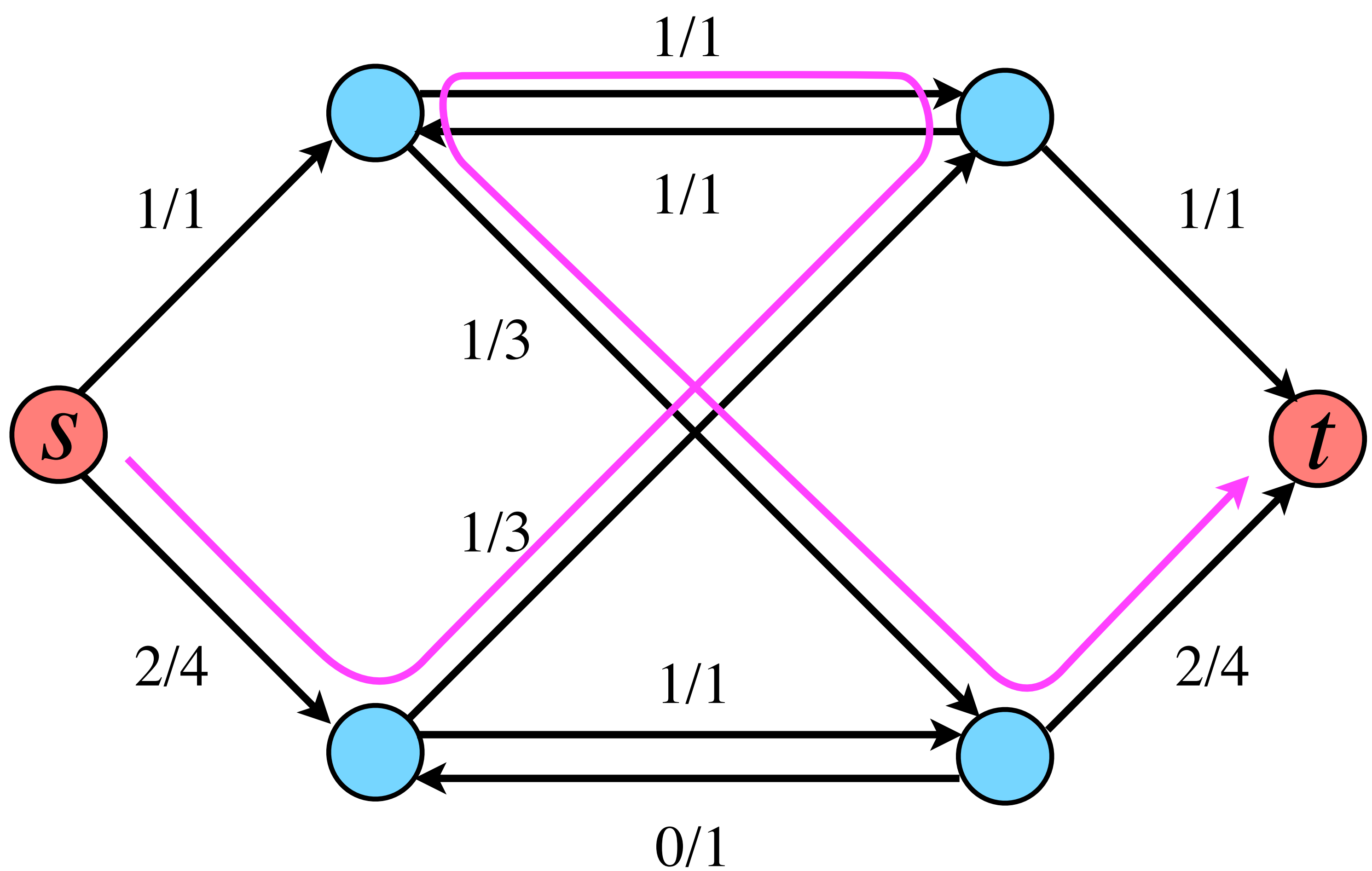
minimum cut

s-t cut: $S \subset V$ $s \in S, t \notin S$ $\displaystyle\sum_{u \in S, v \notin S, (u,v) \in E} c_{uv}$

Fundamental Facts about Flows

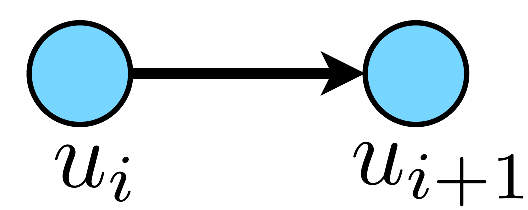
- Integrality does not affect the optimal solution.
- max-flow = min-cut

Augmenting Path

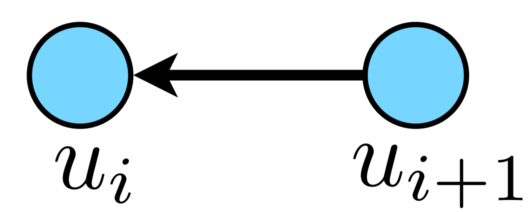


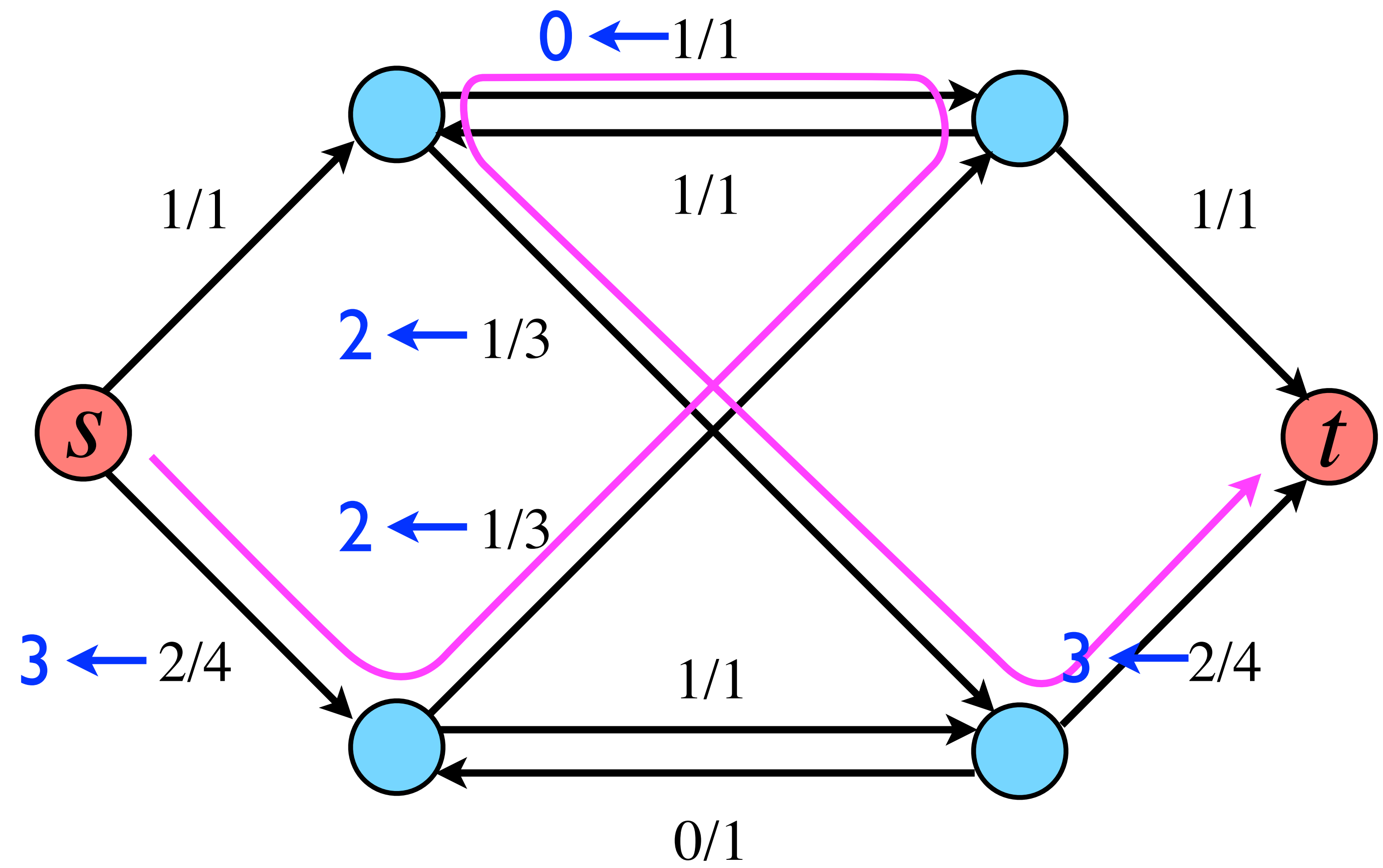
augmenting path: $s = u_0 u_1 \cdots u_k = t$

$f(u_i u_{i+1}) < c(u_i u_{i+1})$ if



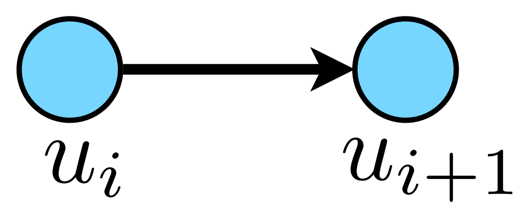
$f(u_{i+1} u_i) > 0$ if





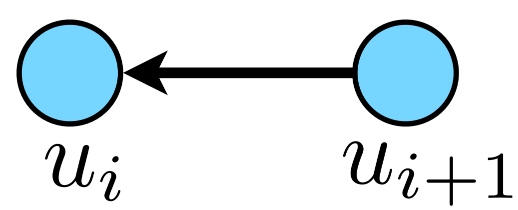
augmenting path: $s = u_0 u_1 \cdots u_k = t$ flow increased

$f(u_i u_{i+1}) < c(u_i u_{i+1})$ if



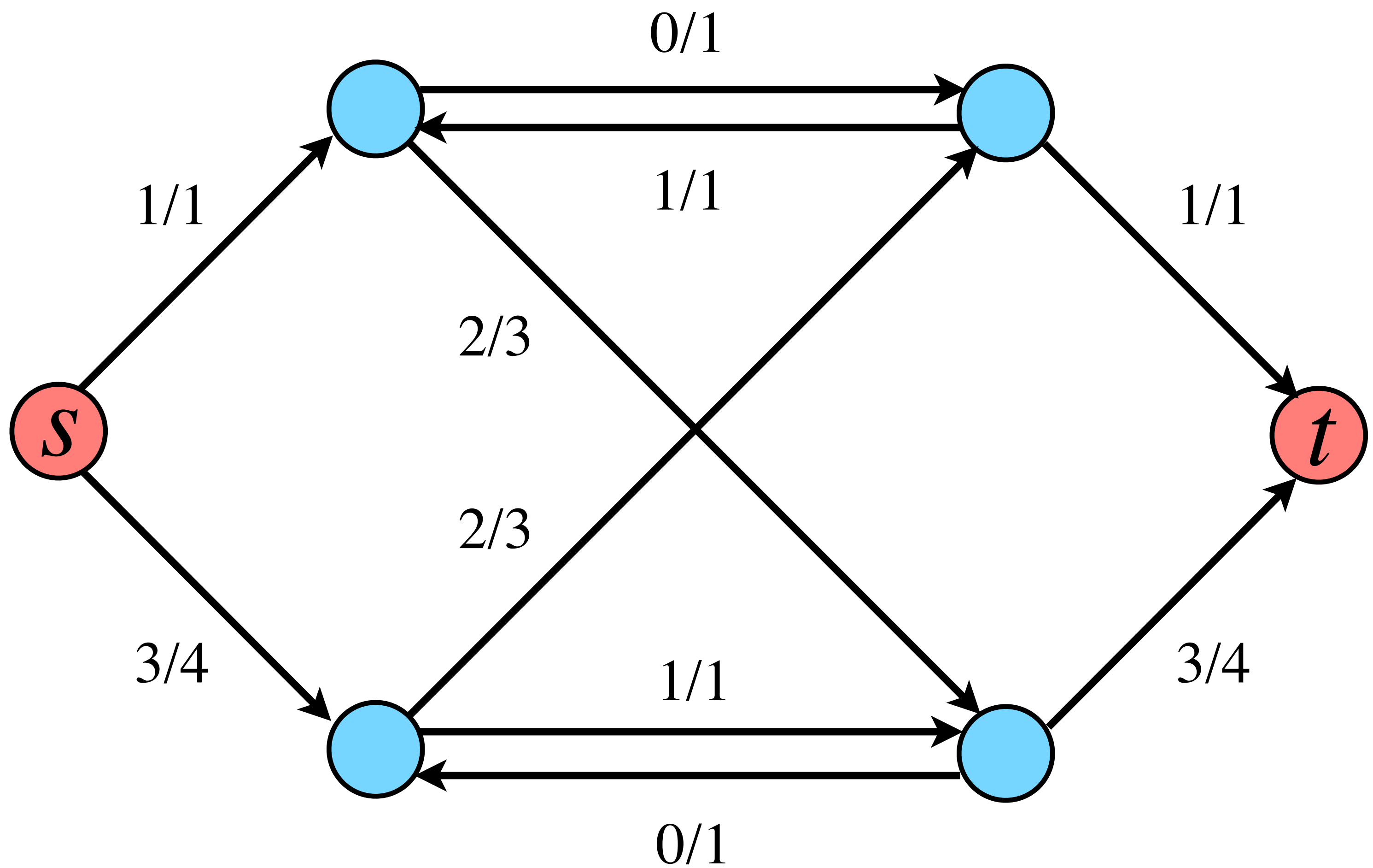
$f(u_i u_{i+1}) + \epsilon$

$f(u_{i+1} u_i) > 0$ if

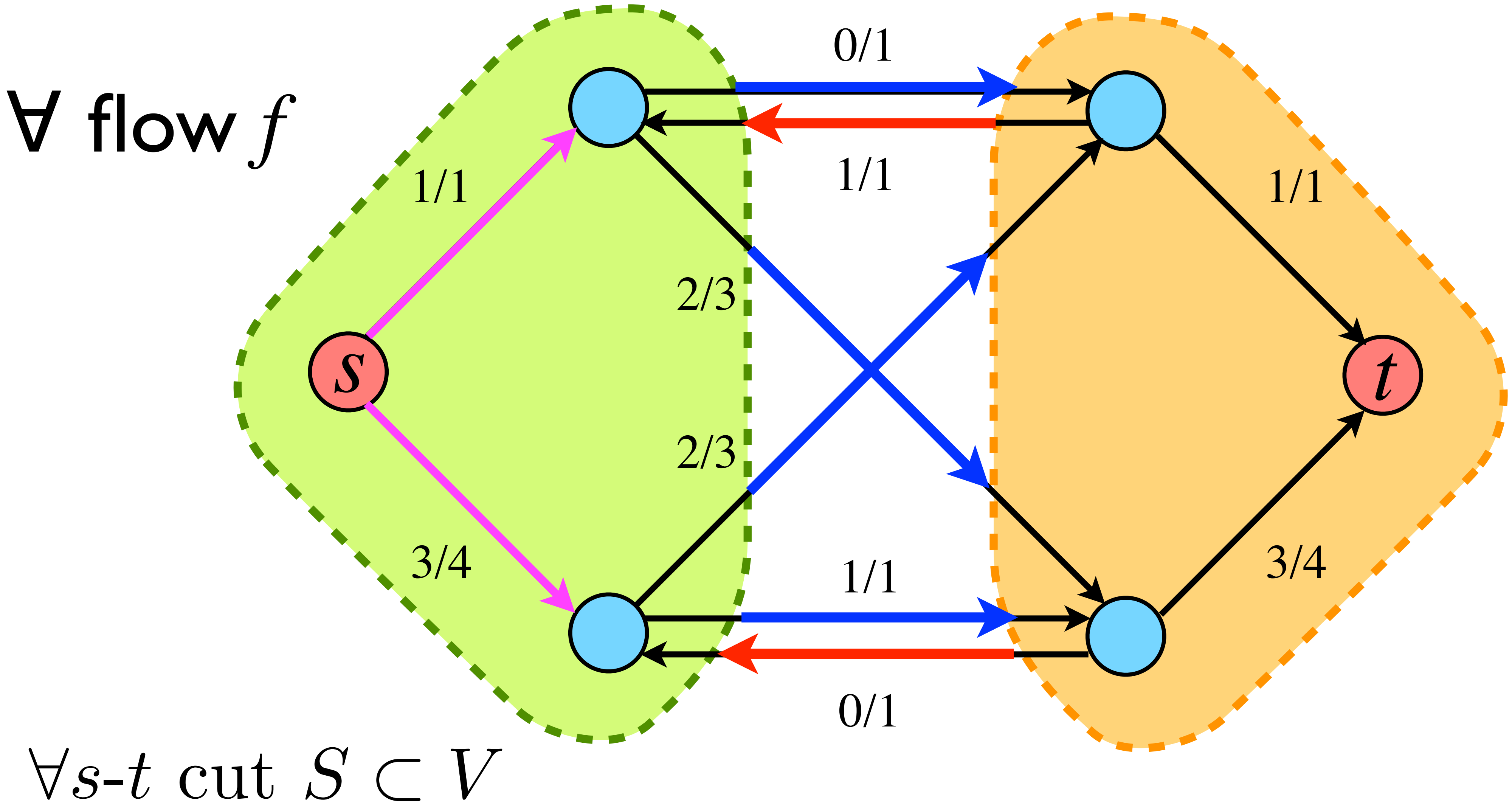


$f(u_{i+1} u_i) - \epsilon$

maximum flow ⇒ no augmenting path



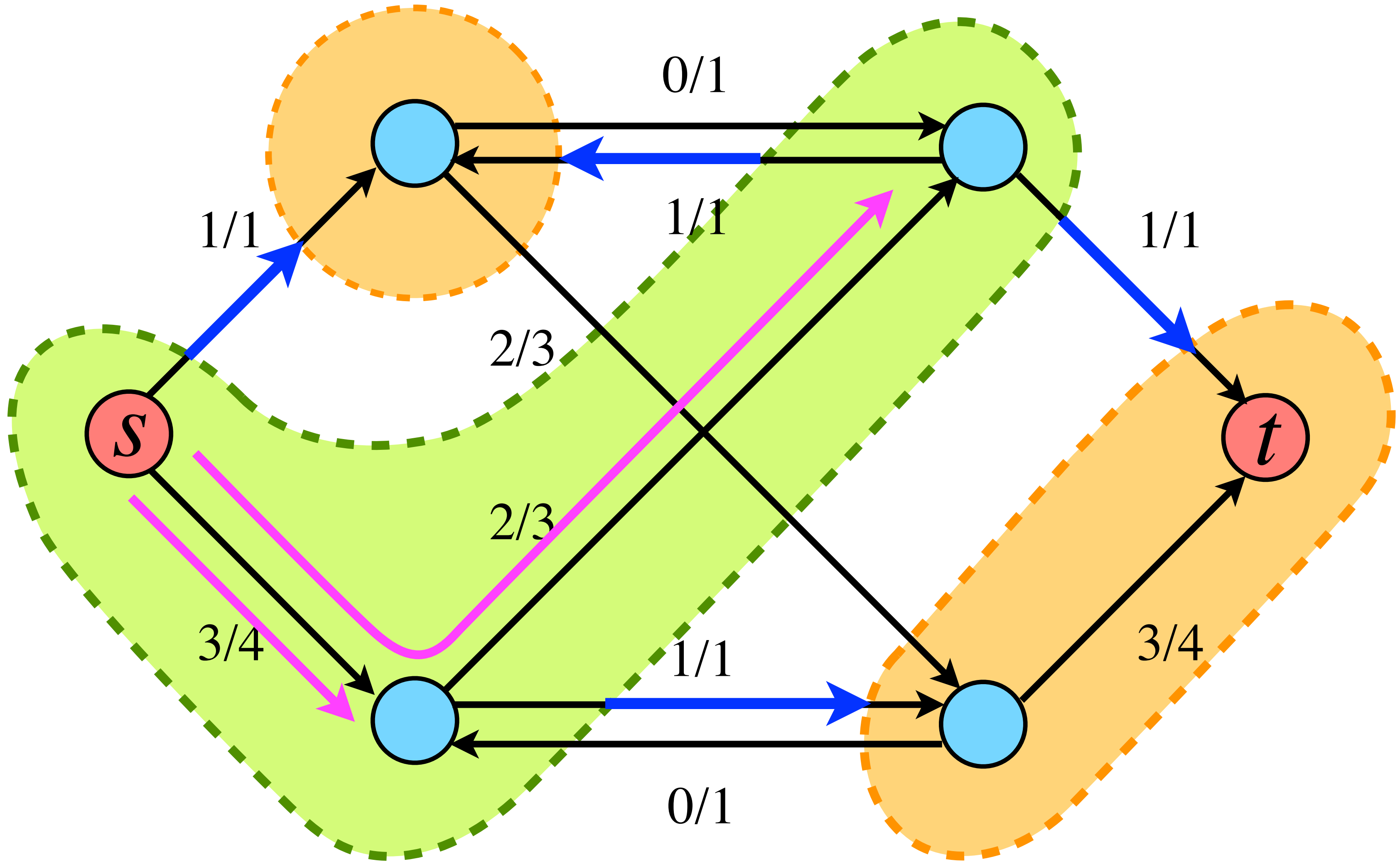
maximum flow ⟺ no augmenting path



$$\sum_{u:(s,u)\in E} f_{su} = \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} f_{uv} - \sum_{\substack{u\in S, v\notin S \\ (v,u)\in E}} f_{vu}$$

$$\sum_{u:(s,u)\in E} f_{su} \leq \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} f_{uv} \leq \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} c_{uv}$$

max-flow min-cut



S = { u | ∃ augmenting path from s to u}

no augmenting path ⇨ $s \in S, t \notin S \quad s\text{-}t \text{ cut}$

$$\forall u \in S, v \notin S, (u,v) \in E \quad \begin{cases} f_{uv} = c_{uv} \\ f_{vu} = 0 \end{cases}$$

flow $\displaystyle\sum_{u:(s,u)\in E} f_{su} = \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} f_{uv} - \sum_{\substack{u\in S, v\in S \\ (v,u)\in E}} f_{vu} = \sum_{\substack{u\in S, v\notin S \\ (u,v)\in E}} c_{uv}$ cut maximum flow ⟺ no augmenting path

Max-Flow Min-Cut Theorem

(Ford-Fulkerson 1956, Kotzig 1956)

max-flow = min-cut

Flow Integrality Theorem

If capacities are integers, then

max integral flow = max-flow

in an integral flow f: ∃ augmenting path ⟹

∃ integral augmenting path ⟹ ∃ larger integral flow

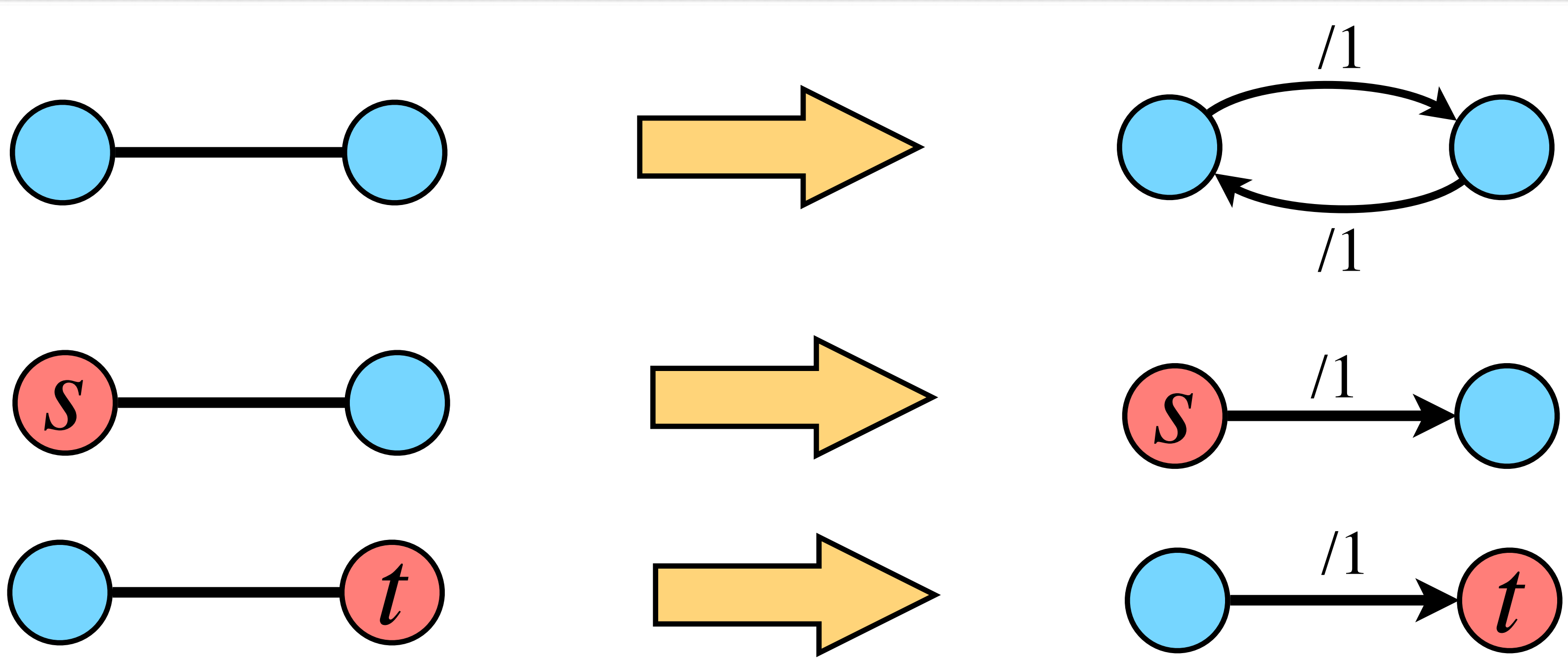
Menger's Theorem

undirected graph: $G(V,E)$ $\forall s,t \in V$

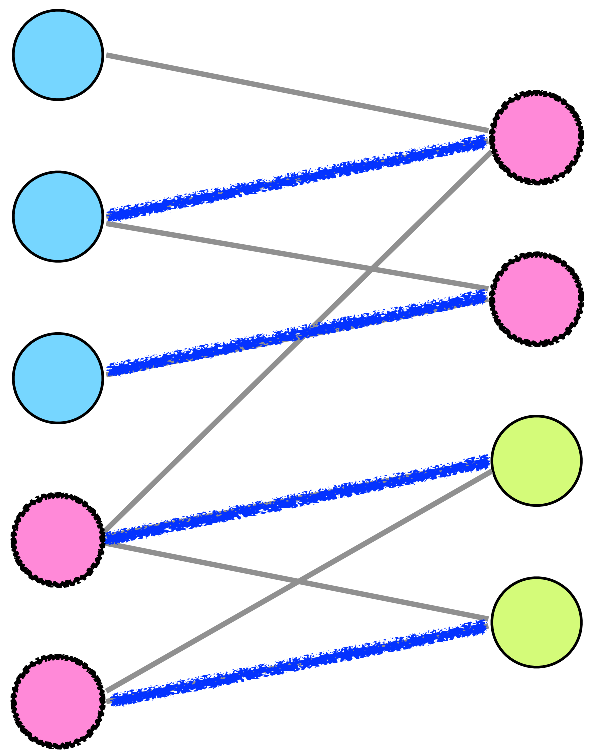
s-t cut $C \subset E$ removing C disconnects s,t

Theorem (Menger 1927)

min s-t cut = max # of disjoint s-t path



Bipartite Matching



matching: $M \subseteq E$

no $e_1, e_2 \in M$ share a vertex

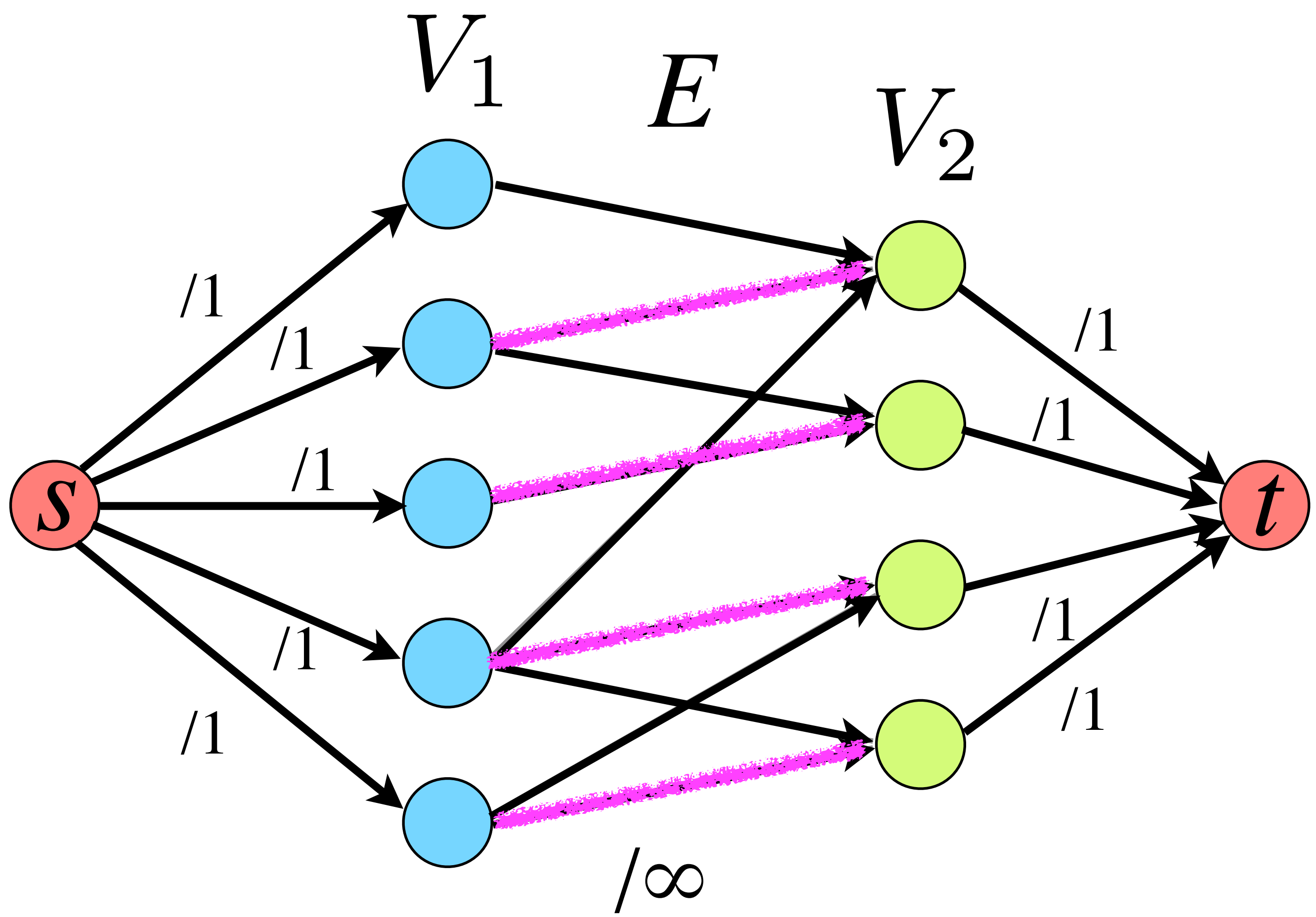
vertex cover: $C \subseteq V$

all $e \in E$ adjacent to some $v \in C$

Theorem (König 1931, Egerváry 1931)

In a bipartite graph,

max matching = min vertex cover.

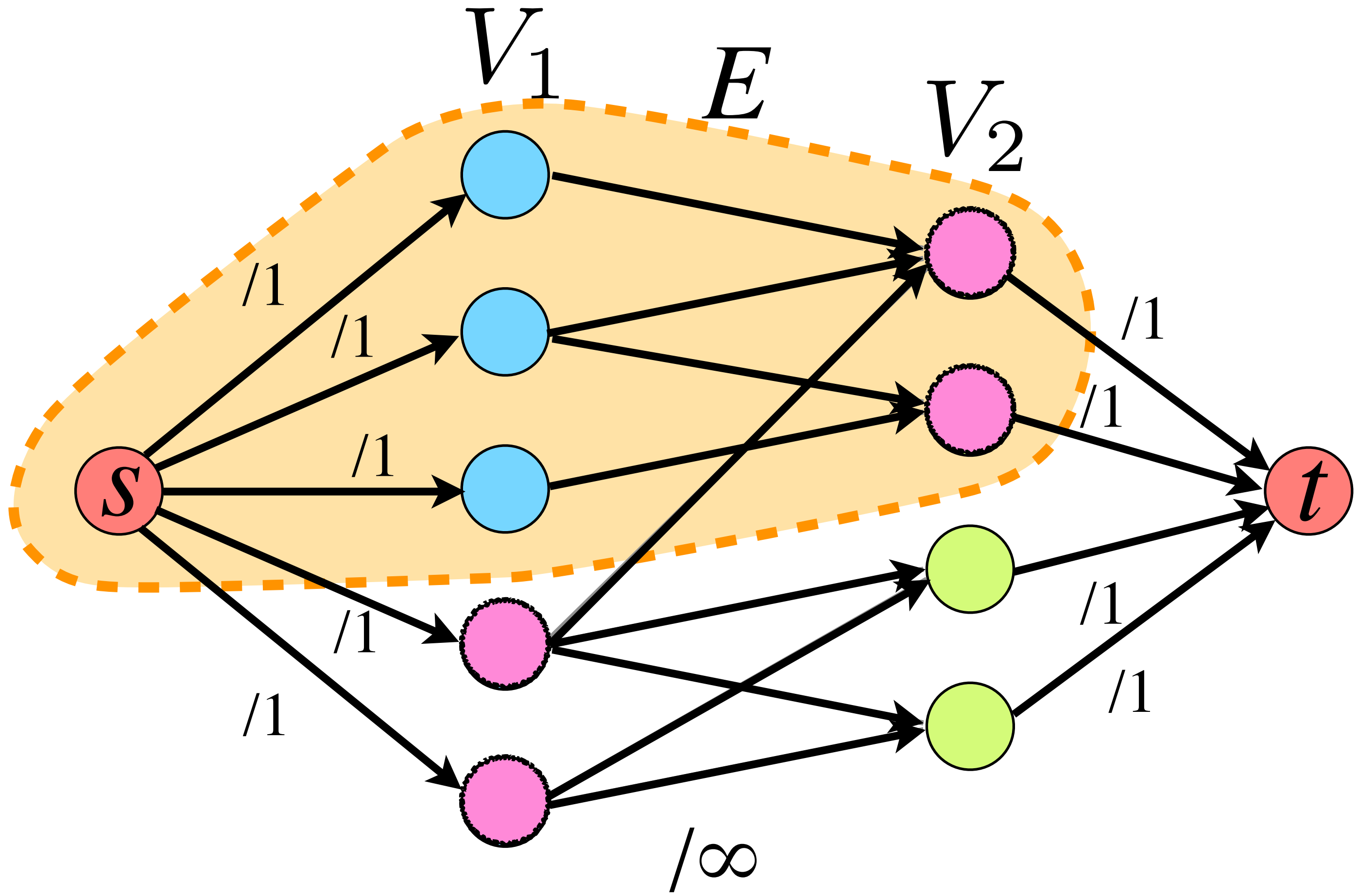


max integral flow = max matching

$$\forall (u,v) \in E \qquad f_{uv} \in \{0,1\}$$

$$\forall u \in V_1, \quad \sum_{v:(u,v)\in E} f_{uv} \leq 1$$

$$\forall v \in V_2, \quad \sum_{u:(u,v)\in E} f_{uv} \leq 1$$



min s-t cut = vertex cover

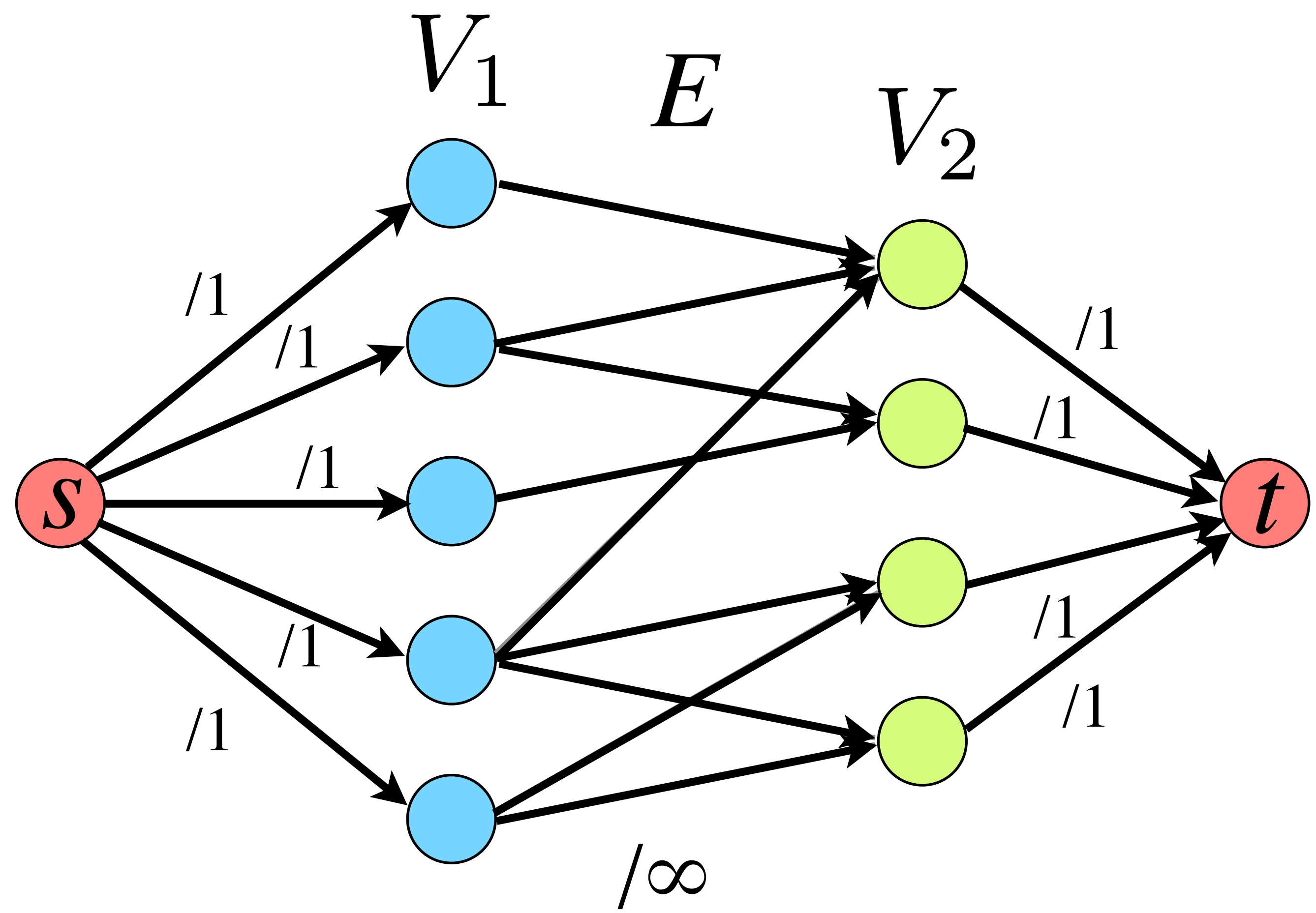
min-cut

$s \in S, t \notin S$ $\Rightarrow$ $\displaystyle\sum_{\substack{u\in S, v\notin S\\(u,v)\in E}} c_{uv} < \infty$ $\Rightarrow$

no edge $(u,v) \in E$ has $u \in V_1 \cap S, v \in V_2 \setminus S$

$\Rightarrow$ $(V_1 \setminus S) \cup (V_2 \cap S)$ **is a vertex cover**

$$|V_1 \setminus S| + |V_2 \cap S| = \sum_{v \in V_1 \setminus S} c_{sv} + \sum_{u \in V_2 \cap S} c_{ut} = \sum_{\substack{u\in S, v\notin S\\(u,v)\in E}} c_{uv}$$



max integral flow = max matching

min s-t cut = vertex cover

Theorem (König 1931, Egerváry 1931)

In a bipartite graph,

max matching = min vertex cover.