Combinatorics

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Extremal Combinatorics

"How large or how small a collection of finite objects can be, if it has to satisfy certain restrictions."

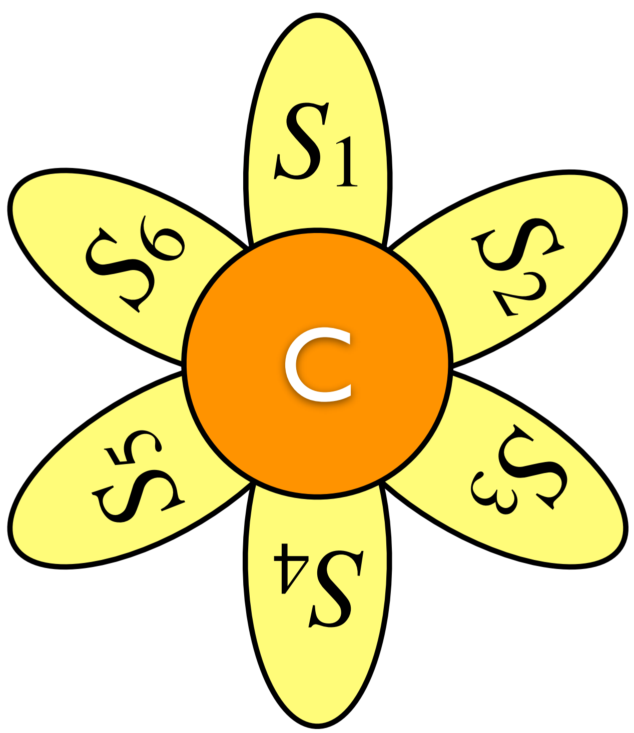
set system $\mathcal{F} \subseteq 2^{[n]}$ with ground set $[n]$

Sunflowers

$\mathcal{F}$ a sunflower of size r with center C:

$$|\mathcal{F}| = r \qquad \forall S, T \in \mathcal{F}, \quad S \cap T = C$$

a sunflower of size 6
with core C



Sunflowers

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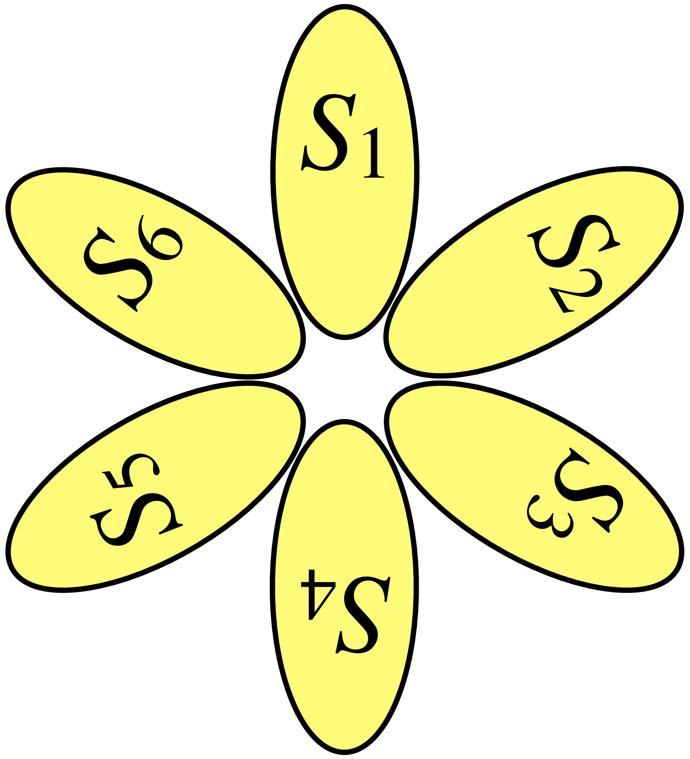
a sunflower of size 6
with core $\emptyset$



Sunflower Lemma (Erdős-Rado 1960)

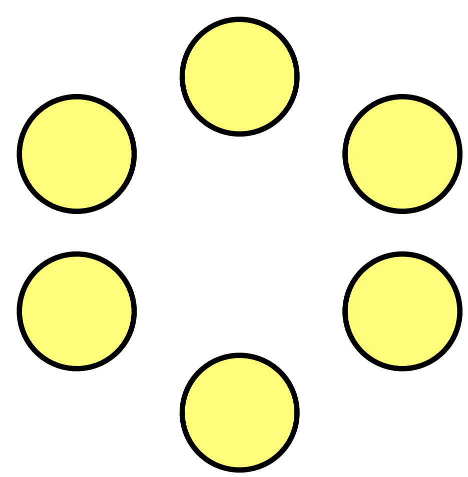
$$\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r-1)^k \quad \Rightarrow$$

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

Induction on k. when k=1

$$\mathcal{F} \subseteq \binom{[n]}{1} \qquad |\mathcal{F}| > r-1$$

$\exists$ r singletons



Sunflower Lemma (Erdős-Rado 1960)

$$\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r-1)^k \Rightarrow$$

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

For $k \geq 2$,

take largest $\mathcal{G} \subseteq \mathcal{F}$ with disjoint members

$$\forall S, T \in \mathcal{G} \text{ that } S \neq T, \ S \cap T = \emptyset$$

case.1: $|\mathcal{G}| \geq r,$ $\mathcal{G}$ is a sunflower of size r

case.2: $|\mathcal{G}| \leq r-1,$

Goal: find a popular $x \in [n]$

Sunflower Lemma (Erdős-Rado 1960)

$$\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r-1)^k \Rightarrow$$

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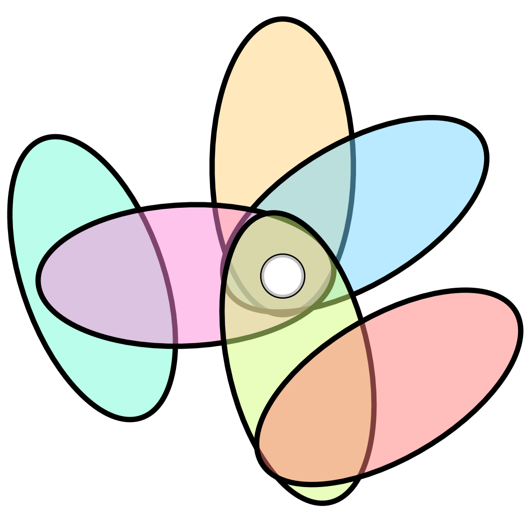
$|\mathcal{G}| \leq r-1$, Goal: find a popular $x \in [n]$



consider

$$\{S \in \mathcal{F} \mid x \in S\}$$

remove x

$$\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$$

$\mathcal{H} \subseteq \binom{[n]}{k-1}$ if $|\mathcal{H}| > (k-1)!(r-1)^{k-1}$ I.H.

$$\mathcal{F} \subseteq \binom{[n]}{k}. \qquad |\mathcal{F}| > k!(r-1)^k$$

take largest $\mathcal{G} \subseteq \mathcal{F}$ with disjoint members

$|\mathcal{G}| \le r-1,$ let $Y = \bigcup_{S\in\mathcal{G}} S$ $\quad |Y| \le k(r-1)$

claim: Y intersects all $S \in \mathcal{F}$

if otherwise: $\exists T \in \mathcal{F},\ T \cap Y = \emptyset$

T is disjoint with all $S \in \mathcal{G}$

contradiction!

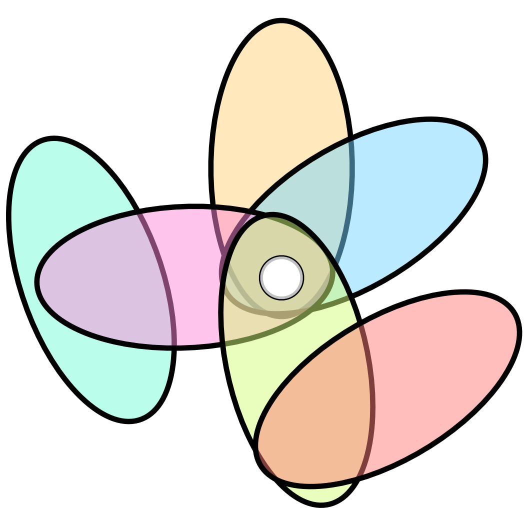
$$\mathcal{F} \subseteq \binom{[n]}{k}. \qquad |\mathcal{F}| > k!(r-1)^k$$

take maximal $\mathcal{G} \subseteq \mathcal{F}$ with disjoint members

$|\mathcal{G}| \le r-1,$ let $Y = \bigcup_{S \in \mathcal{G}} S$ $\quad |Y| \le k(r-1)$

Y intersects all $S \in \mathcal{F}$

pigeonhole: $\exists\, x \in Y$, # of $S \in \mathcal{F}$ contain x



$$|\{S \in \mathcal{F} \mid x \in S\}| \ge \frac{|\mathcal{F}|}{|Y|} \ge \frac{k!(r-1)^k}{k(r-1)}$$

$$= (k-1)!(r-1)^{k-1}$$

$$\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$$

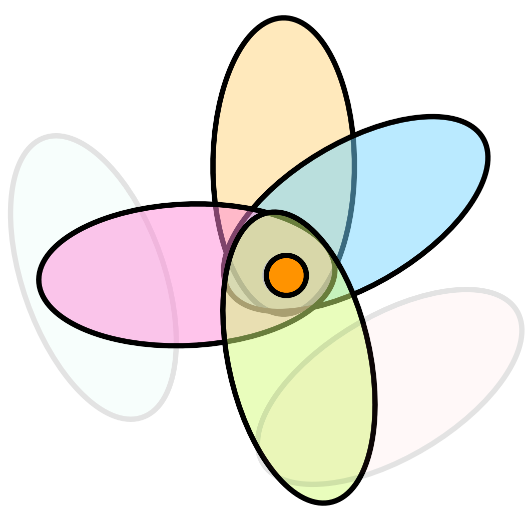
$$\mathcal{H} \subseteq \binom{[n]}{k-1} \qquad |\mathcal{H}| > (k-1)!(r-1)^{k-1}$$

Sunflower Lemma (Erdős-Rado 1960)

$$\mathcal{F} \subseteq \binom{[n]}{k}. \quad |\mathcal{F}| > k!(r-1)^k \Rightarrow$$

$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$\exists\, x \in Y$, let $\mathcal{H} = \{S \setminus \{x\} \mid S \in \mathcal{F} \wedge x \in S\}$



$$\mathcal{H} \subseteq \binom{[n]}{k-1}$$

$$|\mathcal{H}| > (k-1)!(r-1)^{k-1}$$

I.H.: $\mathcal{H}$ contains a sunflower of size r

adding x back, it is a sunflower in $\mathcal{F}$

Sunflower Conjecture

$\mathcal{F} \subseteq \binom{[n]}{k}$. $|\mathcal{F}| > c(r)^k$ $\Rightarrow$

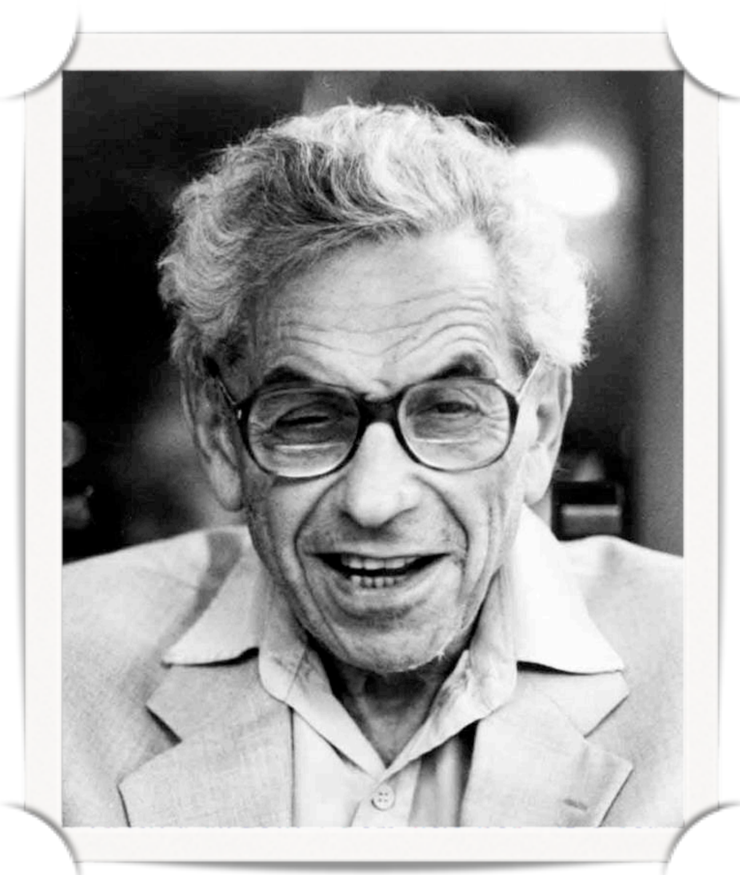
$\exists$ a sunflower $\mathcal{G} \subseteq \mathcal{F}$, such that $|\mathcal{G}| = r$

$c(r)$: constant depending only on r

Alon-Shpilka-Umans 2012:

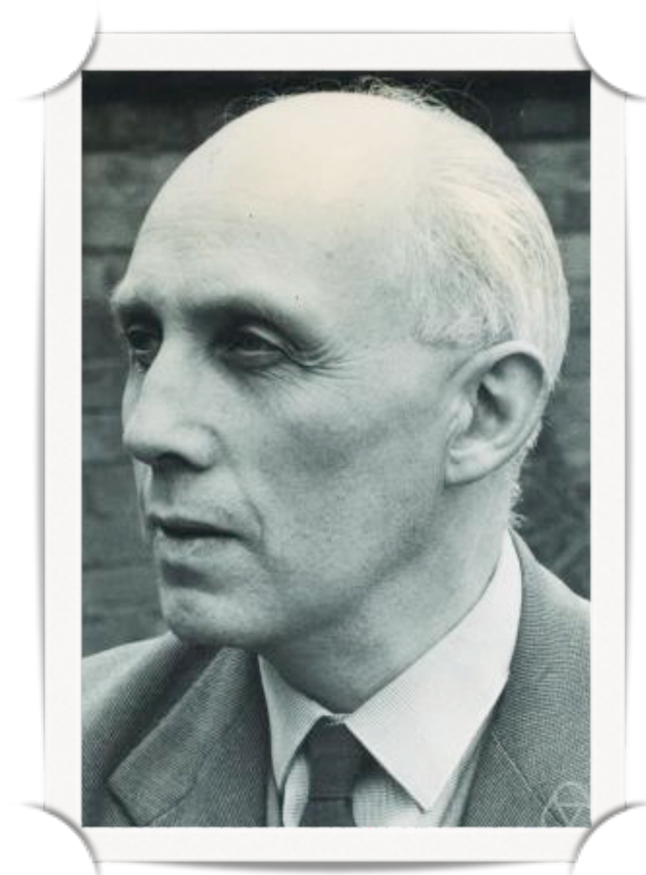
if sunflower conjecture is true
then matrix multiplication is slow

Erdős-Ko-Rado Theorem

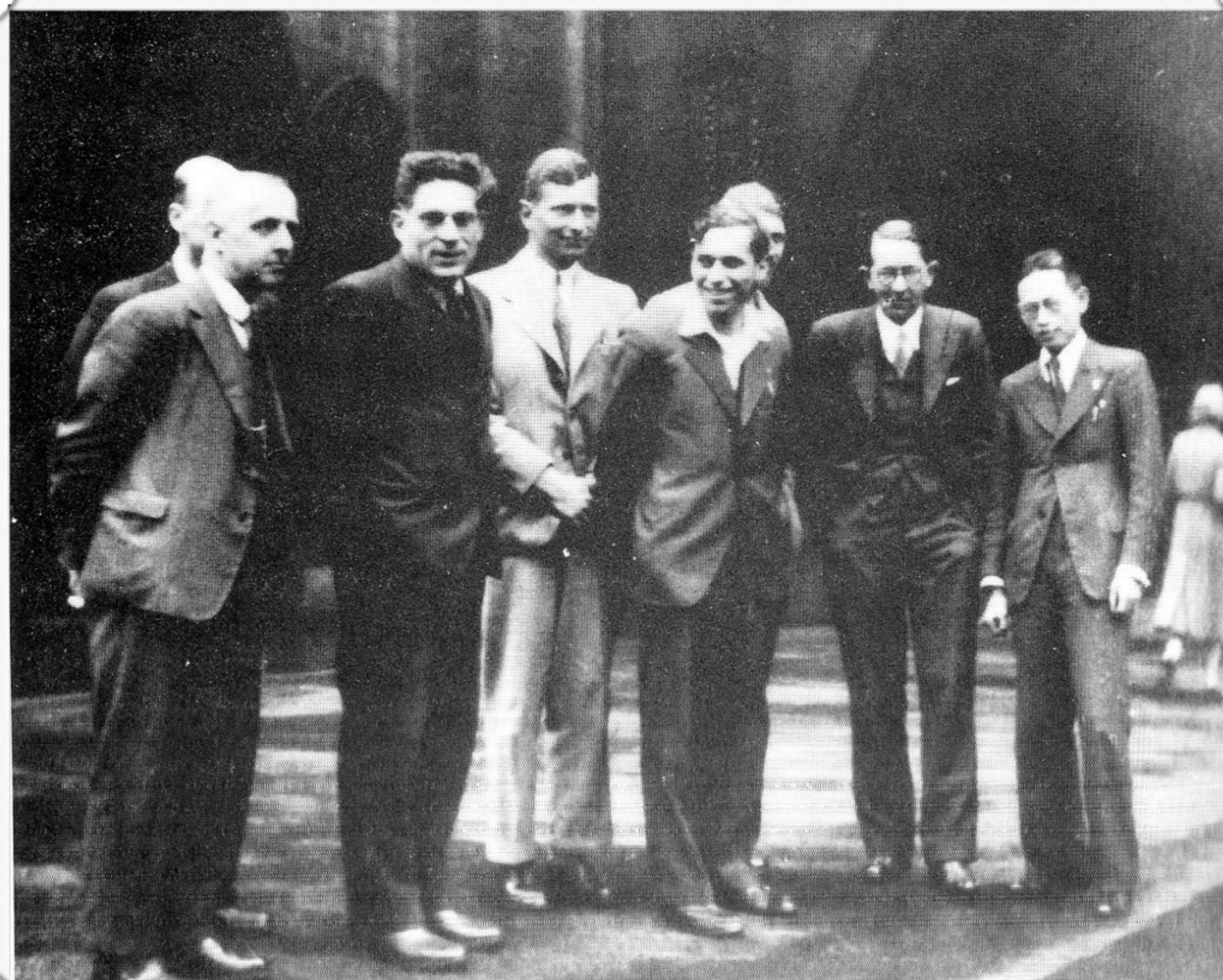


Paul Erdős
(1913-1996)

柯召
(1910-2002)

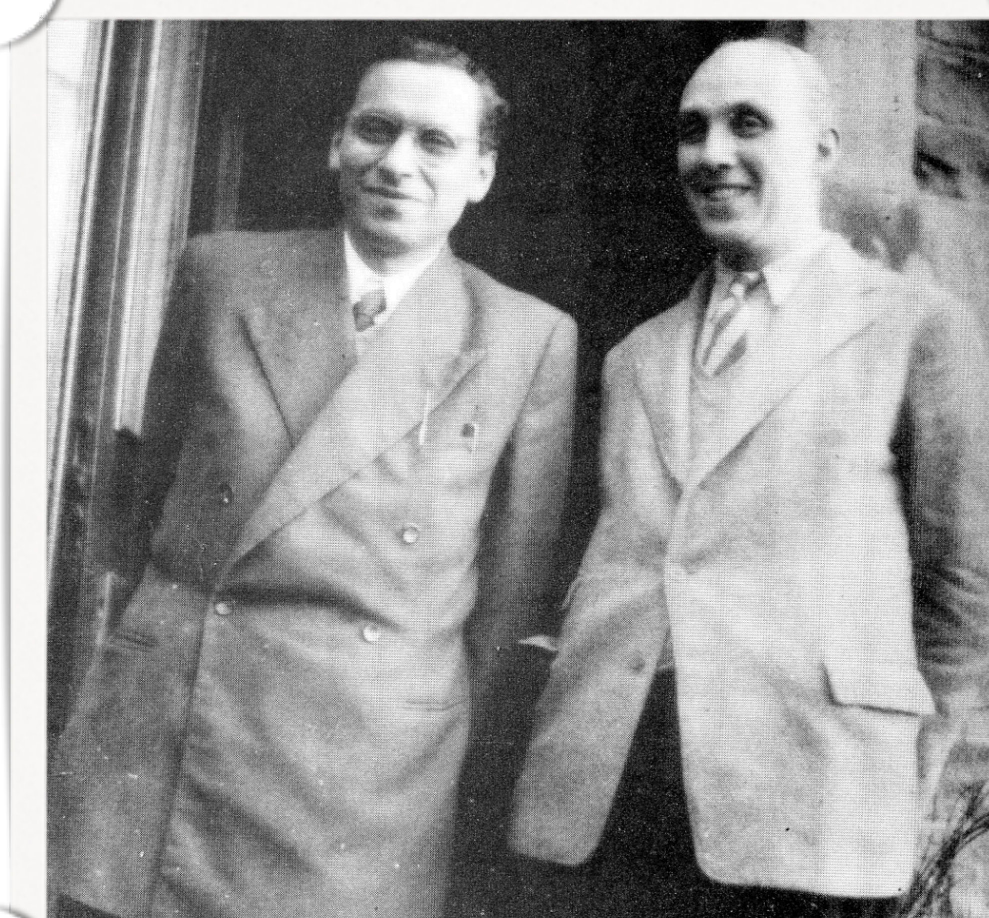


Richard Rado
(1906-1989)



Erdős 柯召

Erdős Rado



Intersecting Families

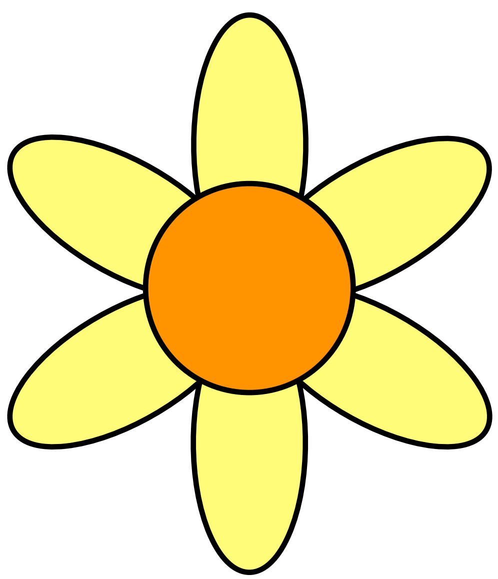
$$\mathcal{F} \subseteq \binom{[n]}{k}$$

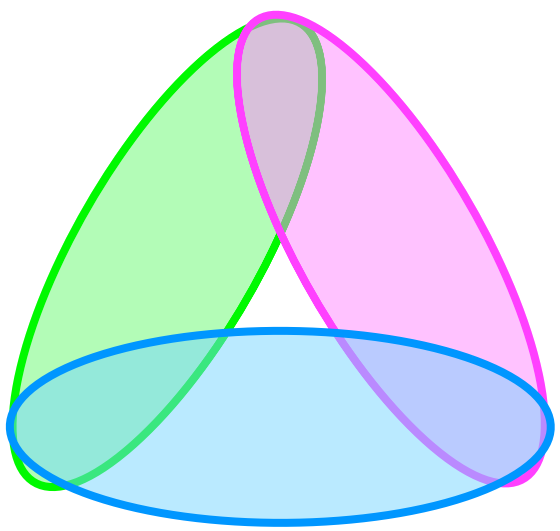
intersecting:

$$\forall S, T \in \mathcal{F}, \quad S \cap T \neq \emptyset$$

trivial case: $n < 2k$

nontrivial examples:





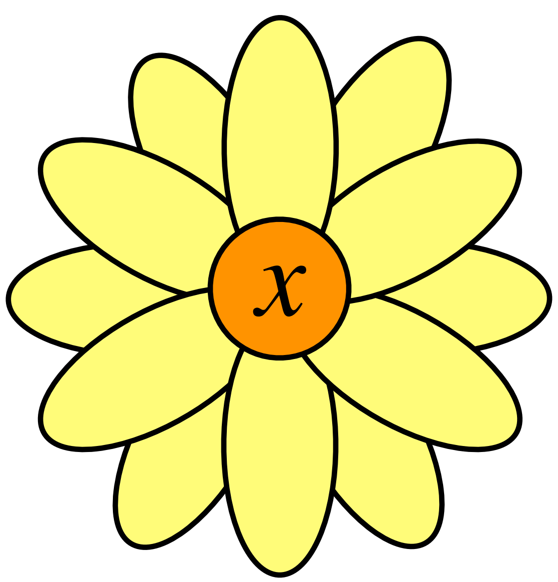
"How large can a nontrivial intersecting family be?"

Erdős-Ko-Rado Theorem

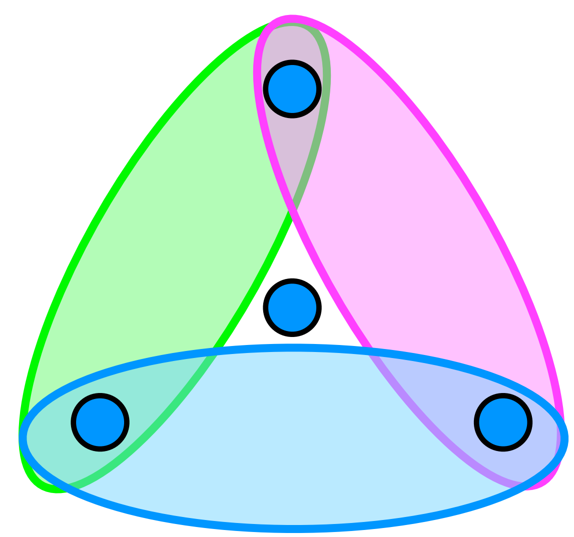
Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$\mathcal{F}$ is intersecting $\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset$ $\Rightarrow$ $|\mathcal{F}| \leq \binom{n-1}{k-1}$

proved in 1938; published in 1961;



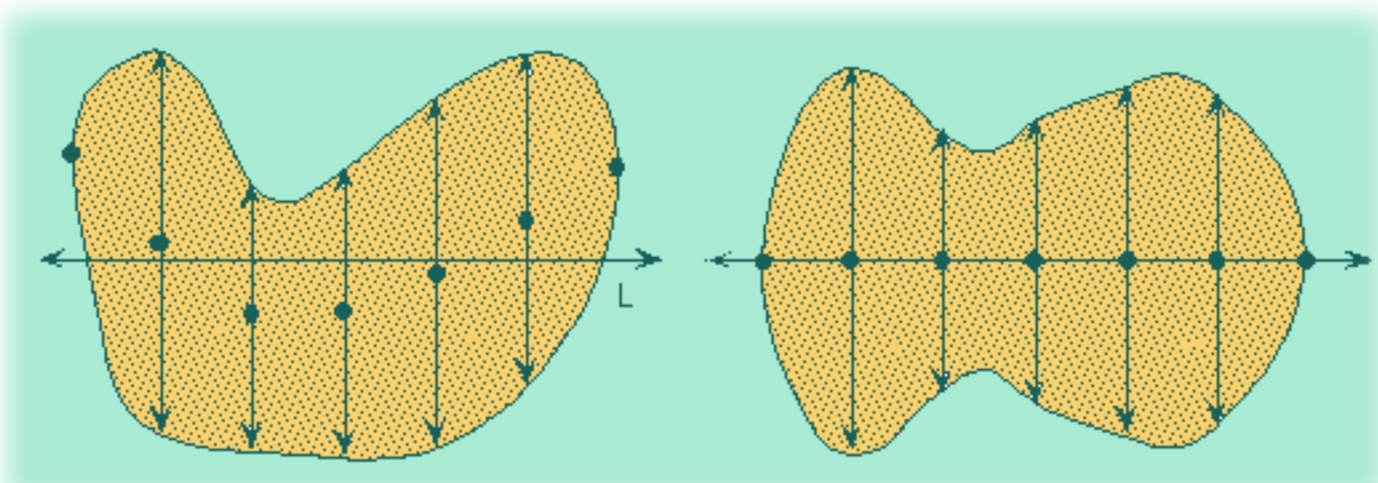
all $S \ni x$



Shifting

Isoperimetric problem:

With fixed perimeter,
what plane figure has the largest area?



Steiner symmetrization

Shifting

Isoperimetric problem:

With fixed area,
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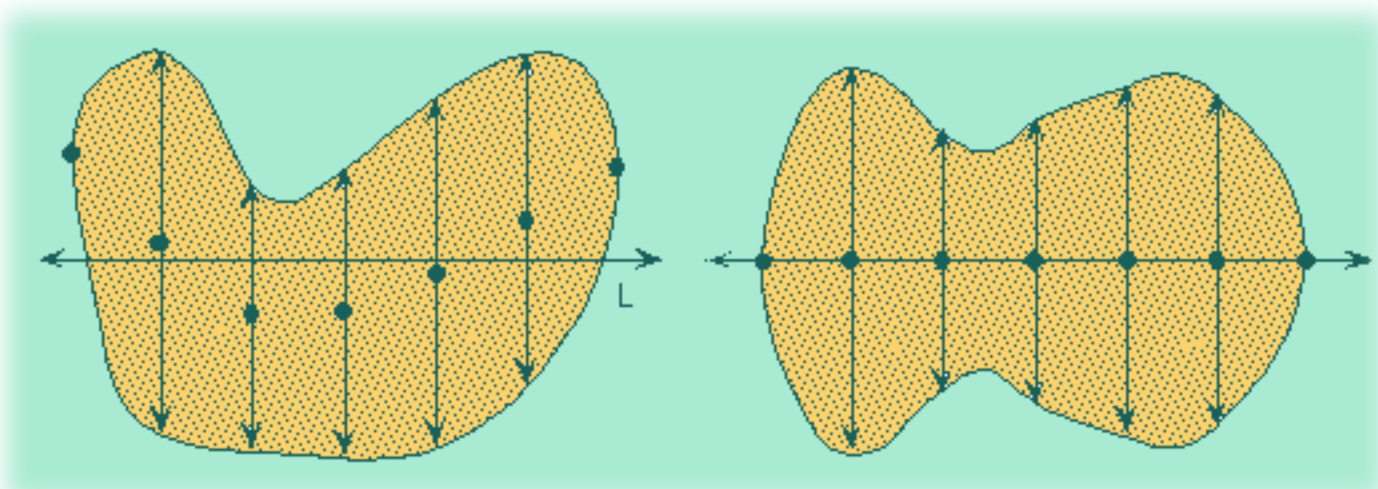


Steiner symmetrization

Erdős-Ko-Rado Theorem

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

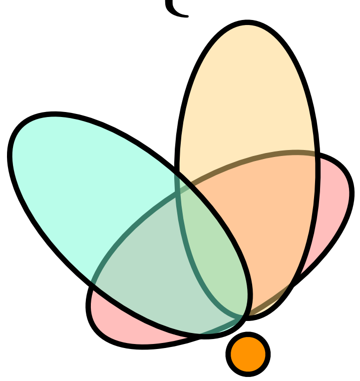
$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1}$

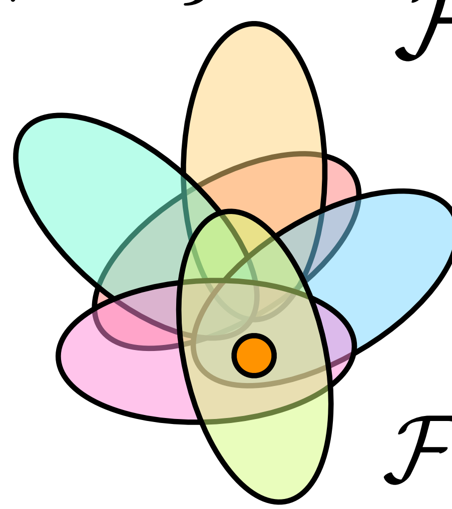
induction on n and k

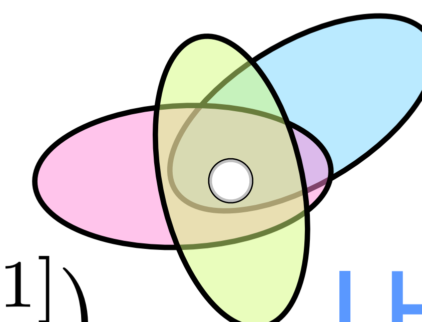
$$\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$$

$$\mathcal{F}_0 = \{S \in \mathcal{F} \mid n \notin S\}$$

$$\mathcal{F}'_1 = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$$







$\mathcal{F}_0 \subseteq \binom{[n-1]}{k}$ intersecting

I.H. $|\mathcal{F}_0| \leq \binom{n-2}{k-1}$

$\mathcal{F}'_1 \subseteq \binom{[n-1]}{k-1}$ intersecting?

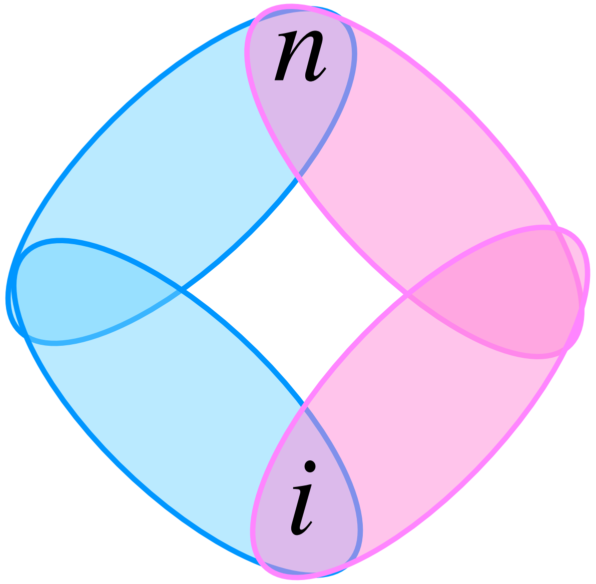
I.H. $|\mathcal{F}'_1| \leq \binom{n-2}{k-2}$

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}'_1| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$$

Shifting (compression)

special $\mathcal{F} \subseteq \binom{[n]}{k}$

$\mathcal{F}$ remains intersecting after deleting n



Shifting (compression)

$\mathcal{F} \subseteq 2^{[n]}$ for $1 \le i < j \le n$

$$\forall T \in \mathcal{F},\ \text{write } T_{ij} = (T \setminus \{j\}) \cup \{i\}$$

(i, j)-shift: $S_{ij}(\cdot)$

$\forall T \in \mathcal{F},$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

$1 \leq i < j \leq n \quad \forall T \in \mathcal{F}, \;$ write $T_{ij} = (T \setminus \{j\}) \cup \{i\}$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

1. $|S_{ij}(T)| = |T|$ **and** $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$
2. $\mathcal{F}$ intersecting $\Rightarrow$ $S_{ij}(\mathcal{F})$ intersecting

(2) the only bad case: $A, B \in \mathcal{F} \quad A \cap B = \{j\}$

$A_{ij} = A \setminus \{j\} \cup \{i\} \in \mathcal{F} \quad B_{ij} = B \setminus \{j\} \cup \{i\} \notin \mathcal{F} \quad i \notin B$

$\Rightarrow \quad A_{ij} \cap B = \emptyset$ contradiction!

$1 \le i < j \le n \quad \forall T \in \mathcal{F}, \;$ write $T_{ij} = (T \setminus \{j\}) \cup \{i\}$

$$S_{ij}(T) = \begin{cases} T_{ij} & \text{if } j \in T, i \notin T, \text{ and } T_{ij} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_{ij}(\mathcal{F}) = \{S_{ij}(T) \mid T \in \mathcal{F}\}$$

1. $|S_{ij}(T)| = |T|$ **and** $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$
2. $\mathcal{F}$ intersecting $\Rightarrow$ $S_{ij}(\mathcal{F})$ intersecting

repeat applying (i,j)-shifting $S_{ij}(\mathcal{F})$ for $1 \le i < j \le n$

eventually, $\mathcal{F}$ is unchanged by any $S_{ij}(\mathcal{F})$

called: $\mathcal{F}$ is shifted

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Longrightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$$

Erdős-Ko-Rado's proof:

true for $k{=}1$;

when $n = 2k$,

$\forall S \in \binom{[n]}{k}$ at most one of S and $\bar{S}$ is in $\mathcal{F}$

$$|\mathcal{F}| \leq \frac{1}{2}\binom{n}{k} = \frac{n!}{2 \cdot k!(n-k)!}$$

$$= \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Longrightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$$

arbitrary intersecting $\mathcal{F}$ $\xrightarrow[\text{keep intersecting}]{|\mathcal{F}| = |\mathcal{F}'|}$ shifted $\mathcal{F}'$

$$\Downarrow$$

$$|\mathcal{F}| \leq \binom{n-1}{k-1} \Longleftarrow |\mathcal{F}'| \leq \binom{n-1}{k-1}$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\boxed{\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}}$$

when $n > 2k$, induction on n

WLOG: $\mathcal{F}$ is shifted

$\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$ $\qquad$ $\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$

$\boxed{\mathcal{F}_1' \text{ is intersecting}}$

otherwise, $\exists A, B \in \mathcal{F}$ $\qquad$ $A \cap B = \{n\}$

$|A \cup B| \leq 2k - 1 < n - 1 \Rightarrow \exists i < n,\ i \notin A \cup B$

$C = A \setminus \{n\} \cup \{i\} \in \mathcal{F} \Leftarrow$ $\mathcal{F}$ is shifted

$C \cap B = \emptyset$ $\qquad$ contradiction!

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1}$$

when $n > 2k$, induction on n

WLOG: $\mathcal{F}$ is shifted

$\mathcal{F}_0 = \{S \in \mathcal{F} \mid n \notin S\}$ $\quad$ $\mathcal{F}_1 = \{S \in \mathcal{F} \mid n \in S\}$

$\mathcal{F}_0 \subseteq \binom{[n-1]}{k}$ and intersecting $\xRightarrow{\text{I.H.}}$ $|\mathcal{F}_0| \leq \binom{n-2}{k-1}$

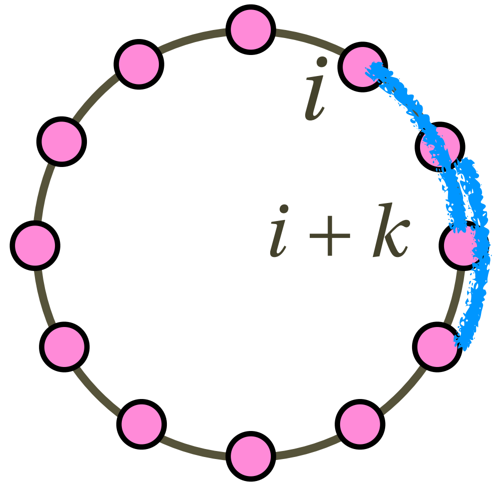
$\mathcal{F}_1' = \{S \setminus \{n\} \mid S \in \mathcal{F}_1\}$

$\mathcal{F}_1' \subseteq \binom{[n-1]}{k-1}$ and intersecting $\xRightarrow{\text{I.H.}}$ $|\mathcal{F}_1'| \leq \binom{n-2}{k-2}$

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| = |\mathcal{F}_0| + |\mathcal{F}_1'| \leq \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$$

Katona's proof (1972)

n-cycle:



k-arc: length k path on cycle

intersecting arcs: share edges

Lemma
$n \geq 2k$. Suppose $A_1, A_2, \ldots, A_t$ are distinct pairwise intersecting k-arcs. Then $t \leq k$.

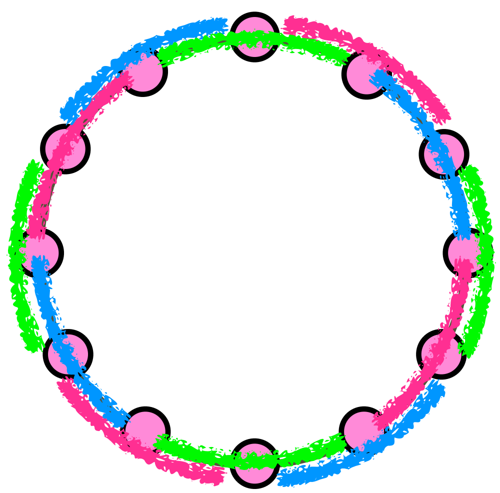
every node can be endpoint of at most 1 arc

take A_1: A_1 has k+1 nodes

2 endpoints of itself

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$$



take an n-cycle π of $[n]$

family of all k-arcs in π

$$\mathcal{G}_\pi = \{\{\pi_{(i+j) \bmod n} \mid j \in [k]\} \mid i \in [n]\}$$

double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

each n-cycle π an n-cycle has ≤k intersecting k-arcs

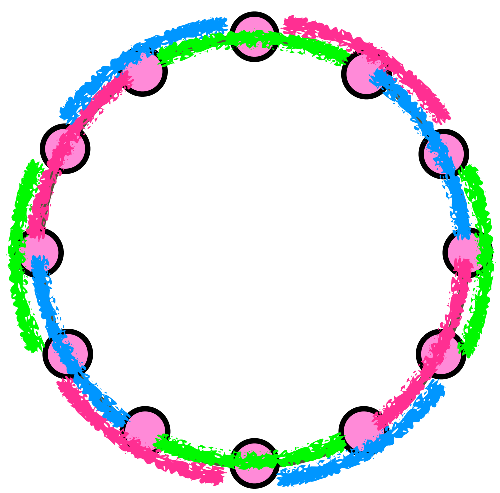
$$|\mathcal{F} \cap \mathcal{G}_\pi| \leq k$$

of n-cycles: $(n-1)!$

$$|X| = \sum_{n\text{-cycle } \pi} |\mathcal{F} \cap \mathcal{G}_\pi| \leq k(n-1)!$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \implies |\mathcal{F}| \leq \binom{n-1}{k-1}$$



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double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

$$|X| \leq k(n-1)!$$

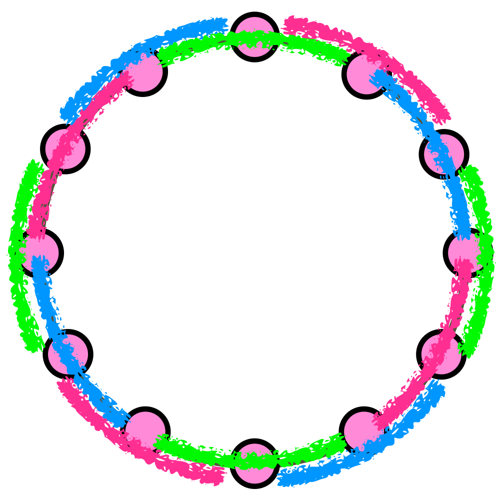
each S is a k-arc in

$k!(n\text{-}k)!$ cycles

$$|X| = \sum_{S \in \mathcal{F}} |\{\pi \mid S \in \mathcal{G}_\pi\}| = |\mathcal{F}| k!(n-k)!$$

Let $\mathcal{F} \subseteq \binom{[n]}{k}$, $n \geq 2k$.

$$\forall S, T \in \mathcal{F},\ S \cap T \neq \emptyset \Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$$



take an n-cycle π of $[n]$

family of all k-arcs in π

$$\mathcal{G}_\pi = \{\{\pi_{(i+j) \bmod n} \mid j \in [k]\} \mid i \in [n]\}$$

double counting: $X = \{(S, \pi) \mid S \in \mathcal{F} \cap \mathcal{G}_\pi\}$

$|X| \leq k(n-1)!$

$|X| = |\mathcal{F}| k!(n-k)!$

$$|\mathcal{F}| \leq \frac{k(n-1)!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}$$

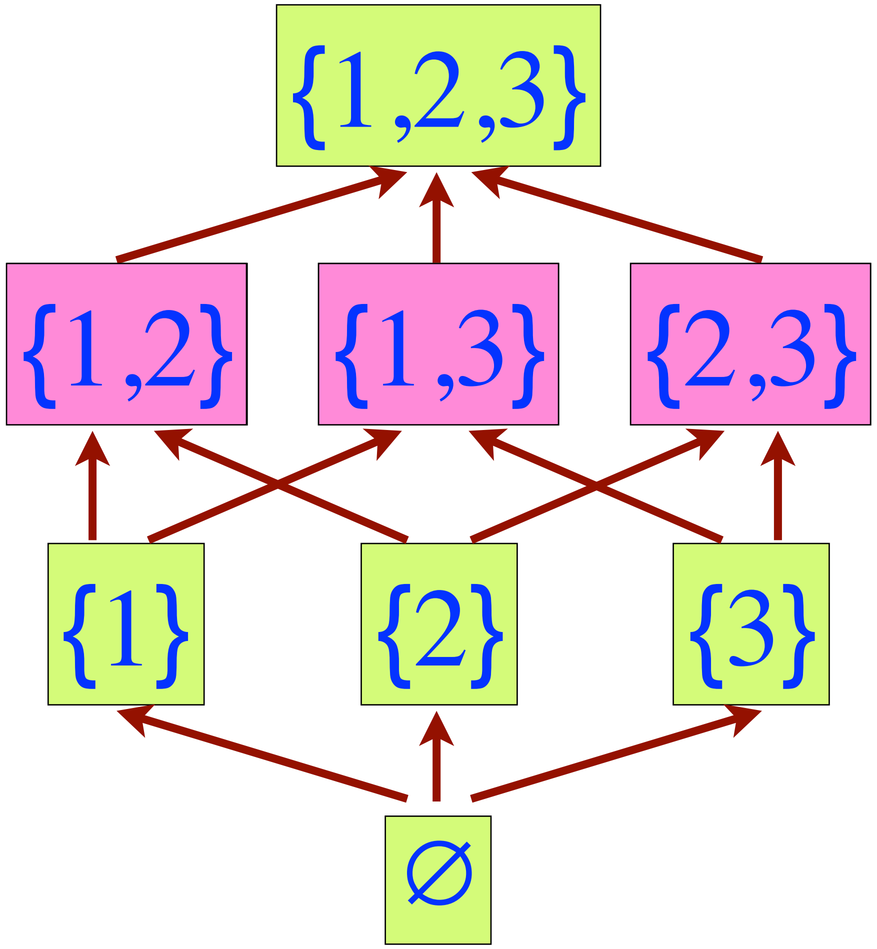
Antichains

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain

$\forall A, B \in \mathcal{F}, \quad A \not\subseteq B$

$\binom{[n]}{k}$ is antichain

largest size: $\binom{n}{\lfloor n/2 \rfloor}$



"Is this the largest size for all antichains?"

Sperner's Theorem

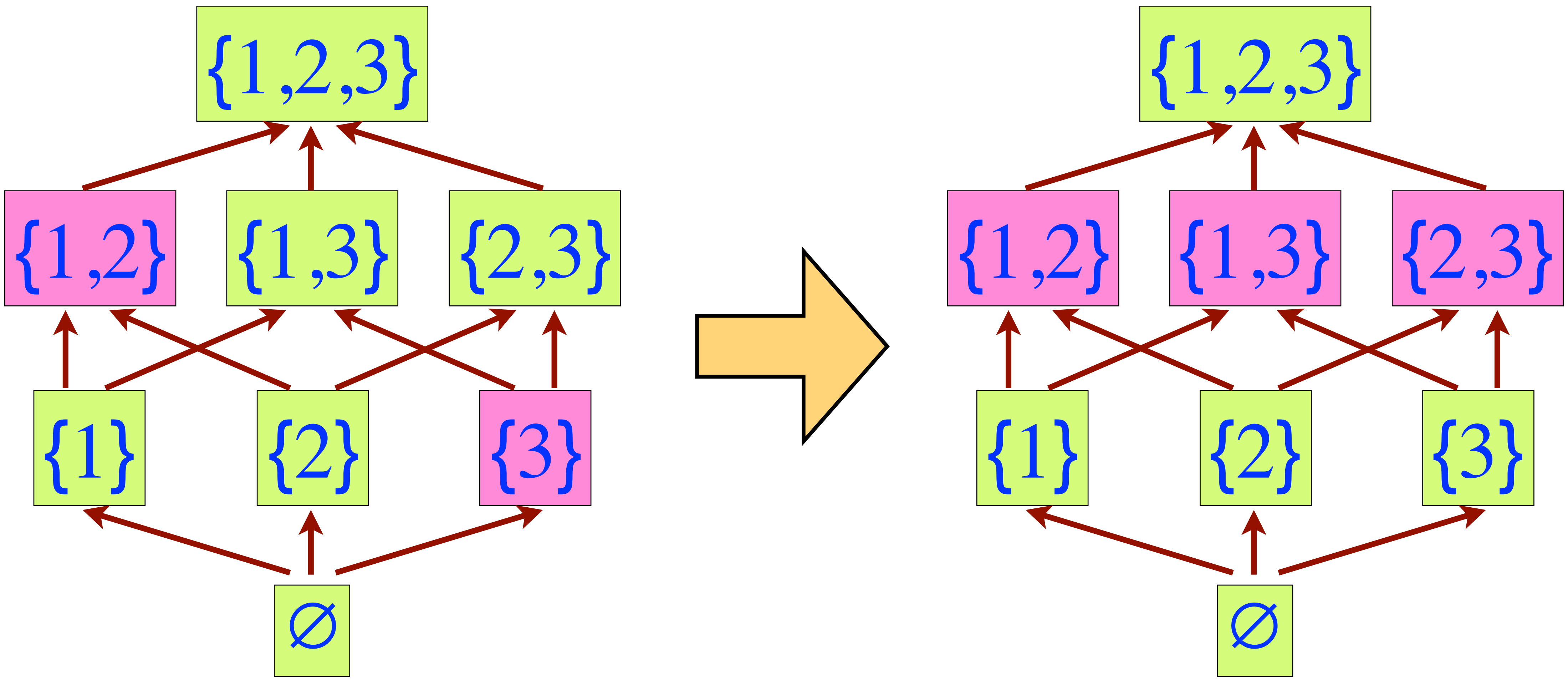
Theorem (Sperner 1928)

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain.

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Emanuel Sperner
(1905 - 1980)

Sperner's proof



$$\mathcal{F} \subseteq \binom{[n]}{k}$$

shade: $\nabla\mathcal{F} = \left\{ T \in \binom{[n]}{k+1} \mid \exists S \in \mathcal{F}, S \subset T \right\}$

shadow: $\Delta\mathcal{F} = \left\{ T \in \binom{[n]}{k-1} \mid \exists S \in \mathcal{F}, T \subset S \right\}$

[n] = {1,2,3,4,5}

$\mathcal{F} =$ { {1,2,3}, {1,3,4}, {2,3,5} }

$\nabla\mathcal{F} =$ { {1,2,3,4}, {1,2,3,5}, {1,3,4,5}, {2,3,4,5} }

$\Delta\mathcal{F} =$ {{1,2}, {2,3}, {1,3}, {3,4}, {1,4}, {2,5}, {3,5}}

Lemma (Sperner)

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$|\nabla\mathcal{F}| \geq \frac{n-k}{k+1}|\mathcal{F}| \qquad \text{(for } k < n)$$

$$|\Delta\mathcal{F}| \geq \frac{k}{n-k+1}|\mathcal{F}| \qquad \text{(for } k > 0)$$

double counting

$$\mathcal{R} = \{(S,T) \mid S \in \mathcal{F}, T \in \nabla\mathcal{F}, S \subset T\}$$

$$\forall S \in \mathcal{F}, \qquad n-k\ T \in \binom{[n]}{k+1} \text{ have } T \supset S$$

$$|\mathcal{R}| = (n-k)|\mathcal{F}|$$

$$\forall T \in \nabla\mathcal{F}, \quad T \text{ has } \binom{k+1}{k} = k+1 \text{ many } k\text{-subsets}$$

$$|\mathcal{R}| \leq (k+1)|\nabla\mathcal{F}|$$

Lemma (Sperner)

Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then

$$|\nabla \mathcal{F}| \geq \frac{n-k}{k+1} |\mathcal{F}| \qquad \text{(for } k < n\text{)}$$

$$|\Delta \mathcal{F}| \geq \frac{k}{n-k+1} |\mathcal{F}| \qquad \text{(for } k > 0\text{)}$$

Corollary:

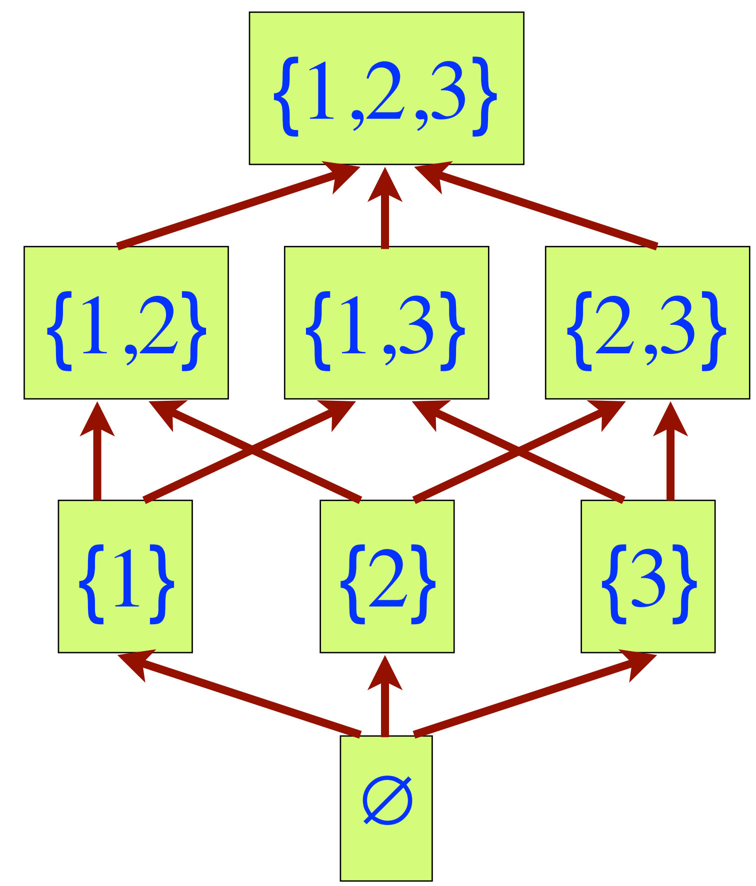
If $k \leq \frac{1}{2}(n-1)$, then $|\nabla \mathcal{F}| \geq |\mathcal{F}|$.

If $k \geq \frac{1}{2}(n+1)$, then $|\Delta \mathcal{F}| \geq |\mathcal{F}|$.

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$



If $k \leq \frac{1}{2}(n-1)$, then $|\nabla\mathcal{F}| \geq |\mathcal{F}|$.

If $k \geq \frac{1}{2}(n+1)$, then $|\Delta\mathcal{F}| \geq |\mathcal{F}|$.

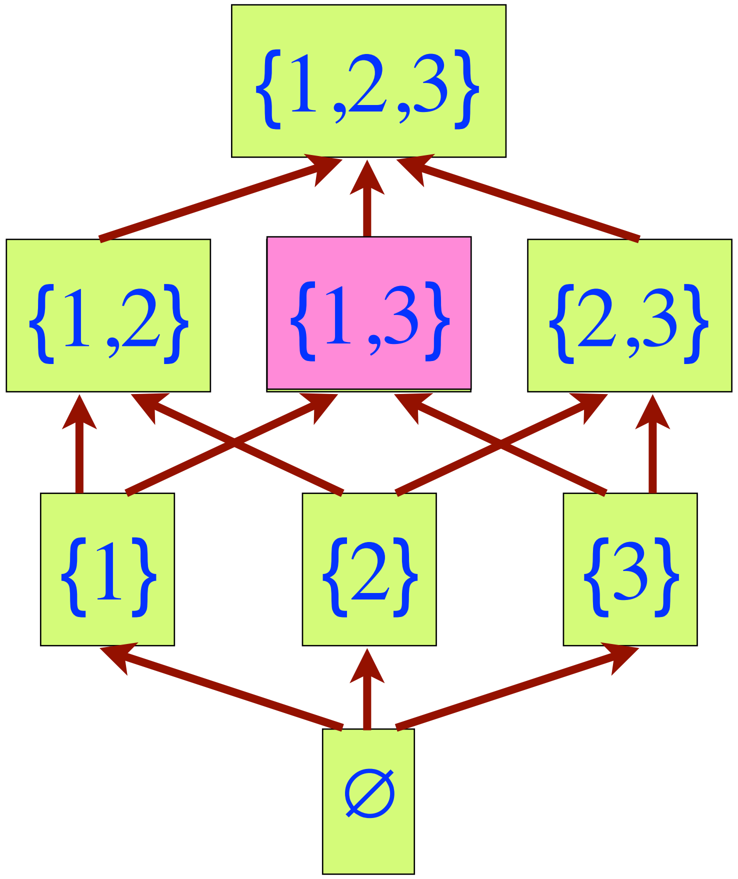
replace $\mathcal{F}_k$ by $\begin{cases} \nabla\mathcal{F}_k & \text{if } k < \frac{1}{2}(n-1) \\ \Delta\mathcal{F}_k & \text{if } k \geq \frac{1}{2}(n+1) \end{cases}$ still antichain!

repeat until $\mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor}$ with no decreasing of $|\mathcal{F}|$

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)



maximal chain:

$$\emptyset \subset S_1 \subset \cdots \subset S_{n-1} \subset [n]$$

# of maximal chains in $2^{[n]}$: $n!$

$\forall S \subseteq [n]$,

# of maximal chains containing S: $|S|!(n-|S|)!$

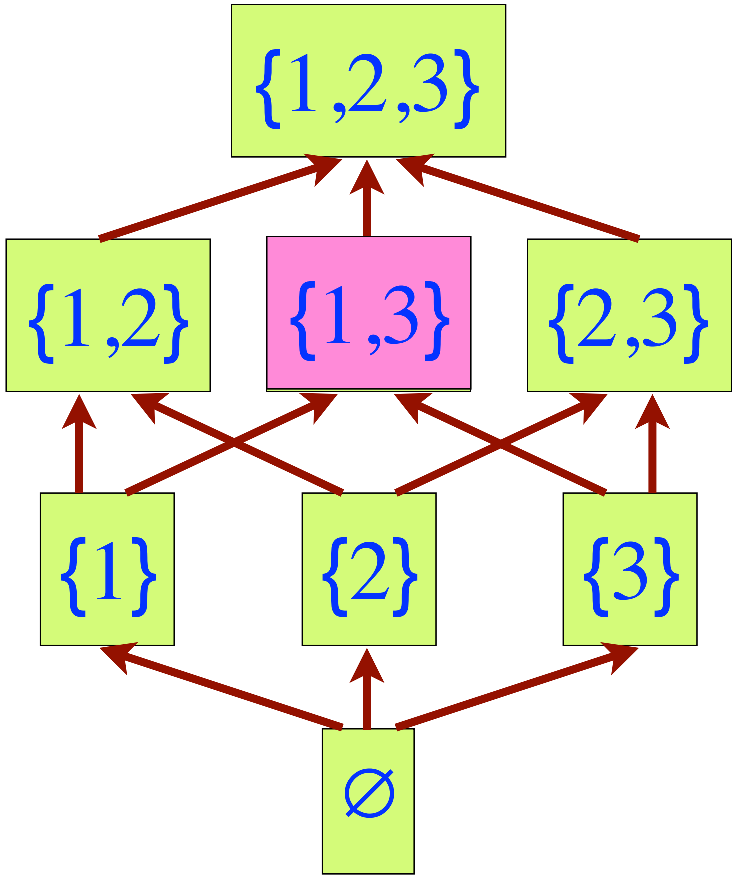
$\mathcal{F}$ is antichain ⇒ $\forall$ chain C, $|\mathcal{F} \cap C| \leq 1$

maximal chains crossing $\mathcal{F}$ ≤ # all maximal chains

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)



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# of maximal chains containing S: $|S|!(n-|S|)!$

$\mathcal{F}$ is antichain $\Rightarrow$ $\forall$ chain C, $|\mathcal{F} \cap C| \le 1$

$$\sum_{S \in \mathcal{F}} |S|!(n-|S|)! \le n!$$

Sperner's Theorem

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lubell's proof (double counting)

$$\sum_{S \in \mathcal{F}} |S|!(n - |S|)! \leq n!$$

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} = \sum_{S \in \mathcal{F}} \frac{|S|!(n - |S|)!}{n!} \leq 1$$

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

LYM Inequality

(Lubell-Yamamoto 1954, Meschalkin 1963)

LYM inequality

$\mathcal{F} \subseteq 2^{[n]}$ is an antichain.

$$\sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

$$\mathcal{F} \subseteq 2^{[n]} \text{ is an antichain.} \quad \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

Alon's proof (the probabilistic method)

let π be a random permutation $[n]$

$$\mathcal{C}_\pi = \{\{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\}\}$$

$$\forall S \in \mathcal{F}, \qquad X_S = \begin{cases} 1 & S \in \mathcal{C}_\pi \\ 0 & \text{otherwise} \end{cases}$$

let $X = \sum_{S \in \mathcal{F}} X_S = |\mathcal{F} \cap \mathcal{C}_\pi|$

$$\mathbf{E}[X_S] = \Pr[S \in \mathcal{C}_\pi] = \frac{1}{\binom{n}{|S|}}$$

$\mathcal{C}_\pi$ contains precisely 1 $|S|$-set

uniform over all $|S|$-sets

$$\mathcal{F} \subseteq 2^{[n]} \text{ is an antichain.} \quad \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1$$

Alon's proof (the probabilistic method)

let π be a random permutation $[n]$

$$\mathcal{C}_\pi = \{\{\pi_1\}, \{\pi_1, \pi_2\}, \ldots, \{\pi_1, \ldots, \pi_n\}\}$$

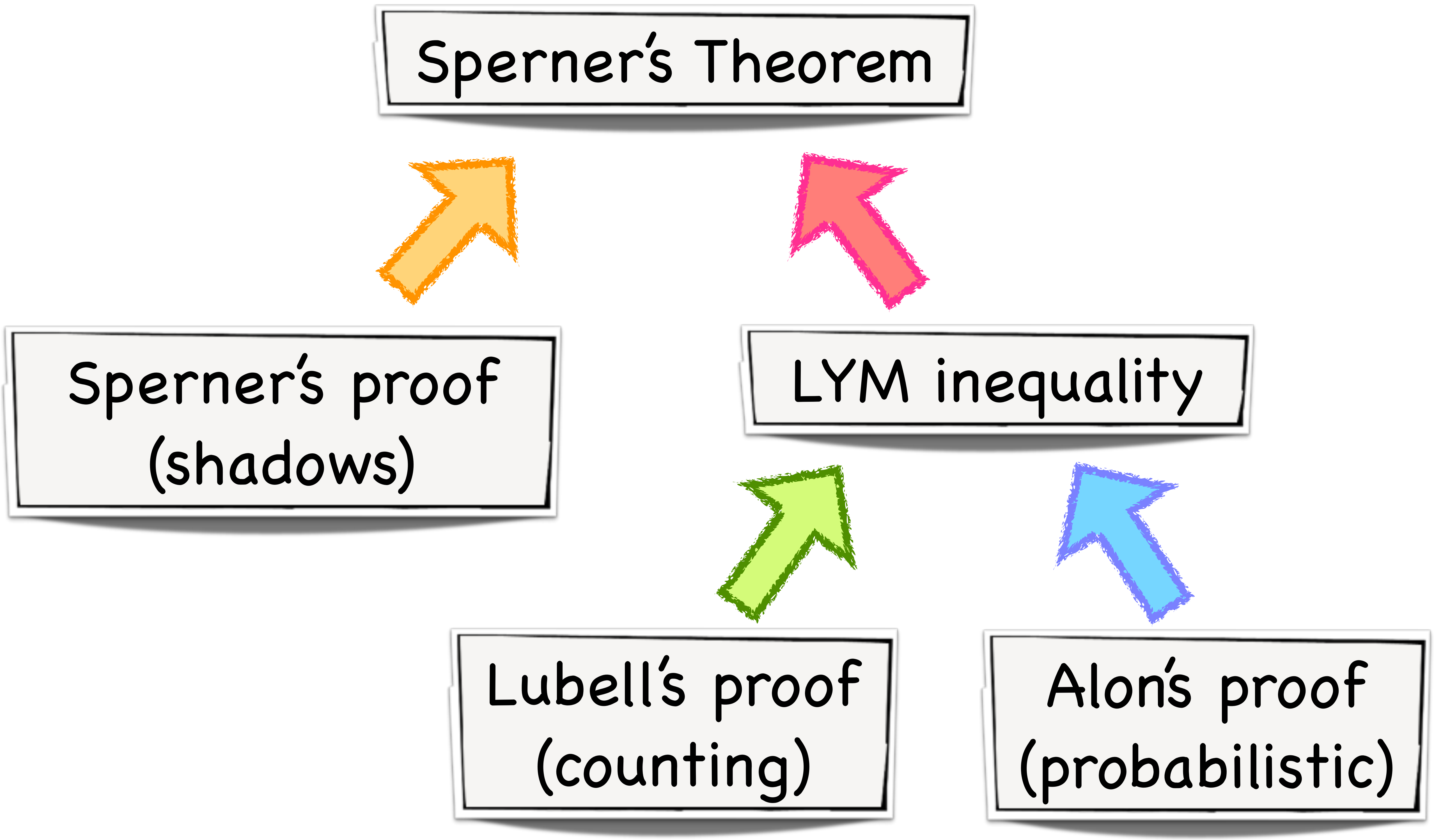
$$X = \sum_{S \in \mathcal{F}} X_S = |\mathcal{F} \cap \mathcal{C}_\pi| \leq 1$$

$\mathcal{F}$ is antichain

$\mathcal{C}_\pi$ is chain

$$\mathbf{E}[X_S] = \frac{1}{\binom{n}{|S|}}$$

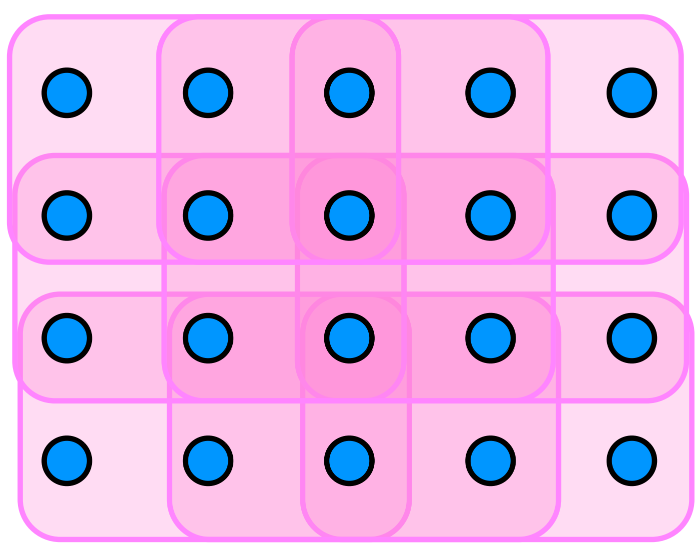
$$1 \geq \mathbf{E}[X] = \sum_{S \in \mathcal{F}} \mathbf{E}[X_S] = \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}}$$


Sperner's Theorem
Sperner's proof (shadows)
LYM inequality
Lubell's proof (counting)
Alon's proof (probabilistic)

Shattering

$\mathcal{F} \subseteq 2^{[n]}$ $\qquad$ $R \subseteq [n]$

trace $\mathcal{F}|_R$: $\qquad$ $\mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$



$\mathcal{F}$ shatters R $\qquad$ $\mathcal{F}|_R = 2^R$

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k},\ \mathcal{F} \text{ shatters } R$$

Sauer; Shelah-Perles; Vapnik-Cervonenkis;

VC-dimension of $\mathcal{F}$

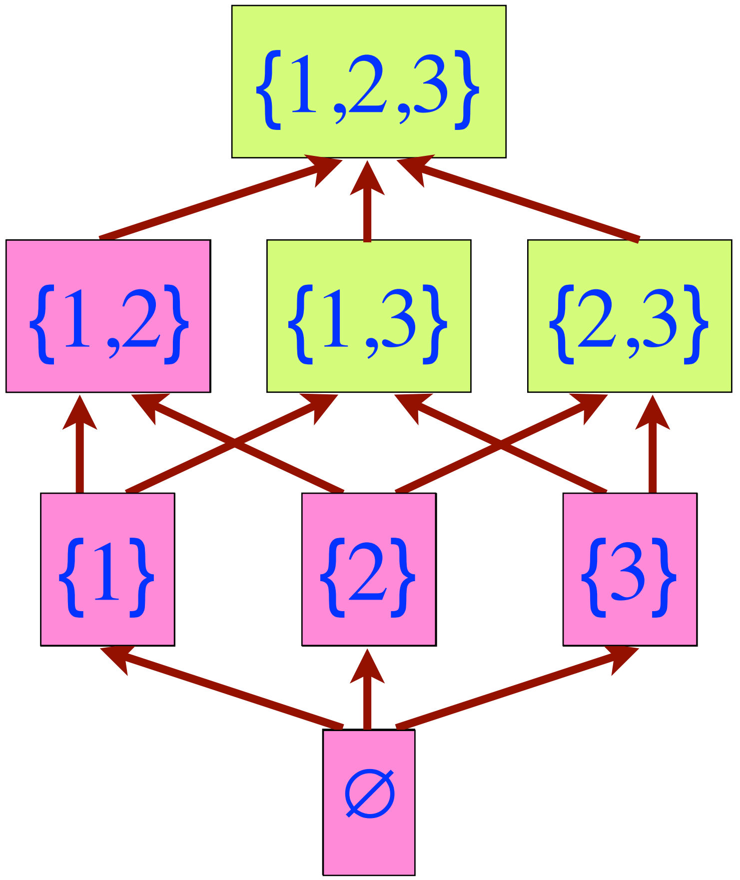
size of the largest R shattered by $\mathcal{F}$

$$\mathcal{F} \subseteq 2^{[n]} \qquad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$$

$$\text{VC-dim}(\mathcal{F}) = \max\left\{|R| \mid R \subseteq [n], \mathcal{F}|_R = 2^R\right\}$$

Heredity (ideal, simplicial complex)

$\mathcal{F}$ is hereditary if $\forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F}$



Heredity (ideal, simplicial complex)

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \tbinom{[n]}{k},\ \mathcal{F} \text{ shatters } R$$

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \mathcal{F}, |R| \ge k$$

for hereditary $\mathcal{F}$: $\quad \forall B \subseteq A \in \mathcal{F}, \quad B \in \mathcal{F}$

$$R \in \mathcal{F} \Rightarrow \mathcal{F} \text{ shatters } R$$

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k},\ \mathcal{F} \text{ shatters } R$$

$$|\mathcal{F}| \le |\mathcal{F}'|$$

arbitrary $\mathcal{F}$ $\longrightarrow$ hereditary $\mathcal{F}'$

$$\text{VC-dim}(\mathcal{F}) \ge \text{VC-dim}(\mathcal{F}')$$

$\Downarrow$

$\mathcal{F}$ shatters a k-set $\Leftarrow$ $\mathcal{F}'$ shatters a k-set

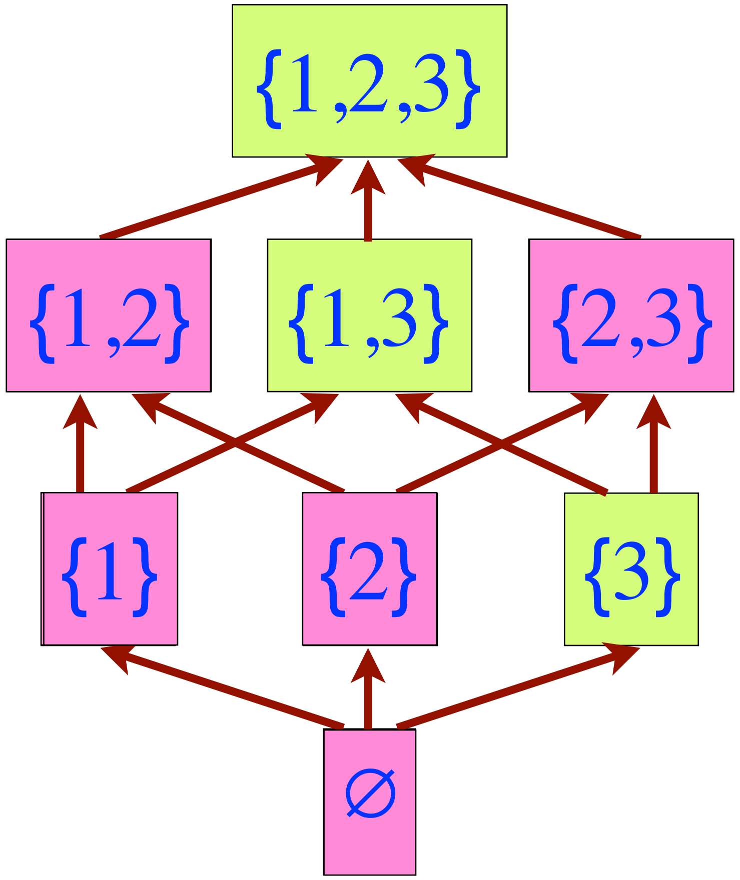
Down Shift

$\mathcal{F} \subseteq 2^{[n]}$ **for** $i \in [n]$

down-shift: $S_i(\cdot)$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$



$\mathcal{F} \subseteq 2^{[n]}$ $\quad \mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$ $\quad$ **for** $i \in [n]$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$

1. $|S_i(\mathcal{F})| = |\mathcal{F}|$ ✓
2. $|S_i(\mathcal{F})|_R| \leq |\mathcal{F}|_R|$ for all $R \subseteq [n]$

$$S_i(\mathcal{F})|_R \subseteq S_i(\mathcal{F}|_R)$$

by-case analysis

$$A \in S_i(\mathcal{F}) \Rightarrow \left\{ \begin{array}{l} A = S_i(A \cup \{i\}) \\ \\ A = S_i(A) \end{array} \right\} \Rightarrow A \cap R \in S_i(\mathcal{F}|_R)$$

$\mathcal{F} \subseteq 2^{[n]}$ $\mathcal{F}|_R = \{S \cap R \mid S \in \mathcal{F}\}$ for $i \in [n]$

$$S_i(T) = \begin{cases} T \setminus \{i\} & \text{if } i \in T \in \mathcal{F}, \text{ and } T \setminus \{i\} \notin \mathcal{F}, \\ T & \text{otherwise.} \end{cases}$$

$$S_i(\mathcal{F}) = \{S_i(T) \mid T \in \mathcal{F}\}$$

1. $|S_i(\mathcal{F})| = |\mathcal{F}|$
2. $|S_i(\mathcal{F})|_R| \leq |\mathcal{F}|_R|$ for all $R \subseteq [n]$

repeat applying down-shifting $S_i(\mathcal{F})$ for $i \in [n]$

eventually, $\mathcal{F}$ is unchanged by any $S_i(\mathcal{F})$

$\forall A \in \mathcal{F}$ if $B \subseteq A$ $\Rightarrow$ $B \in \mathcal{F}$

$\mathcal{F}$ is hereditary

Sauer's Lemma

$$|\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \Rightarrow \exists R \in \binom{[n]}{k},\ \mathcal{F} \text{ shatters } R$$

repeat down-shift $\mathcal{F}$ until unchanged

$$\left.\begin{array}{l} \mathcal{F} \text{ is } \textbf{hereditary} \\ |\mathcal{F}| > \sum_{0 \le i < k} \binom{n}{i} \end{array}\right\} \Rightarrow \begin{array}{c} \exists S \in \binom{[n]}{\ell} \text{ with } \ell \ge k \\ 2^S \subseteq \mathcal{F} \end{array}$$

take any $R \in \binom{S}{k}$ $\mathcal{F}$ shatters R