Combinatorics

南京大学
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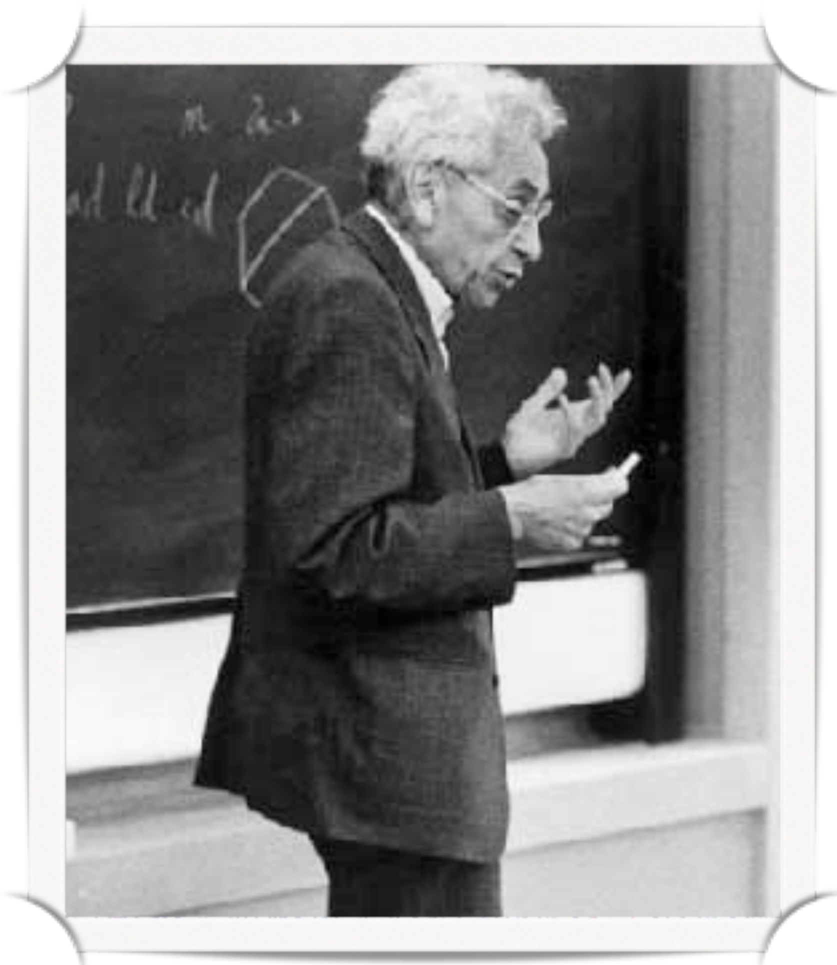


Paul Erdős

The Probabilistic Method

Ramsey Number

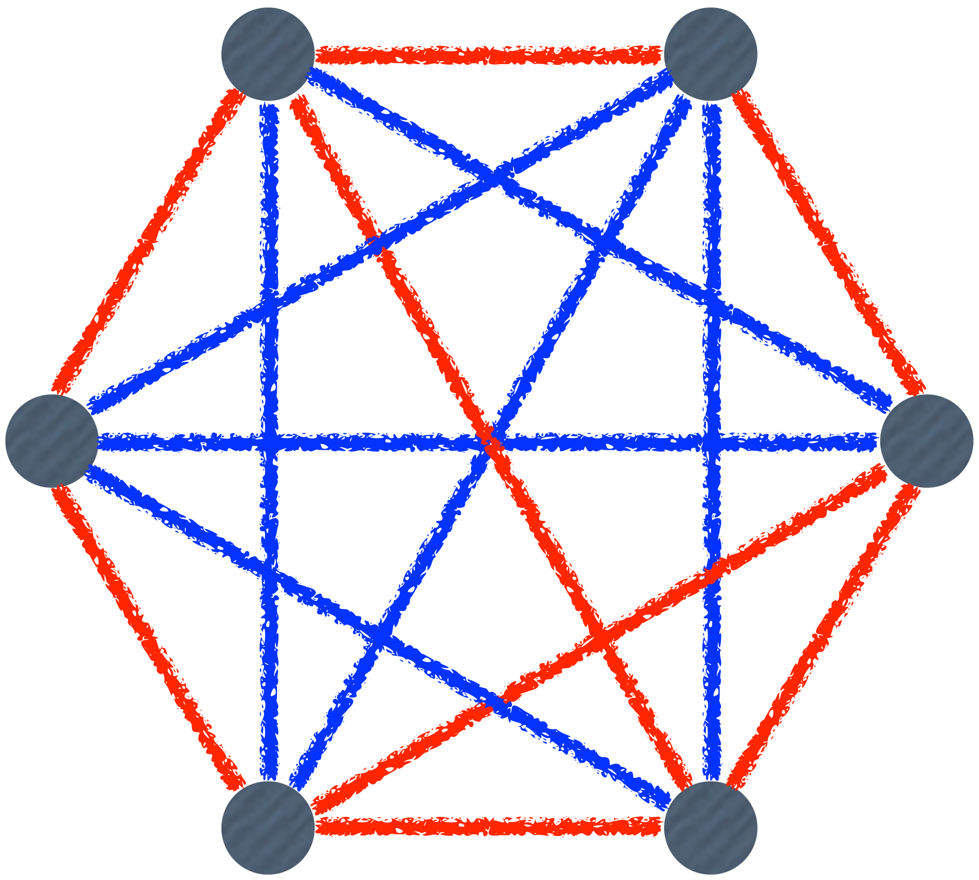
"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

- For any two coloring of K_6, there is a *monochromatic* K_3.

Ramsey's Theorem

If $n \geq R(k,k)$, for any two coloring of K_n, there is a monochromatic K_k.

Ramsey number: $R(k,k)$



Theorem (Erdős 1947)

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with two colors so that there is no monochromatic K_k subgraph.

For each edge $e \in K_n$,

e is colored
- with prob 1/2
- with prob 1/2

For a particular K_k, $\binom{k}{2}$ edges

$$\Pr[\ K_k \text{ or } K_k\ ] = 2^{1-\binom{k}{2}}$$

Theorem (Erdős 1947)

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with two colors so that there is no monochromatic K_k subgraph.

For a particular K_k,

$$\Pr[\text{the } K_k \text{ is monochromatic}] = 2^{1-\binom{k}{2}}$$

number of K_k in K_n: $\binom{n}{k}$

$$\Pr[\exists \text{ a monochromatic } K_k]$$

$$\leq \binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$$

Theorem (Erdős 1947)

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with two colors so that there is no monochromatic K_k subgraph.

For a random two-coloring:

$$\Pr[\exists \text{ a monochromatic } K_k] < 1$$
$$\Pr[\neg\exists \text{ a monochromatic } K_k] > 0$$

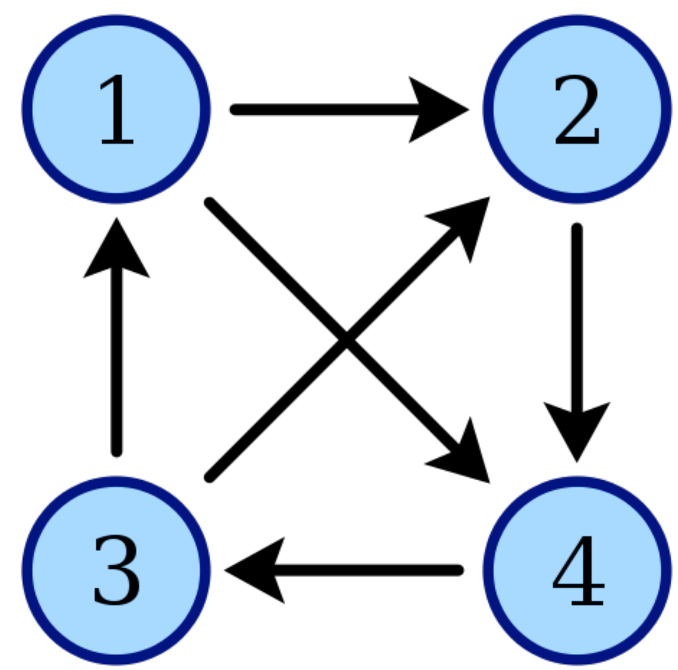
There exists a two-coloring without monochromatic K_k.

Tournament

T(V, E)

n players, each pair has a match.

u points to *v* iff u beats v.



k-paradoxical:

For every k-subset S of V, there is a player in $V \setminus S$ who beats all players in S.

"*Does there exist a k-paradoxical tournament for every finite k?*"

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament *T* on *n* players [*n*].

Fixed any $S \in \binom{[n]}{k}$

Event A_S : no player in $V \backslash S$ beat all players in S.

$$\Pr[A_S] = \left(1-2^{-k}\right)^{n-k}$$

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament *T* on *n* players [*n*].

Event A_S : no player in $V \setminus S$ beat all players in S.

$$\Pr[A_S] = \left(1-2^{-k}\right)^{n-k}$$

$$\Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] \leq \sum_{S\in\binom{[n]}{k}} (1-2^{-k})^{n-k} \quad < 1$$

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament *T* on *n* players [*n*].

Event A_S : no player in $V \setminus S$ beat all players in S.

$$\Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] < 1$$

$$\Pr[T \text{ is } k\text{-paradoxical}] = 1 - \Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] > 0$$

Theorem (Erdős 1963)

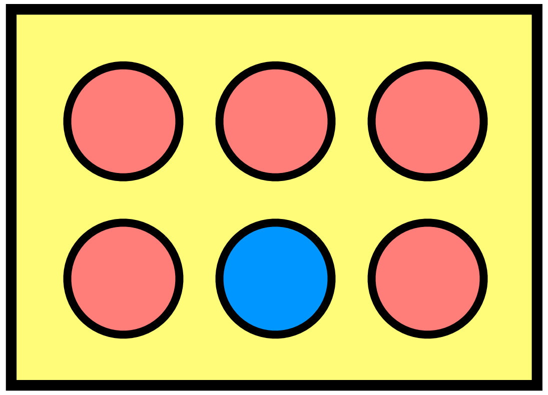
If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament T on n players $[n]$.

$$\Pr[T \text{ is } k\text{-paradoxical}] > 0$$

There is a k-paradoxical tournament on n players.

The Probabilistic Method



- Pick random ball from a box,

  Pr[the ball is blue]>0.

  ⇒ There is a blue ball.

- Define a probability space Ω, and a property P:

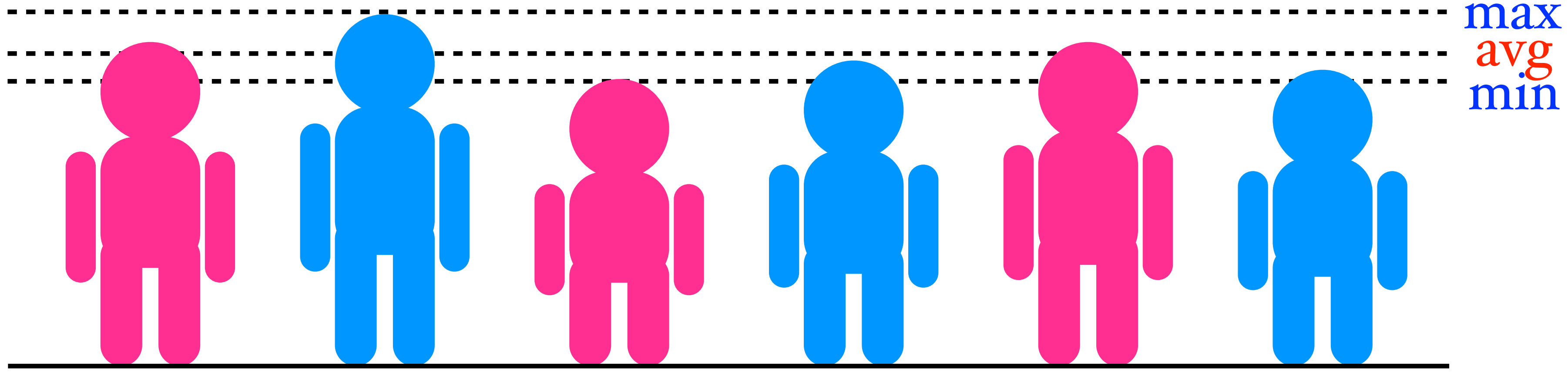
$$\Pr_x[\mathrm{P}(x)] > 0$$

$$\implies \exists x \in \Omega \text{ with the property P.}$$

Averaging Principle

- Average height of the students in class is l.

  $\Rightarrow$ There is a student of height $\geq l$ ($\leq l$)

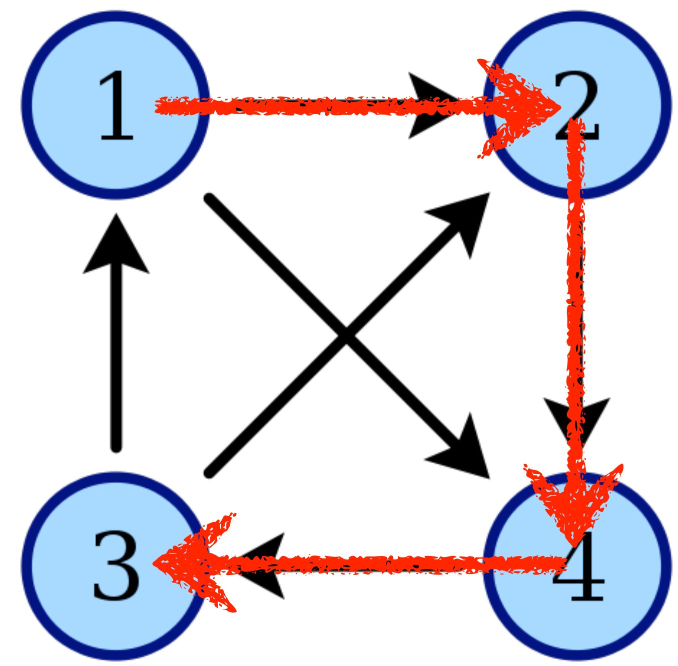


- For a random variable X,
  - $\exists\, x \leq \mathrm{E}[X]$, such that $X = x$ is possible;
  - $\exists\, x \geq \mathrm{E}[X]$, such that $X = x$ is possible.

Hamiltonian paths in tournament

Hamiltonian path:

a path visiting every vertex *exactly* once.



Theorem (Szele 1943)

There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

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There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

Pick a random tournament *T* on *n* players [*n*].

For every permutation π of [*n*],

$$X_\pi = \begin{cases} 1 & \pi \text{ is a Hamiltonian path} \\ 0 & \pi \text{ is } \textit{not} \text{ a Hamiltonian path} \end{cases}$$

\# Hamiltonian paths: $X = \sum_{\pi} X_\pi$

$$\mathrm{E}[X_\pi] = \Pr[X_\pi = 1] = 2^{-(n-1)}$$

Theorem (Szele 1943)

There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

Pick a random tournament T on n players $[n]$.

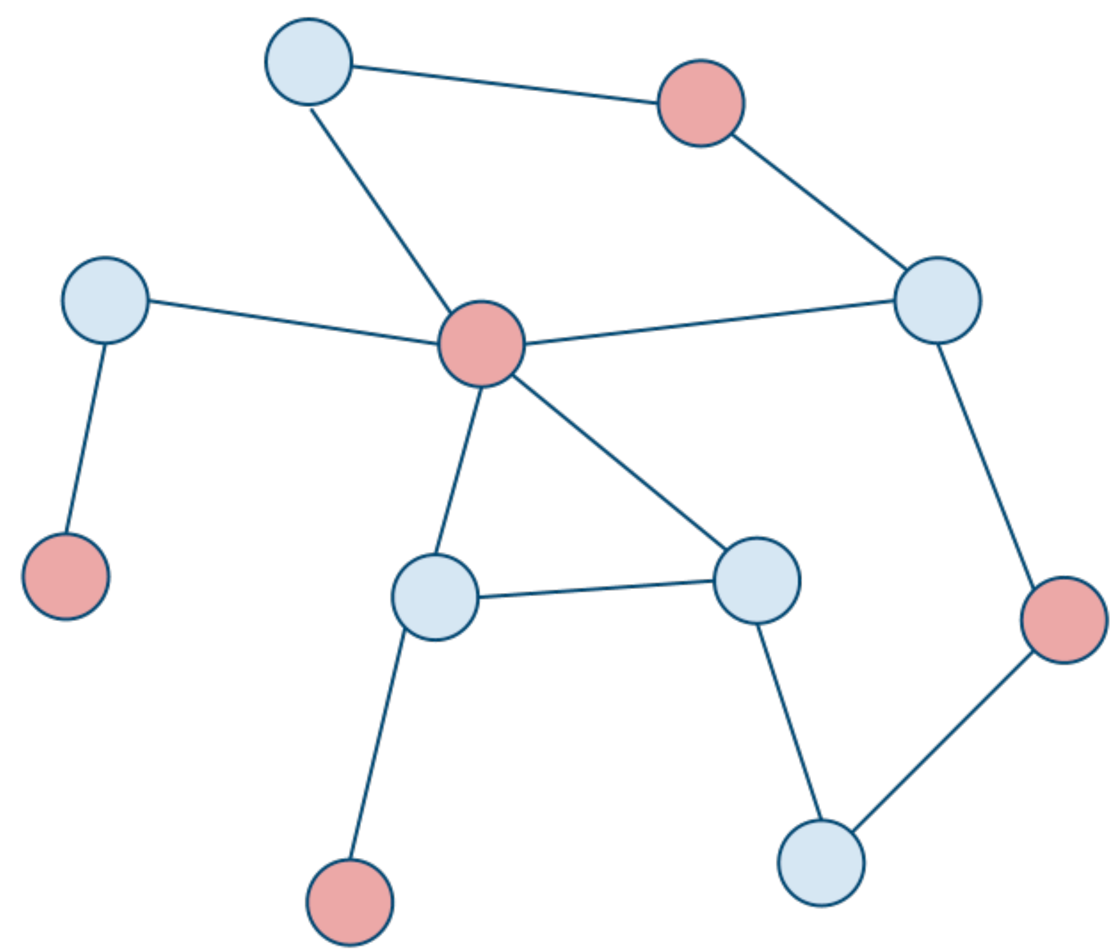
Hamiltonian paths: $X = \sum_{\pi} X_{\pi}$

$$E[X_\pi] = \Pr[X_\pi = 1] = 2^{-(n-1)}$$

$$E[X] = \sum_{\pi} E[X_\pi] = n!2^{-(n-1)}$$

Large Independent Set

- Graph G(V,E)
- independent set $S \subseteq V$
  - no adjacent vertices in S
- max independent set is NP-hard



Theorem

G has *n* vertices and *m* edges

$\Rightarrow$ $\exists$ an independent set *S* of size $\dfrac{n^2}{4m}$

~~uniformly sample *S* ?~~

A uniform S is very unlikely to be an independent set!

$G(V,E)$: n vertices, m edges

1. sample a random S : each vertex is chosen *independently* with probability p

2. modify S to S^* : independent set!

$\forall uv \in E$ if $u, v \in S$

delete one of u,v from S

Y: # of edges in S $\quad Y = \sum_{uv \in E} Y_{uv} \quad Y_{uv} = \begin{cases} 1 & u, v \in S \\ 0 & \text{o.w.} \end{cases}$

$$\mathbf{E}[|S^*|] \geq \mathbf{E}[|S|] - \mathbf{E}[Y]$$

$$\mathbf{E}[|S|] = np \qquad \mathbf{E}[Y] = \sum_{uv \in E} \mathbf{E}[Y_{uv}] = mp^2$$

G(*V*,*E*) : *n* vertices, *m* edges

1. sample a random *S* : each vertex is chosen *independently* with probability *p*

2. modify *S* to S^* : independent set!

$\forall uv \in E$ if $u, v \in S$

delete one of u,v from S

$$\mathbf{E}[|S^*|] \geq np - mp^2 = \frac{n^2}{4m}$$

when $p = \frac{n}{2m}$

G(*V*,*E*) : *n* vertices, *m* edges

average degree $d = \frac{2m}{n}$

random S^* : $\mathbf{E}[|S^*|] \geq \frac{n^2}{4m} = \frac{n}{2d}$

Theorem

G has *n* vertices and *m* edges

⇒ ∃ an independent set *S* of size $\frac{n^2}{4m}$

Markov's Inequality

Markov's Inequality:

For *nonnegative* X, for any $t > 0$,

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

Proof:

$$\text{Let } Y = \begin{cases} 1 & \text{if } X \geq t, \\ 0 & \text{otherwise.} \end{cases} \Rightarrow Y \leq \left\lfloor \frac{X}{t} \right\rfloor \leq \frac{X}{t},$$

$$\Pr[X \geq t] = \mathbf{E}[Y] \leq \mathbf{E}\left[\frac{X}{t}\right] = \frac{\mathbf{E}[X]}{t}.$$

QED

Markov's Inequality

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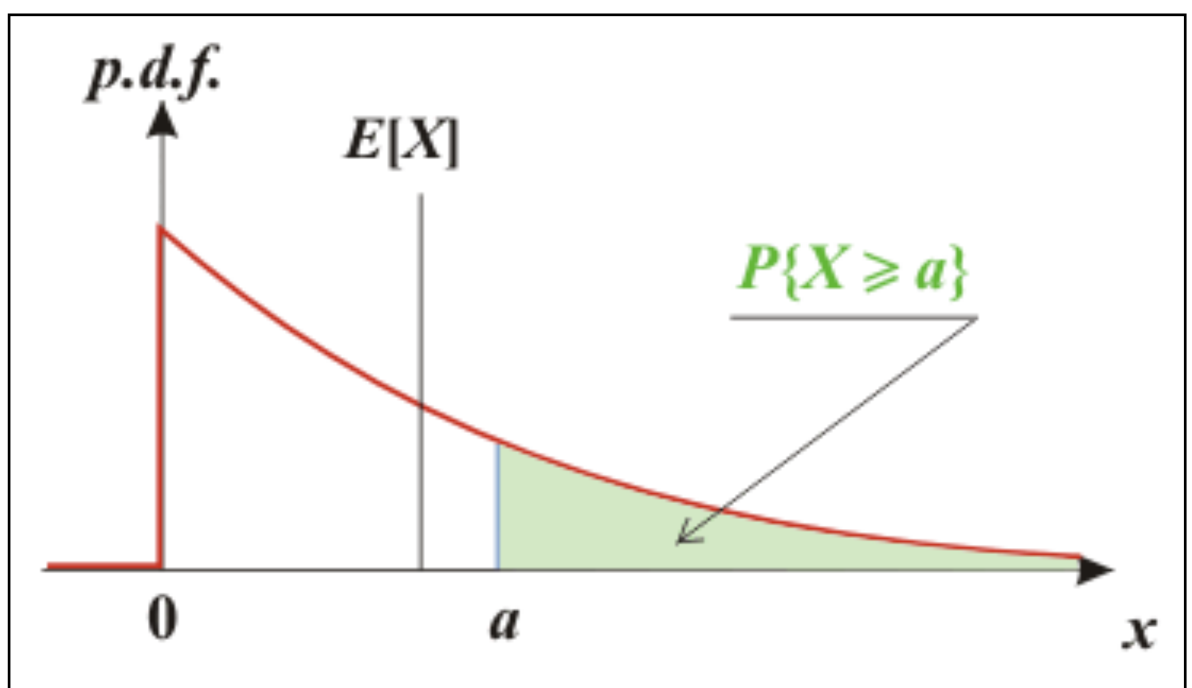
For *nonnegative* X, for any $t > 0$,

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

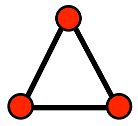


Graph $G(V,E)$

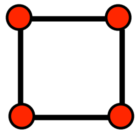
girth $g(G)$: length of the shortest cycle

chromatic number $\chi(G)$:

minimum number of color to properly color the vertices of G.



$g(G) = 3$ $\chi(G)=3$



$g(G) = 4$ $\chi(G)=2$

Intuition: Large cycles are easy to color!

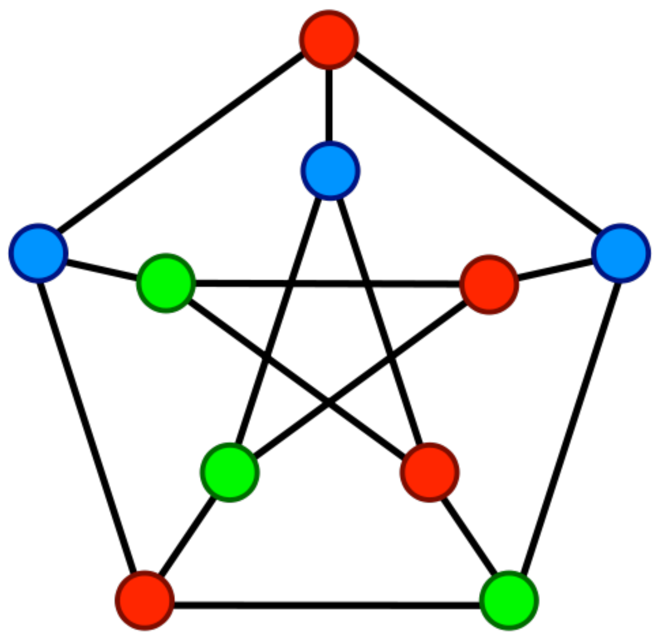
Theorem (Erdős 1959)

For all k, ℓ, there exists a finite graph G with $\chi(G) \geq k$ and $g(G) \geq \ell$.

coloring classes:

equivalence classes of vertices

"Independent sets!"



independence number $\alpha(G)$:

size of the largest independent set in G.

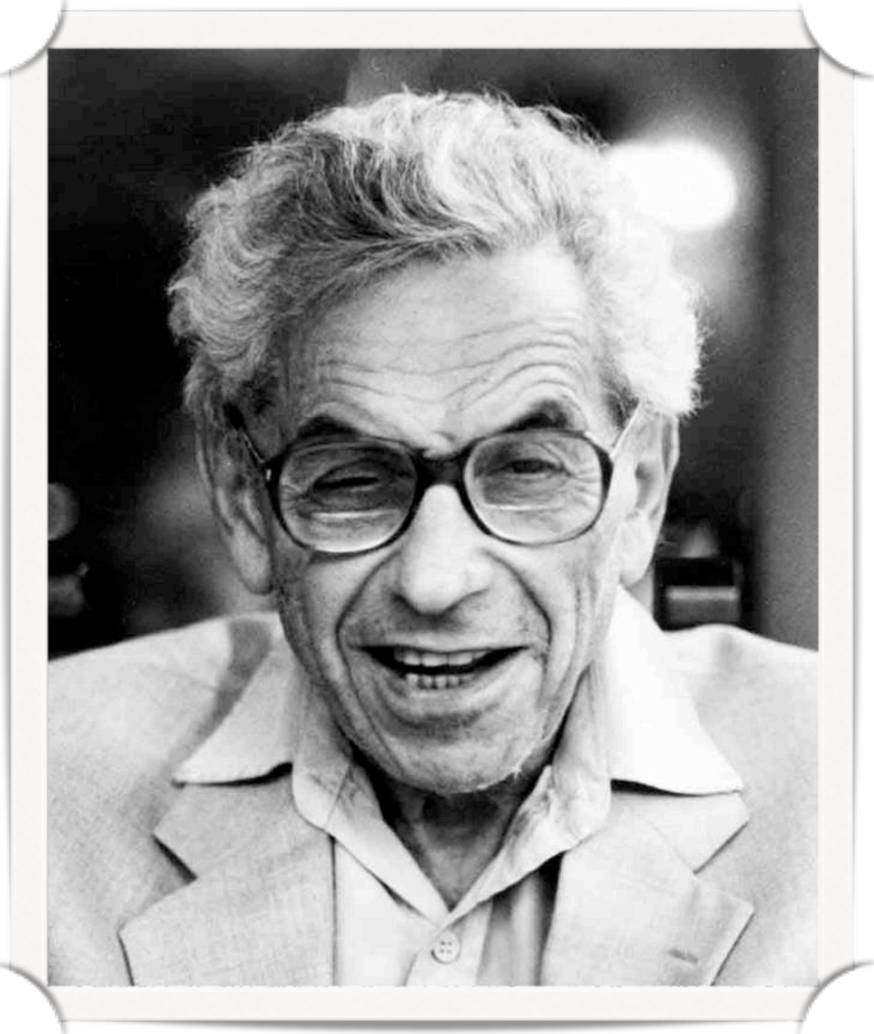
n vertices $\quad \chi(G) \geq \frac{n}{\alpha(G)} \geq k$, with $\alpha(G) \leq \frac{n}{k}$

For all k, ℓ, there exists a graph G on n vertices with $\alpha(G) \le \frac{n}{k}$ and $g(G) \ge \ell$.

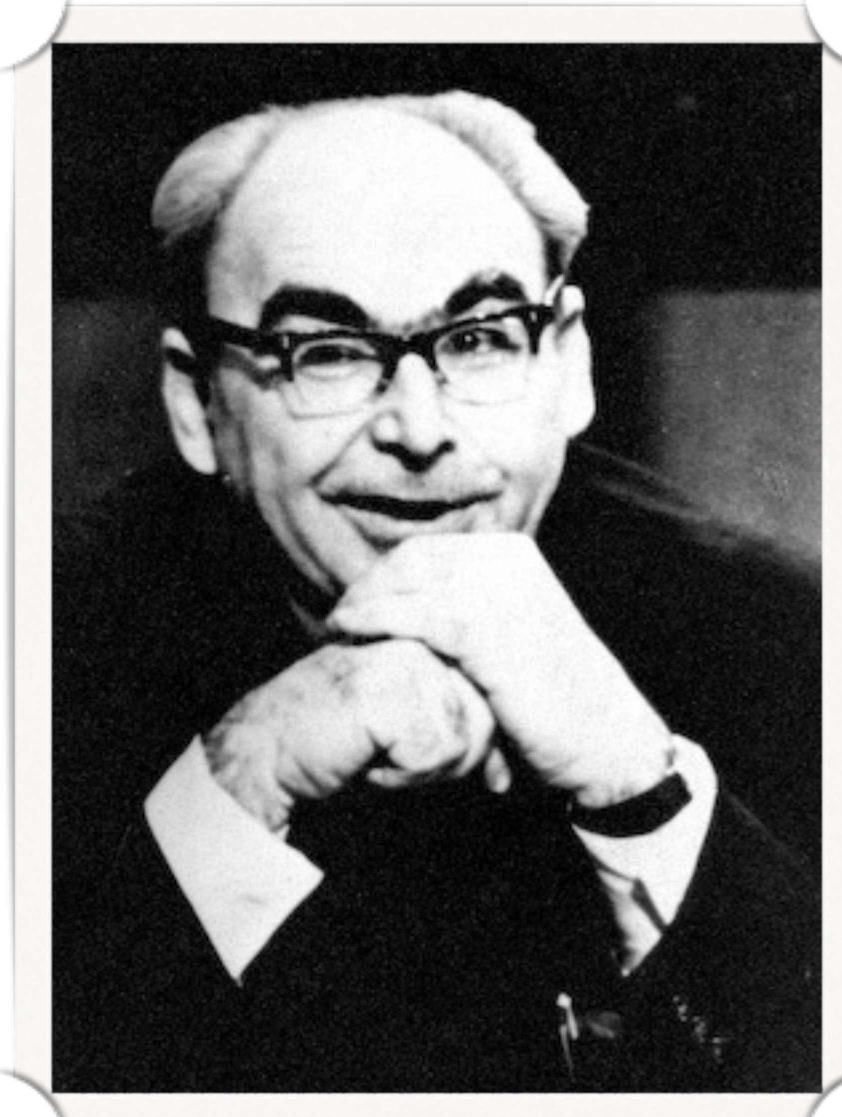
$|V| = n \qquad \forall \{u, v\} \in \binom{V}{2}$

independently $\Pr[\{u, v\} \in E] = p$

Random Graphs



Paul Erdős
(1913 - 1996)



Alfréd Rényi
(1921 - 1970)

Erdős-Rényi 1960 paper:

ON THE EVOLUTION OF RANDOM GRAPHS

by

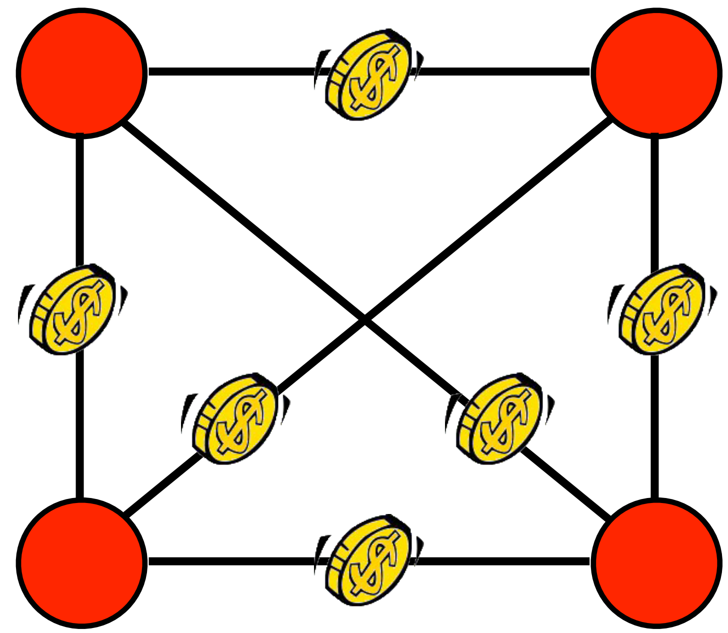
P. ERDÖS and A. RÉNYI

Institute of Mathematics
Hungarian Academy of Sciences, Hungary

1. Definition of a random graph

Let $E_{n,N}$ denote the set of all graphs having n given labelled vertices $V_1, V_2, \cdots, V_n$ and N edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing N out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1/\binom{\binom{n}{2}}{N}$. There is however an other slightly different point of view, which has some advantages. *We may consider the formation of a random graph as a stochastic process* defined as follows: At time $t=1$ we choose one out of the $\binom{n}{2}$ possible edges connecting the points $V_1, V_2, \cdots, V_n$,

$G(n, p)$



$|V| = n$ $\forall u, v \in V$

independently $\Pr[\,\{u, v\} \in E\,] = p$

uniform random graph: $G(n, \frac{1}{2})$

For all k, ℓ, there exists a graph G on n vertices with $\alpha(G) \le \frac{n}{k}$ and $g(G) \ge \ell$.

fix any large k, l exists n

$G \sim G(n,p)$

Plan:

union bound

$\Pr[\ \alpha(G) > n/k\ ] < 1/2$

$\Pr[\ g(G) < l\ ] < 1/2$

$\Rightarrow$

$\Pr[\ \alpha(G) > n/k \vee g(G) < l\ ] < 1$

$\Pr[\ \alpha(G) \le n/k \wedge g(G) \ge l\ ] > 0$

$G \sim G(n,p)$

$$\Pr[\alpha(G) \geq n/k] \quad \leq \Pr[\exists \text{ind. set of size } n/k]$$

$$\leq \Pr\left[\exists S \in \binom{[n]}{n/k} \forall \{u,v\} \in \binom{S}{2}, uv \notin G\right]$$

$$\leq \sum_{S \in \binom{[n]}{n/k}} \Pr\left[\forall \{u,v\} \in \binom{S}{2}, uv \notin G\right]$$ union bound

$$= \sum_{S \in \binom{[n]}{n/k}} \prod_{\{u,v\} \in \binom{S}{2}} \Pr[uv \notin G] \quad = \binom{n}{n/k} (1-p)^{\binom{n/k}{2}}$$

$$\leq n^{n/k} (1-p)^{\binom{n/k}{2}}$$

$G \sim G(n, p)$ $\Pr[\alpha(G) \geq n/k] \leq n^{n/k}(1-p)^{\binom{n/k}{2}}$

$\Pr[\, g(G) > l \,] < ?$

for each *i*-cycle $\sigma : u_1 \to u_2 \to \ldots \to u_i \to u_1$

$\Pr[\sigma \text{ is a cycle in } G] = p^i$

$$X_\sigma = \begin{cases} 1 & \sigma \text{ is a cycle in } G \\ 0 & \text{otherwise} \end{cases}$$

\# of length$\leq l$ cycles in G $X = \sum_{i=3}^{\ell} \sum_{\sigma : |\sigma| = i} X_\sigma$

$$\mathbb{E}[X] = \sum_{i=3}^{\ell} \sum_{\sigma : |\sigma| = i} \mathbb{E}[X_\sigma] = \sum_{i=3}^{\ell} \sum_{\sigma : |\sigma| = i} p^i$$

$$= \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i \leq \sum_{i=3}^{\ell} \frac{n^i}{2i} p^i$$

$G \sim G(n,p)$ $\quad k = \frac{np}{3\ln n}$ $\quad n/k = \frac{3\ln n}{p}$

$$\begin{aligned}\Pr[\alpha(G) \geq n/k] &\leq n^{n/k}(1-p)^{\binom{n/k}{2}} \\ &\leq n^{n/k}\mathrm{e}^{-p\binom{n/k}{2}} \\ &= \left(n\mathrm{e}^{-p(n/k-1)/2}\right)^{n/k} = o(1)\end{aligned}$$

X : # of length$\leq l$ cycles in G

$$\mathbb{E}[X] \leq \sum_{i=3}^{\ell} \frac{n^i}{2i} p^i = \sum_{i=3}^{\ell} \frac{n^{\theta i}}{2i} = o(n)$$

$$p = n^{\theta-1} \qquad \theta < \frac{1}{2\ell}$$

$$\Pr[X \geq \frac{n}{2}] \leq \frac{2\mathbb{E}[X]}{n} = o(1)$$

Markov

$G \sim G(n,p)$

$$p = n^{\theta - 1} \qquad \theta < \frac{1}{2\ell} \qquad k = \frac{np}{3\ln n} = \frac{n^{1/2\ell}}{3\ln n}$$

$$\Pr[\alpha(G) \geq n/k] = o(1)$$

X : # of length$\leq l$ cycles in G

$$\Pr[X \geq \frac{n}{2}] = o(1)$$

$\exists$ G: $\alpha(G) < n/k$

# of length$\leq l$ cycles in $G < n/2$

delete 1 vertex per each length$\leq l$ cycle in G G'

$g(G') > l$ $\qquad \alpha(G') \leq \alpha(G) < n/k$

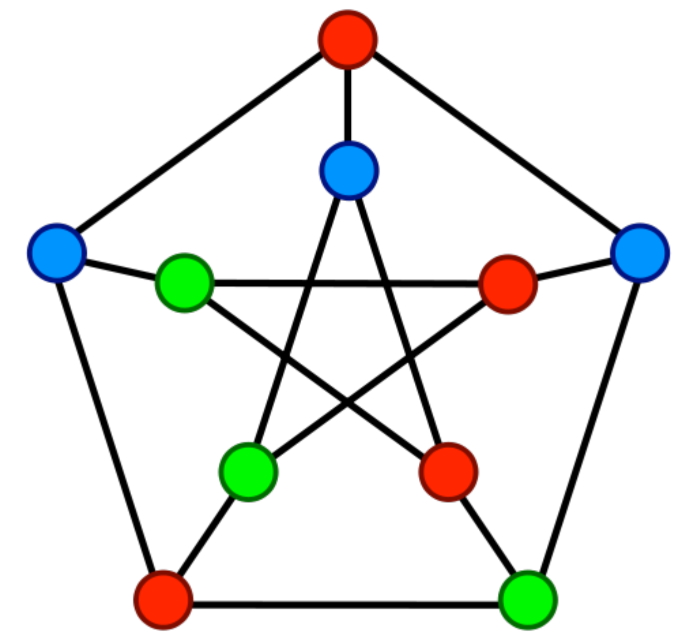
Theorem (Erdős 1959)

For all k, ℓ, there exists a finite graph G with $\chi(G) \geq k$ and $g(G) \geq \ell$.

coloring classes:

equivalence classes of vertices

"Independent sets!"



independence number $\alpha(G)$:

size of the largest independent set in G.

n vertices $\qquad \chi(G) \geq \dfrac{n}{\alpha(G)} \geq k \qquad \alpha(G) \leq \dfrac{n}{k}$

Ramsey Number

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- For any two coloring of K_6, there is a *monochromatic* K_3.

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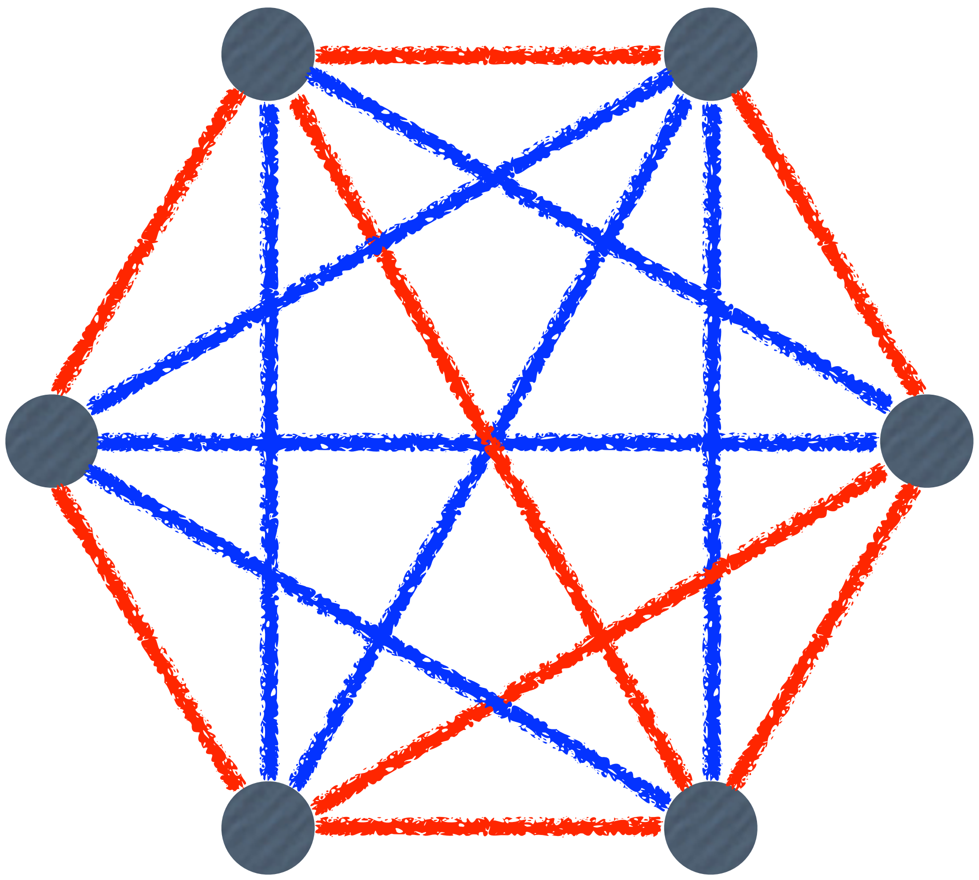


Ramsey number: $R(k,k)$

$R(k,k) > ?$

"∃ a 2-coloring of K_n, no monochromatic K_k."

The Probabilistic Method:

a random 2-coloring of K_n

$\forall S \in \binom{[n]}{k}$

event A_S: S is a monochromatic K_k

To prove:

$$\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$$

Dependency!

Lovász Sieve

- Bad events: $A_1, A_2, \ldots, A_n$
- None of the bad events occurs:

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right]$$

- The probabilistic method: being good is possible

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

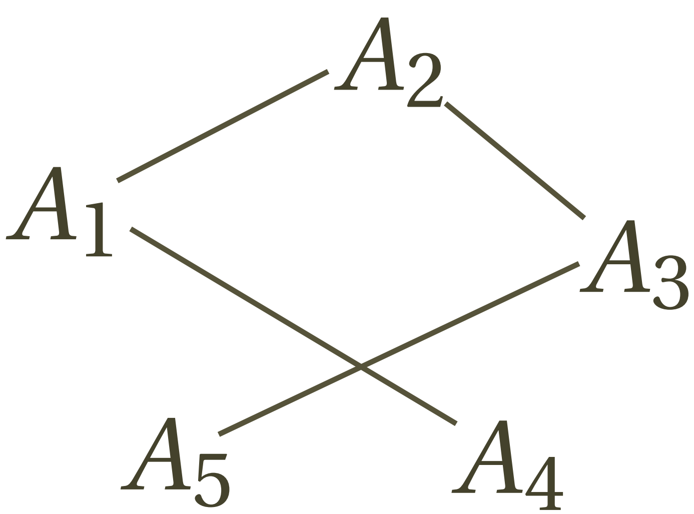
events: $A_1, A_2, \ldots, A_n$

dependency graph: $D(V,E)$

$$V = \{1, 2, \ldots, n\}$$

$ij \in E \iff A_i$ and A_j are dependent

d : max degree of dependency graph



$A_1(X_1, X_4)$ $\quad A_4(X_4)$

$A_2(X_1, X_2)$ $\quad A_5(X_3)$

$A_3(X_2, X_3)$

$X_1, \ldots, X_4$ mutually independent

events: $A_1, A_2, \ldots, A_n$

d : max degree of dependency graph

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \leq p$
- $ep(d+1) \leq 1$

$\Rightarrow$ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \leq x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow$ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \geq \prod_{i=1}^{n} (1 - x_i)$

$$R(k,k) \geq n$$

"$\exists$ a 2-coloring of K_n, no monochromatic K_k."

a random 2-coloring of K_n :

$\forall \{u,v\} \in K_n$, *uniformly* and *independently* $\begin{cases} uv \\ uv \end{cases}$

$\forall S \in \binom{[n]}{k}$ event A_S : S is a monochromatic K_k

$$\Pr[A_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent $\Leftrightarrow$ $|S \cap T| \geq 2$

max degree of dependency graph $d \leq \binom{k}{2}\binom{n}{k-2}$

To prove: $\Pr\left[\bigwedge_{S \in \binom{[n]}{k}} \overline{A_S}\right] > 0$

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \le p$
- $ep(d+1) \le 1$

⇨ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

$$\Pr[A_S] = 2^{1-\binom{k}{2}}$$

$$d \le \binom{k}{2}\binom{n}{k-2}$$

⇨ for some $n = ck2^{k/2}$ with constant c

$$e2^{1-\binom{k}{2}}(d+1) \le 1$$

To prove: $\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$

$$R(k,k) \ge n = \Omega(k2^{k/2})$$

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n} \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right] = \prod_{i=1}^{n} \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right]\right)$$

Lemma For any $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$,

$$\Pr\left[\bigwedge_{i=1}^{n} \mathcal{E}_i\right] = \prod_{k=1}^{n} \Pr\left[\mathcal{E}_k \mid \bigwedge_{i<k} \mathcal{E}_i\right].$$

proof:

$$\Pr\left[\mathcal{E}_n \mid \bigwedge_{i=1}^{n-1} \mathcal{E}_i\right] = \frac{\Pr\left[\bigwedge_{i=1}^{n} \mathcal{E}_i\right]}{\Pr\left[\bigwedge_{i=1}^{n-1} \mathcal{E}_i\right]}$$

recursion!

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow$

$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

I.H.

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1} \text{ for any } \{i_1, \ldots, i_m\}$$

induction on m:

$m=1$, trivial

events: $A_1, A_2, \ldots, A_n$

$$\exists x_1, \ldots, x_n \in [0,1)$$
$$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$$

I.H. $\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1}$ for any $\{i_1, \ldots, i_m\}$

suppose i_1 adjacent to $i_2, \ldots, i_k$

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] = \frac{\Pr\left[A_{i_1} \overline{A_{i_2}} \cdots \overline{A_{i_k}} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right]}{\Pr\left[\overline{A_{i_2}} \cdots \overline{A_{i_k}} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right]}$$

$$\le \Pr\left[A_{i_1} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right] = \Pr\left[A_{i_1}\right] \le x_{i_1} \prod_{j=2}^{k} (1 - x_{i_j})$$

$$= \prod_{j=2}^{k} \Pr\left[\overline{A_{i_j}} \mid \overline{A_{i_{j+1}}} \cdots \overline{A_{i_m}}\right] = \prod_{j=2}^{k} \left(1 - \Pr\left[A_{i_j} \mid \overline{A_{i_{j+1}}} \cdots \overline{A_{i_m}}\right]\right)$$

I.H. $\ge \prod_{j=2}^{k} (1 - x_{i_j})$

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1} \quad \text{for any} \quad \{i_1, \ldots, i_m\}$$

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n} \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right] = \prod_{i=1}^{n} \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right]\right)$$

$$\ge \prod_{i=1}^{n} (1 - x_i) \quad > 0$$

events: $A_1, A_2, \ldots, A_n$

d : max degree of dependency graph

Lovász Local Lemma

- $\forall i, \ \Pr[A_i] \leq p$
- $ep(d+1) \leq 1$

$$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \leq x_i \prod_{j \sim i} (1 - x_j)$

$$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \geq \prod_{i=1}^{n} (1 - x_i)$$

Constraint Satisfaction Problem

- variables: $x_1, x_2, ..., x_n \in D$ (domain)
- constraints: $C_1, C_2, ..., C_m$
  - where $C_i(x_{i_1}, x_{i_2}, \ldots) \in \{\text{true}, \text{false}\}$
- CSP solution: an assignment of variables satisfying *all* constraints
- examples: SAT, graph colorability, ...
- *existence*: When does a solution exist?
- *search*: How to find a solution?