Combinatorics

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Course Info

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Combinatorics

combinatorial ≈ discrete, finite

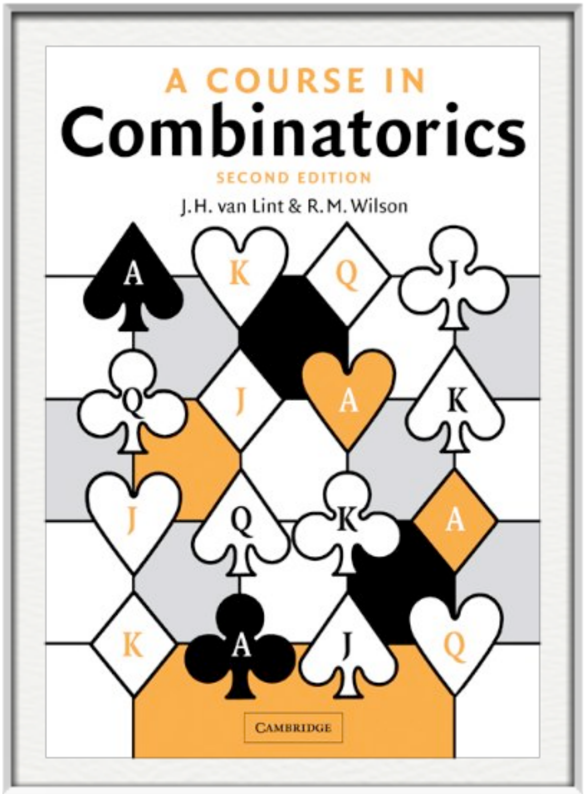
solution: combinatorial object

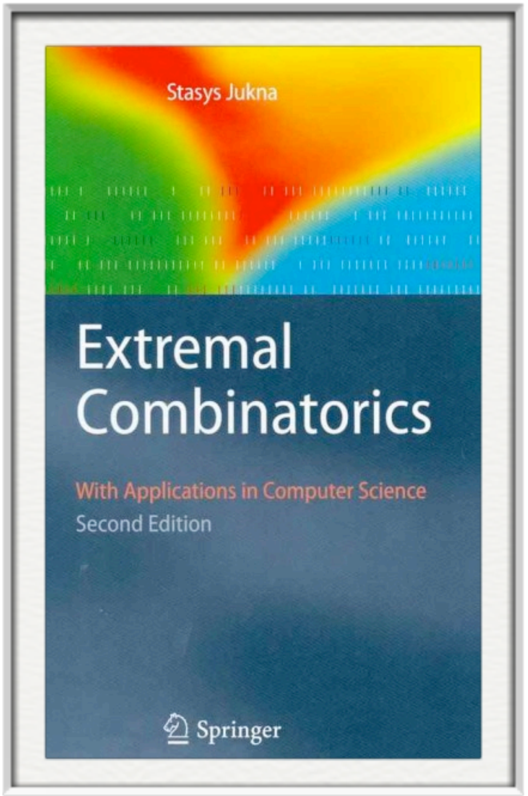
constraint: combinatorial structure

- Enumeration (counting): How many solutions satisfying the constraints?
- Existence: Does there exist a solution?
  - Extremal: How large/small a solution can be to preserve/avoid certain structure?
  - Ramsey: When a solution is sufficiently large, some structure must emerge.
- Optimization: Find the optimal solution.
- Construction (design): Construct a solution.

Textbook

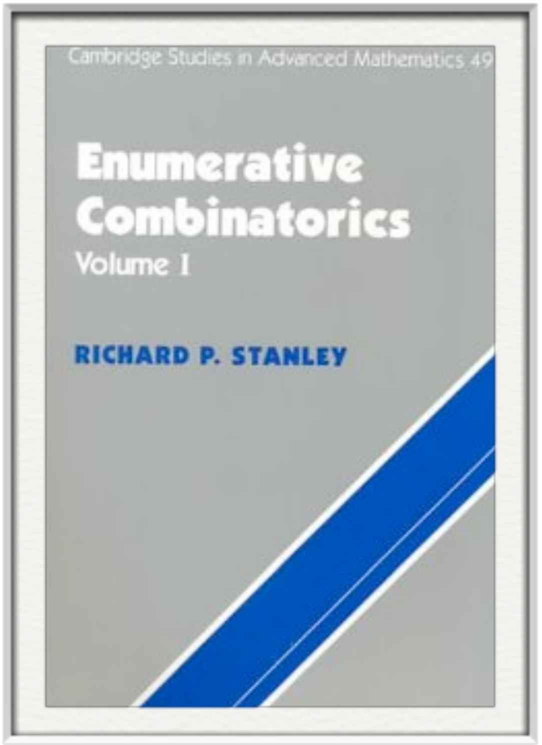
van Lint and Wilson,
A course in Combinatorics, 2nd Edition.



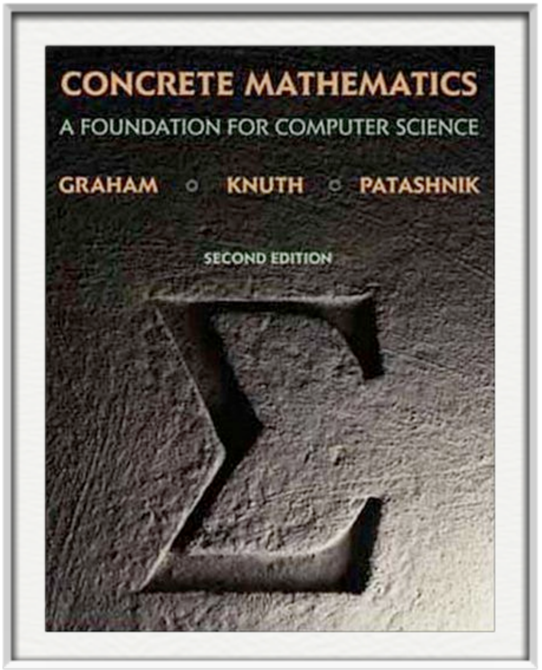


Jukna,
Extremal Combinatorics: with applications in computer science, 2nd Edition.

Reference Books

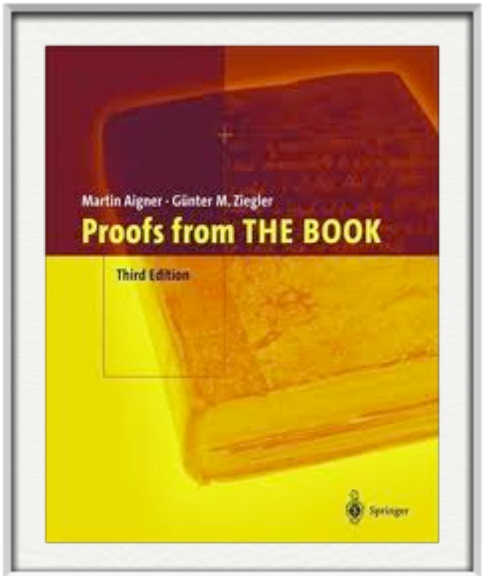


Stanley,
Enumerative Combinatorics,
Volume 1

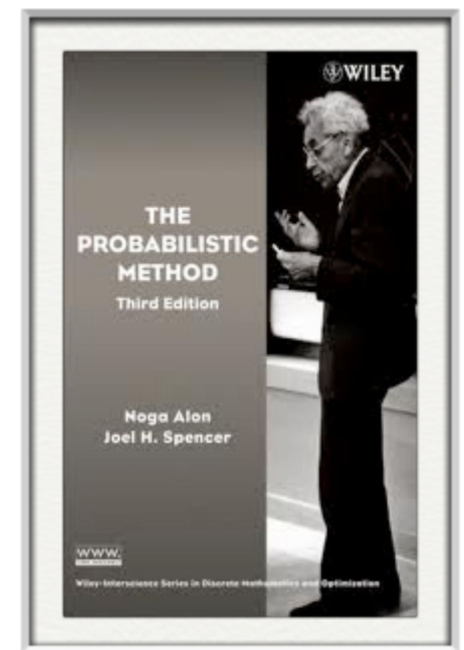


Graham, Knuth, and Patashnik,
Concrete Mathematics: A Foundation for Computer Science

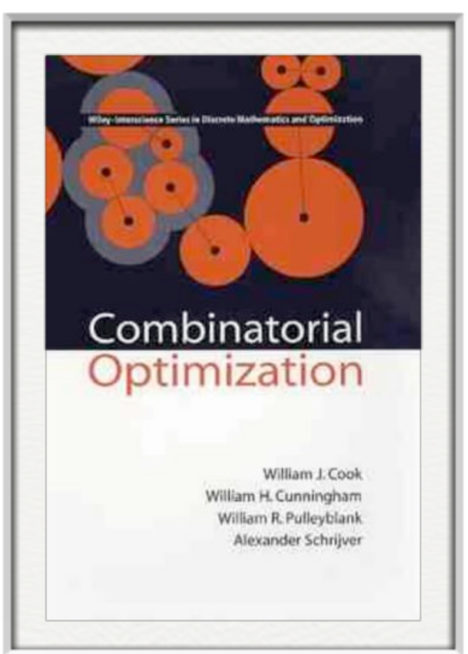
Reference Books



Aigner and Ziegler.
Proofs from THE BOOK.



Alon and Spencer.
The Probabilistic Method.



Cook, Cunningham, Pulleyblank, and Schrijver.
Combinatorial Optimization.

Enumeration

(counting)

How many ways are there:

- to rank *n* people?
- to assign *m* zodiac signs to *n* people?
- to choose *m* people out of *n* people?
- to partition *n* people into *m* groups?
- to distribute *m* yuan to *n* people?
- to partition *m* yuan to *n* parts?
- ... ...

The Twelvefold Way

Gian-Carlo Rota
(1932-1999)

The twelvefold way

$$f : N \to M \qquad |N| = n,\ |M| = m$$

elements of N	elements of M	any f	1-1	on-to
distinct	*distinct*			
identical	*distinct*			
distinct	*identical*			
identical	*identical*			

Knuth's version (in *TAOCP* vol.4A)

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n		
n identical balls, m distinct bins			
n distinct balls, m identical bins			
n identical balls, m identical bins			

Tuples

$$\{1, 2, \ldots, m\}$$

$[m] =$ ~~$\{0, 1, \ldots, m-1\}$~~

$$[m]^n = \underbrace{[m] \times \cdots \times [m]}_{n}$$

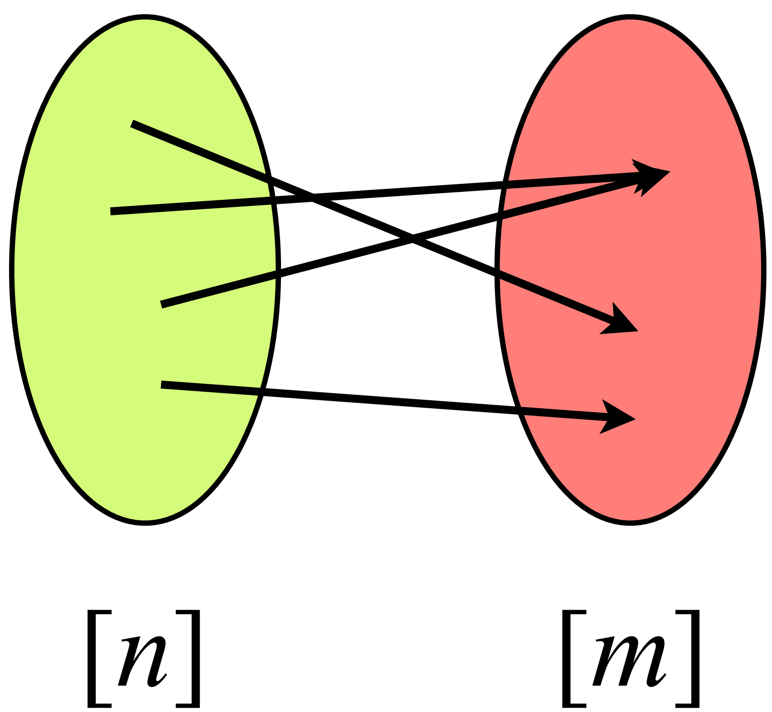
$$|[m]^n| = m^n$$

Product rule:

finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Functions



count the # of functions

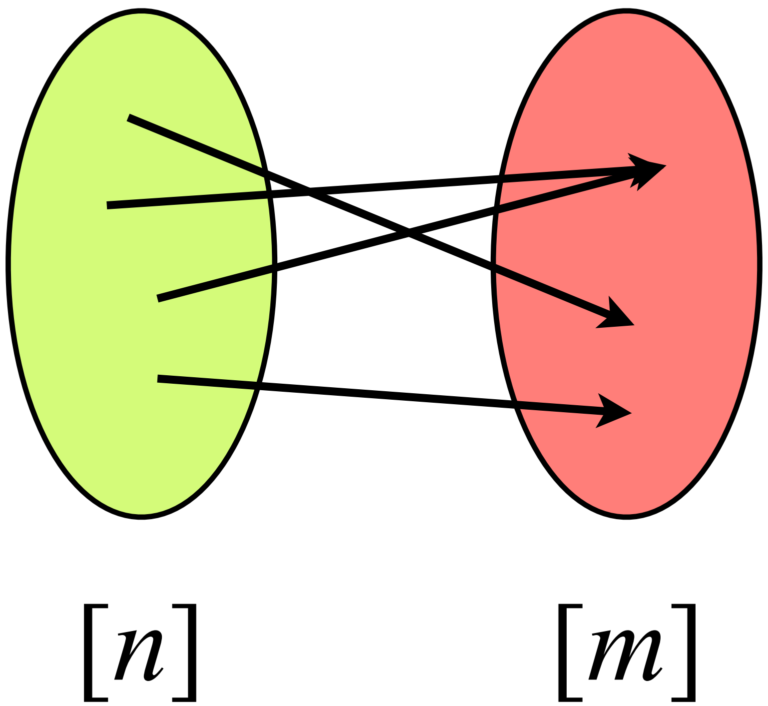
$$f : [n] \to [m]$$

$$(f(1), f(2), \ldots, f(n)) \in [m]^n$$

one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

Functions



count the # of functions

$$f : [n] \to [m]$$

one-one correspondence

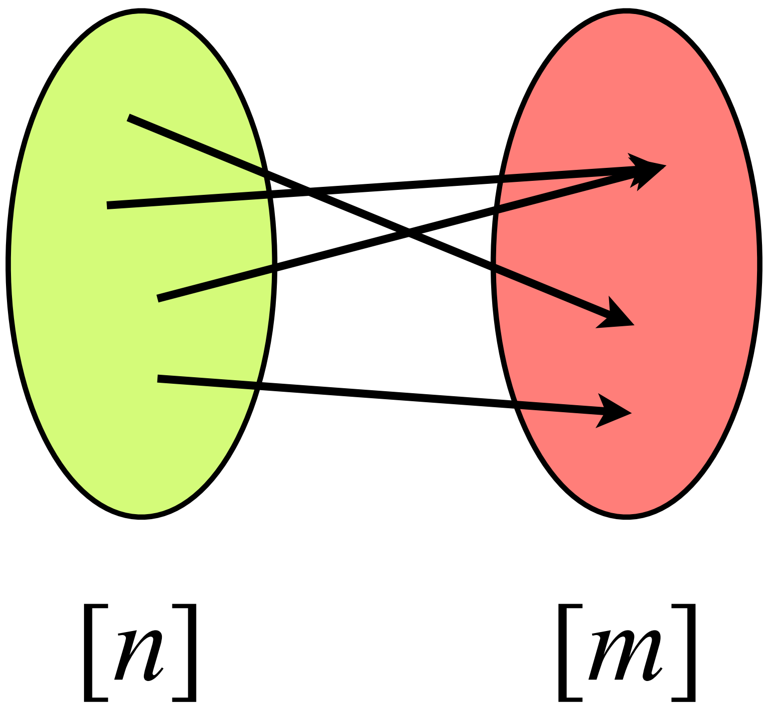
$$[n] \to [m] \Leftrightarrow [m]^n$$

Bijection rule:

finite sets S and T

$$\exists \phi : S \xrightarrow[\text{on}-\text{to}]{1-1} T \implies |S| = |T|$$

Functions



count the # of functions

$$f : [n] \to [m]$$

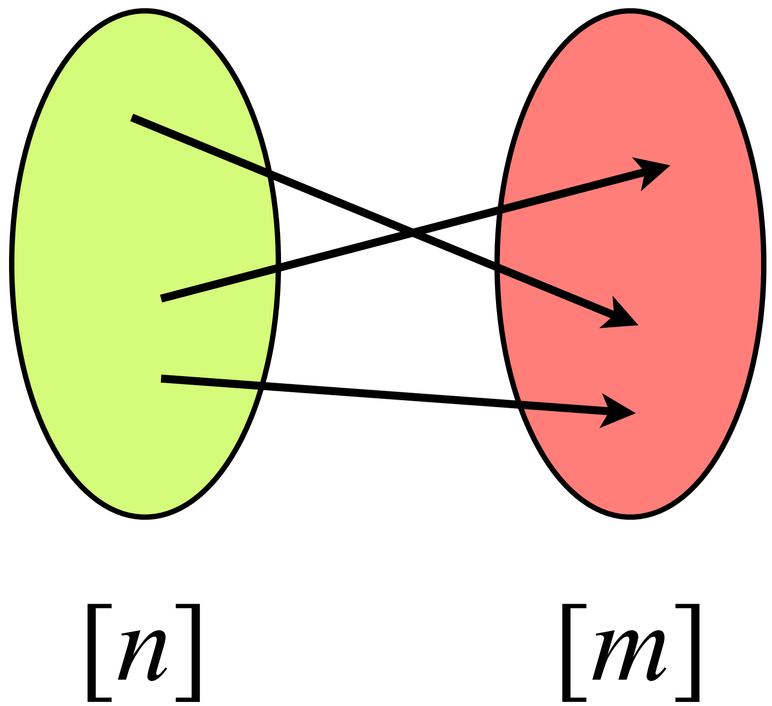
one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

$$|[n] \to [m]| = |[m]^n| = m^n$$

"Combinatorial proof."

Injections



count the # of 1-1 functions

$$f : [n] \xrightarrow{1\text{-}1} [m]$$

one-to-one correspondence

$$\pi = (f(1), f(2), \ldots, f(n))$$

n-permutation: $\pi \in [m]^n$ of distinct elements

$$(m)_n = m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

"*m* lower factorial *n*"

Subsets

subsets of { 1, 2, 3 }:

$\varnothing$,
{1}, {2}, {3},
{1, 2}, {1, 3}, {2, 3},
{1, 2, 3}

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Combinatorial proof:

A subset $S \subseteq [n]$ corresponds to a string of n bit, where bit i indicates whether $i \in S$.

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| = |\{0,1\}^n| = 2^n$$

Combinatorial proof:

$$S \subseteq [n] \Longleftrightarrow \chi_S \in \{0,1\}^n \quad \chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

one-to-one correspondence

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

A not-so-combinatorial proof:

Let $f(n) = \left|2^{[n]}\right|$

$$f(n) = 2f(n-1)$$

$$f(n) = \left|2^{[n]}\right|$$

$$\boxed{f(n) = 2f(n-1)}$$

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \cup \{S \subseteq [n] \mid n \in S\}$$

$$\left|2^{[n]}\right| = \left|2^{[n-1]}\right| + \left|2^{[n-1]}\right| = 2f(n-1)$$

Sum rule:

finite **disjoint** sets S and T

$$|S \cup T| = |S| + |T|$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| = 2^n$$

Let $f(n) = \left|2^{[n]}\right|$

$$\boxed{f(n) = 2f(n-1)}$$

$$f(0) = |2^{\emptyset}| = 1$$

Three rules

Sum rule:

finite **disjoint** sets S and T

$$|S \cup T| = |S| + |T|$$

Product rule:

finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Bijection rule:

finite sets S and T

$$\exists \phi : S \xrightarrow[\text{on}-\text{to}]{1-1} T \implies |S| = |T|$$

Subsets of fixed size

2-subsets of { 1, 2, 3 }: {1, 2}, {1, 3}, {2, 3}

k-uniform $$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$$

$$\binom{n}{k} = \left|\binom{[n]}{k}\right|$$

"*n* choose *k*"

Subsets of fixed size

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}$$

# of ordered k-subsets: $n(n-1)\cdots(n-k+1)$

# of permutations of a *k*-set: $k(k-1)\cdots 1$

Binomial coefficients

Binomial coefficient: $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1. $\binom{n}{k} = \binom{n}{n-k}$

2. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

choose a *k*-subset ⇔

choose its compliment

0-subsets + 1-subsets + ...
+ n-subsets = all subsets

Binomial theorem

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof:

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n}$$

$$\#\text{ of } x^k\text{:}\quad \text{choose } k \text{ factors out of } n$$

Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Let $x = 1$.

$$S = \{x_1, x_2, \ldots, x_n\}$$

$$\begin{aligned} & \# \text{ of subsets of } S \text{ of odd sizes} \\ = \; & \# \text{ of subsets of } S \text{ of even sizes} \end{aligned}$$

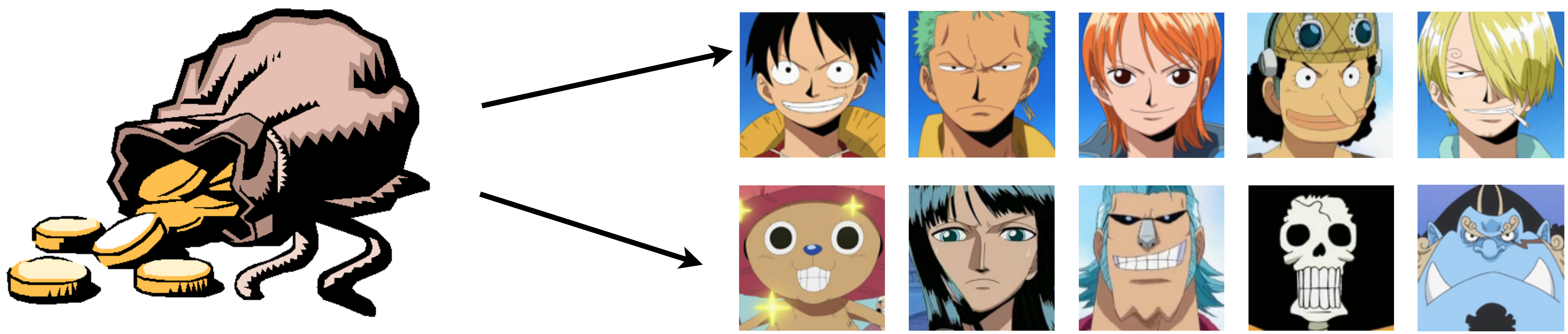
Let $x = -1$.

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins		$\binom{m}{n}$	
n distinct balls, m identical bins			
n identical balls, m identical bins			

Compositions of an integer



n beli

k pirates

How many ways to assign *n* beli to *k* pirates?

How many ways to assign *n* beli to *k* pirates,
so that each pirate receives at least 1 beli?

Compositions of an integer

$n \in \mathbb{Z}^+$

k-composition of *n*:

an ordered sum of *k* positive integers

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

Compositions of an integer

$n \in \mathbb{Z}^+$

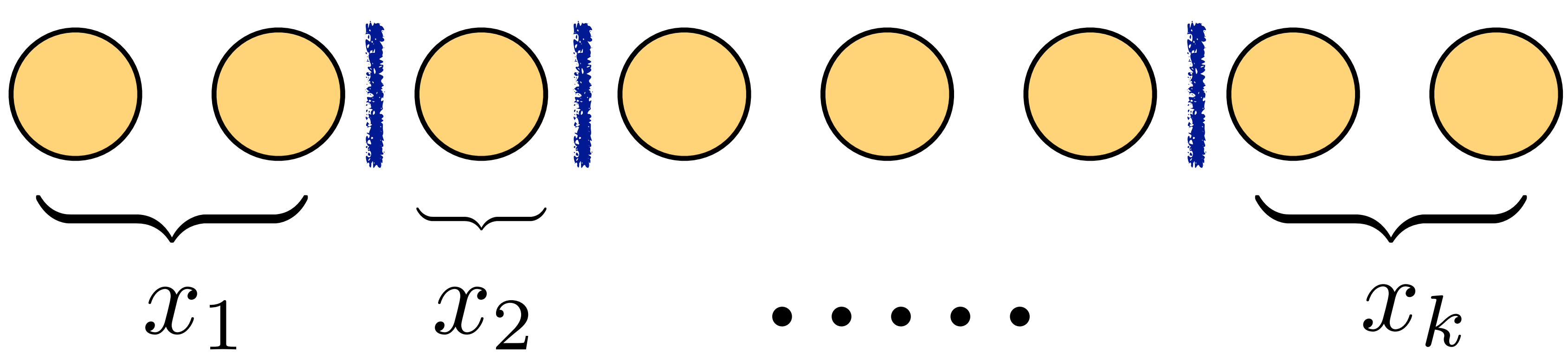
k-composition of *n*:

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

of *k*-compositions of *n*? $\binom{n-1}{k-1}$

n identical balls



Compositions of an integer

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

\# of *k*-compositions of *n*? $\binom{n-1}{k-1}$

$$\phi((x_1, x_2, \ldots, x_k)) = \{x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3,\ \ldots,\ x_1 + x_2 + \cdots + x_{k-1}\}$$

ϕ is a 1-1 correspondence between $\{k\text{-compositions of } n\}$ and $\binom{\{1,2,\ldots,n-1\}}{k-1}$

Compositions of an integer

weak *k*-composition of *n*:

an ordered sum of *k* nonnegative integers

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

Compositions of an integer

weak *k*-composition of *n*:

a *k*-tuple $(x_1, x_2, \cdots, x_k)$

$$x_1 + x_2 + \cdots + x_k = n \text{ and } x_i \in \mathbb{N}$$

\# of weak *k*-compositions of *n*? $\binom{n+k-1}{k-1}$

$$(x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k$$

a *k*-composition of *n*+*k*

1-1 correspondence

Multisets

k-subset of *S* "*k*-combination of *S* without repetition"

3-combinations of { 1, 2, 3, 4 }

without repetition:

{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

with repetition:

{1,1,1}, {1,1,2}, {1,1,3}, {1,1,4}, {1,2,2}, {1,3,3}, {1,4,4}, {2,2,2}, {2,2,3}, {2,2,4}, {2,3,3}, {2,4,4}, {3,3,3}, {3,3,4}, {3,4,4}, {4,4,4}

Multisets

multiset *M* on set *S*:

$$m : S \to \mathbb{N}$$

multiplicity of $x \in S$

$m(x)$: # of repetitions of x in M

cardinality $|M| = \sum_{x \in S} m(x)$

"*k*-combination of *S* with repetition" $\Longleftrightarrow$ *k*-multiset on *S*

$\left(\!\binom{n}{k}\!\right)$: # of *k*-multisets on an *n*-set

Multisets

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

$$k\text{-multiset on } S = \{x_1, x_2, \ldots, x_n\}$$

$$m(x_1) + m(x_2) + \cdots + m(x_n) = k$$

$$m(x_i) \geq 0$$

a weak *n*-composition of *k*

Multinomial coefficients

permutations of a multiset
of size *n* with multiplicities $m_1, m_2 \ldots, m_k$

\# of reordering of "*multinomial*"

permutations of *{a, i,i, l,l, m,m, n, o, t, u}*

assign *n* distinct balls to *k* distinct bins
with the *i-th* bin receiving m_i balls

multinomial coefficient $\binom{n}{m_1, \ldots, m_k}$

$$m_1 + m_2 + \cdots + m_k = n$$

Multinomial coefficients

permutations of a multiset
of size *n* with multiplicities $m_1, m_2 \ldots, m_k$

$$=$$

assign *n* distinct balls to *k* distinct bins
with the *i-th* bin receiving m_i balls

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1! m_2! \cdots m_k!}$$

$$\binom{n}{m, n-m} = \binom{n}{m}$$

Multinomial theorem

Multinomial Theorem

$$(x_1 + x_2 + \cdots + x_k)^n$$

$$= \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

Proof: $(x_1 + x_2 + \cdots + x_k)^n$

$$= \underbrace{(x_1 + x_2 + \cdots + x_k) \cdots\cdots (x_1 + x_2 + \cdots + x_k)}_{n}$$

$\#$ of $x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$:

assign n factors to k groups of sizes $m_1, m_2, \ldots, m_k$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins			
n identical balls, m identical bins			

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins			
n identical balls, m identical bins			

Partitions of a set

n pirates   *k* boats

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

$$A_i \neq \emptyset$$

$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

Partitions of a set

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

$$A_i \neq \emptyset$$

$$A_i \cap A_j = \emptyset$$

$$A_1 \cup A_2 \cup \cdots \cup A_k = S$$

$\left\{ {n \atop k} \right\}$ # of *k*-partitions of an *n*-set

"Stirling number of the second kind"

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}$$ # of partitions of an *n*-set

"Bell number"

Stirling number of the 2nd kind

$\left\{ n \atop k \right\}$ # of *k*-partitions of an *n*-set

$$\left\{ n \atop k \right\} = k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$$

Case.1 {*n*} is not a partition block

n is in one of the k blocks in a k-partition of $[n\text{-}1]$

Case.2 {*n*} is a partition block

the remaining k-1 blocks forms a $(k\text{-}1)$-partition of $[n\text{-}1]$

The twelvefold way

$f : N \to M$ *n* balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins			

Surjections

$$f : [n] \xrightarrow{\text{on-to}} [m]$$

$$\forall i \in [m] \quad f^{-1}(i) \neq \emptyset$$

$$(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m))$$

ordered m-partition of $[n]$

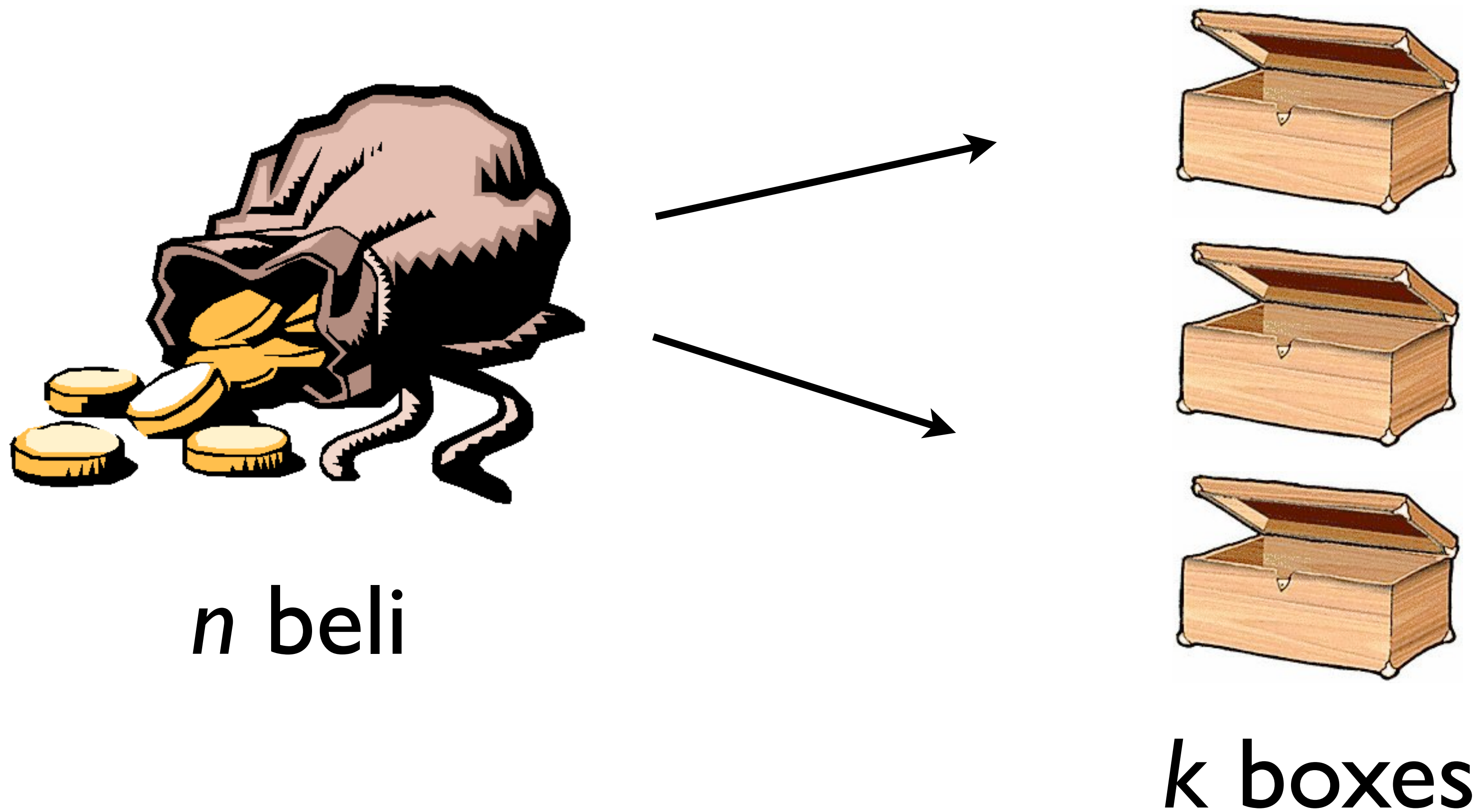
$$m! \left\{ {n \atop m} \right\}$$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	$m! \left\{ {n \atop m} \right\}$
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins			

Partitions of a number



a partition of *n* into *k* parts:

an unordered sum of *k* positive integers

Partitions of a number

a partition of *n* into *k* parts: "positive"

n=7 "unordered"

{7}

{1,6}, {2,5}, {3,4}

{1,1,5}, {1,2,4}, {1,3,3}, {2,2,3}

{1,1,1,4}, {1,1,2,3}, {1,2,2,2}

{1,1,1,1,3}, {1,1,1,2,2}

{1,1,1,1,1,2}

{1,1,1,1,1,1,1}

$p_k(n)$ # of partitions of *n* into *k* parts

$p_k(n)$ # of partitions of *n* into *k* parts

integral solutions to

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$p_k(n) = ?$$

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$\boxed{p_k(n) = p_{k-1}(n-1) + p_k(n-k)}$$

Case.1 $x_k = 1$

$(x_1, \ldots, x_{k-1})$ is a $(k-1)$-partition of $n-1$

Case.2 $x_k > 1$

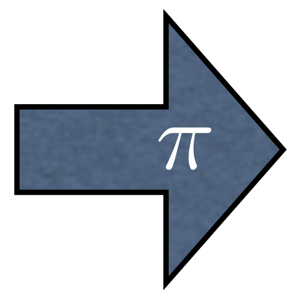
$(x_1 - 1, \ldots, x_k - 1)$ is a k-partition of $n-k$

partition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$

composition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_i \geq 1 \end{cases}$

partition

$\{x_1, \cdots, x_k\}$



permutation

composition

$(x_1, \cdots, x_k)$

“on-to”

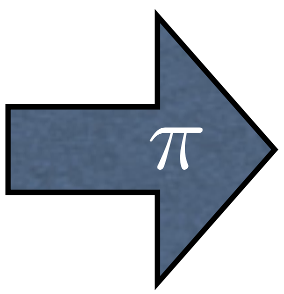
$$k!p_k(n) \geq \binom{n-1}{k-1}$$

partition $\{x_1, \cdots, x_k\}$ $\qquad$ $y_i = x_i + k - i$

$$x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1$$

$+k-1 \quad +k-2 \qquad\qquad +2 \qquad +1$

$$y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1$$



permutation

composition of $n + \frac{k(k-1)}{2}$

$$(y_1, y_2, \ldots, y_k)$$

"1-1"

$$k!p_k(n) \leq \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$$

$$\frac{\binom{n-1}{k-1}}{k!} \leq p_k(n) \leq \frac{\binom{n+\frac{k(k-1)}{2}-1}{k-1}}{k!}$$

If *k* is fixed,

$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!} \qquad \text{as} \quad n \to \infty$$

G. H. Hardy
(1877-1947)

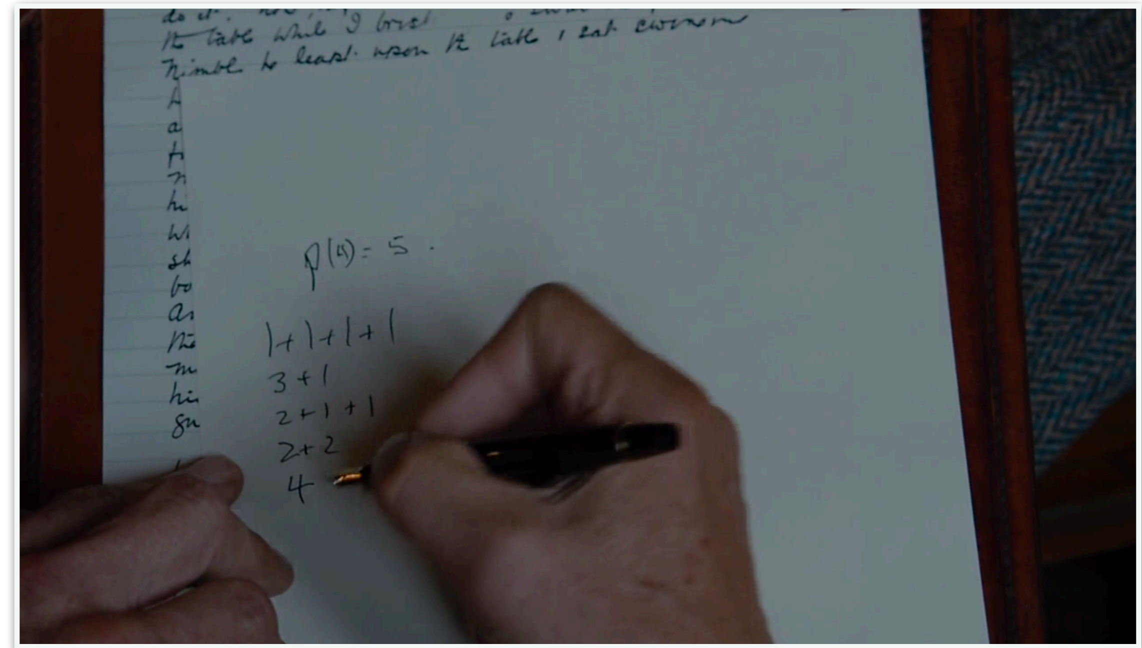
Srinivasa
Ramanujan
(1887-1920)

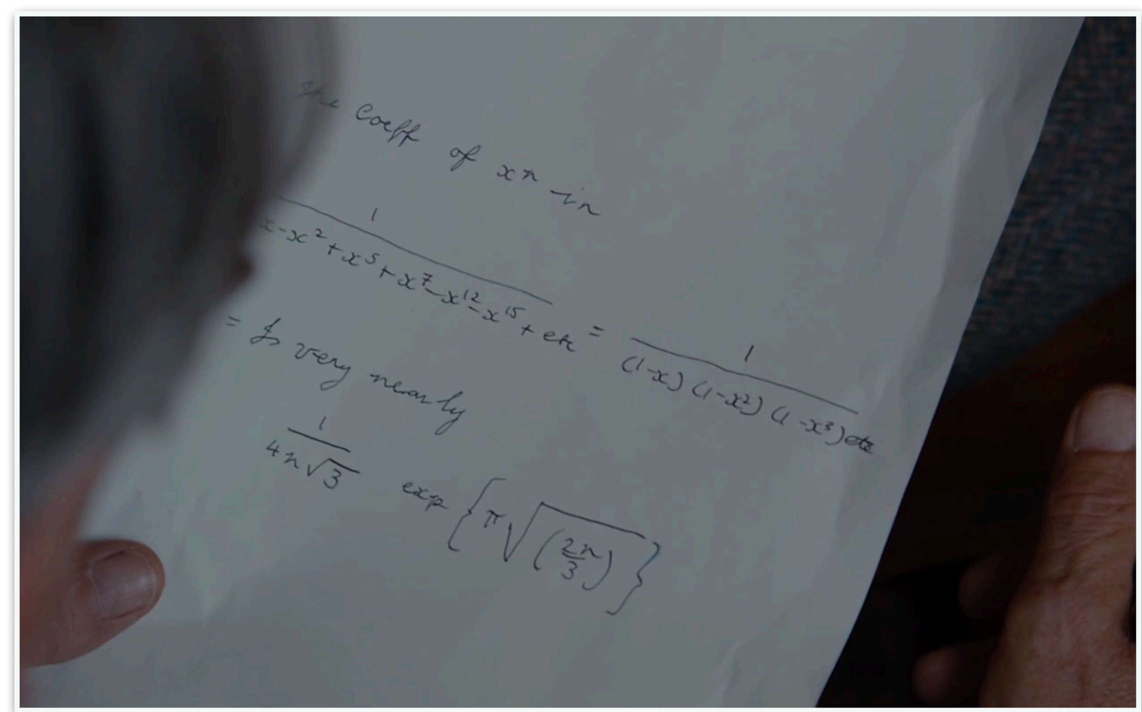
$$p(n) = \sum_{k=1}^{n} p_k(n)$$

$$\approx \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}$$

The Man Who Knew Infinity

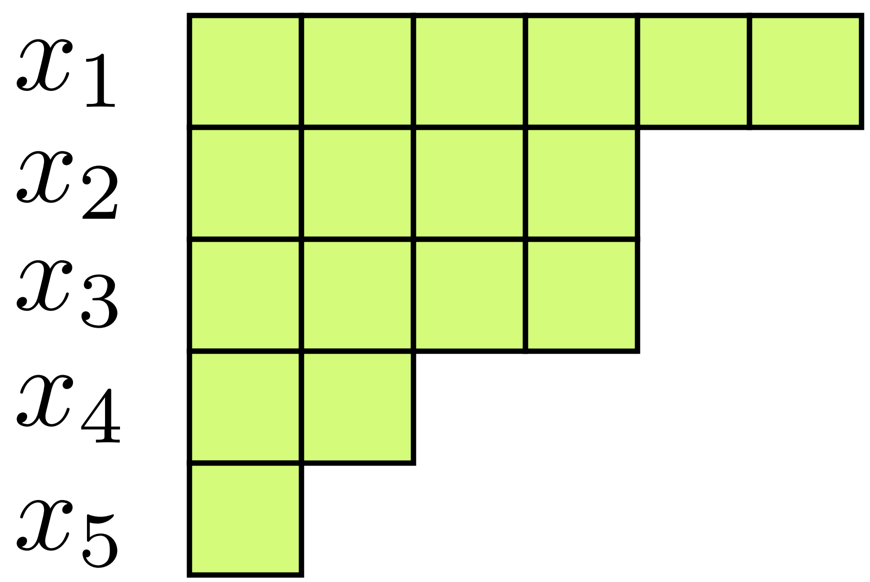
(2015 film)





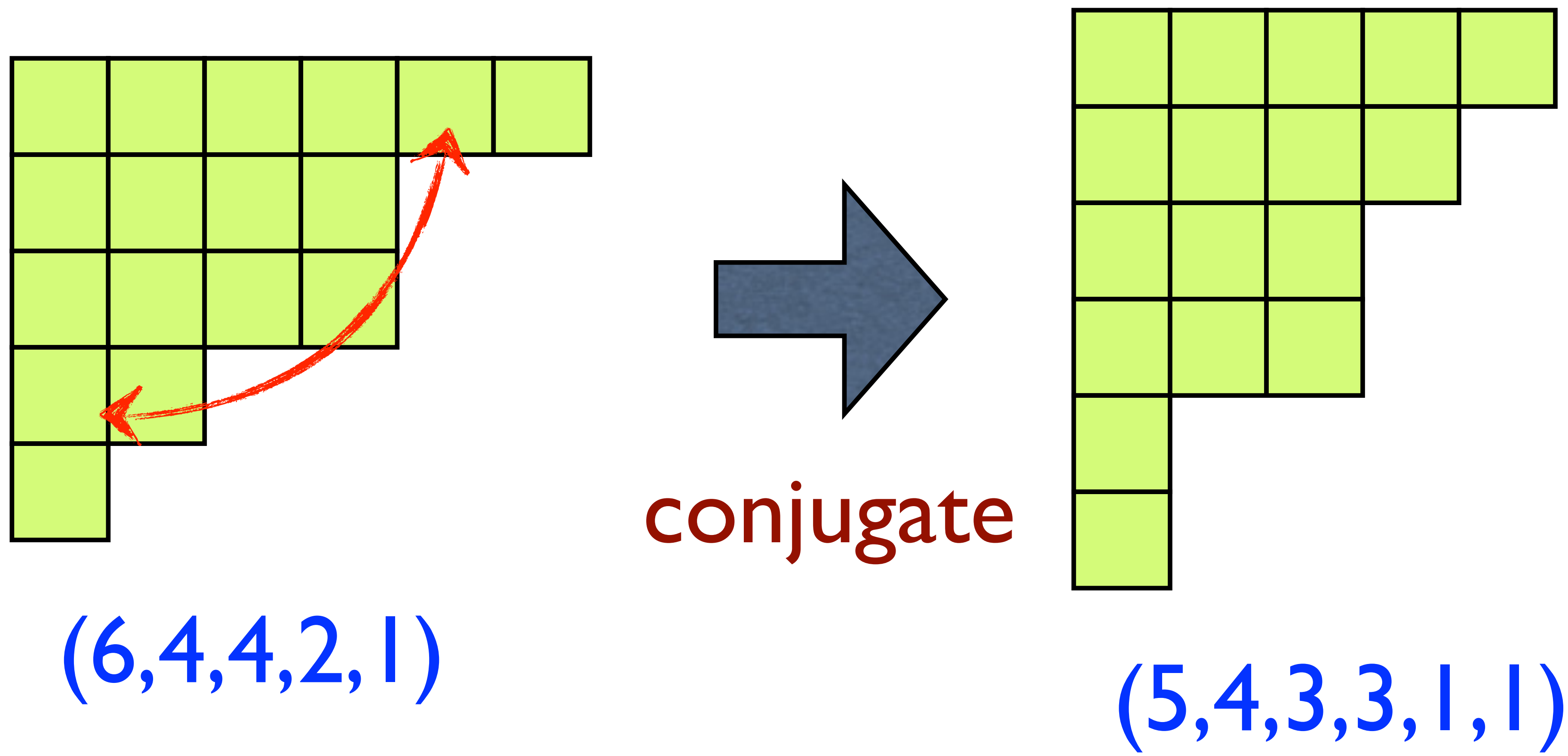
Ferrers diagram

(Young diagram)

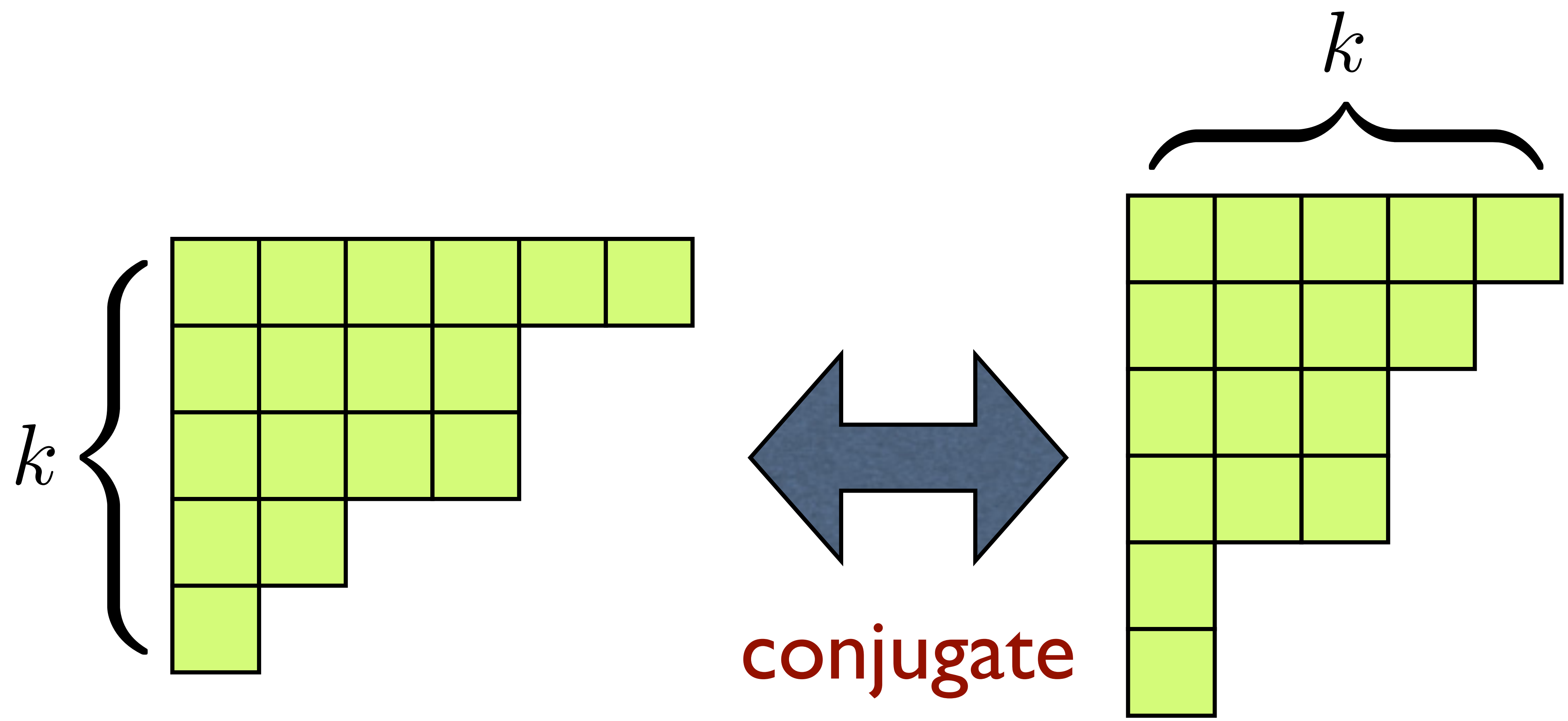


partition

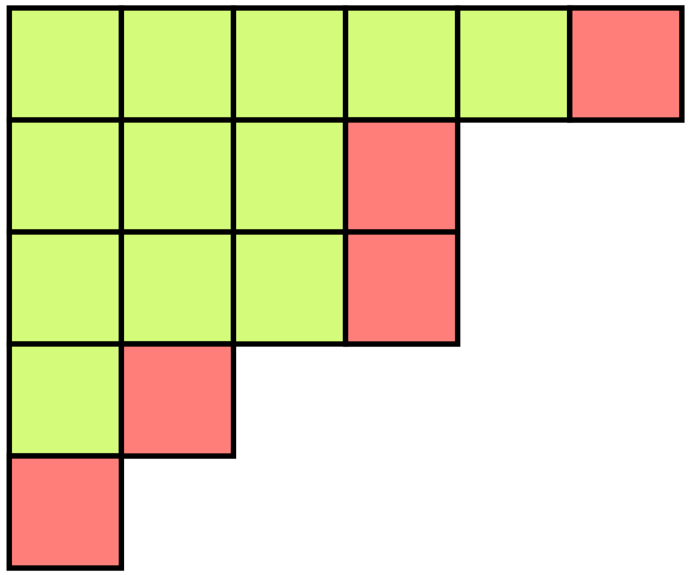
$$
\begin{cases}
x_1 + x_2 + \cdots + x_k = n \\
x_1 \geq x_2 \geq \cdots \geq x_k \geq 1
\end{cases}
$$



one-to-one
correspondence



\# of partitions of *n* into *k* parts = # of partitions of *n* with largest part *k*



# of partitions of *n* into *k* parts = # of partitions of *n*-*k* into at most *k* parts

$$p_k(n) = \sum_{j=1}^{k} p_j(n-k)$$

The twelvefold way

n balls are put into *m* bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	$m! \left\{ {n \atop m} \right\}$
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$