Combinatorics

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Extremal Combinatorics

"how large or how small a collection of finite objects can be, if it has to satisfy certain restrictions"

Extremal Problem:

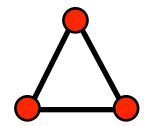
"What is the largest number of edges that an n-vertex *cycle-free* graph can have?"

$$(n\text{-}1)$$

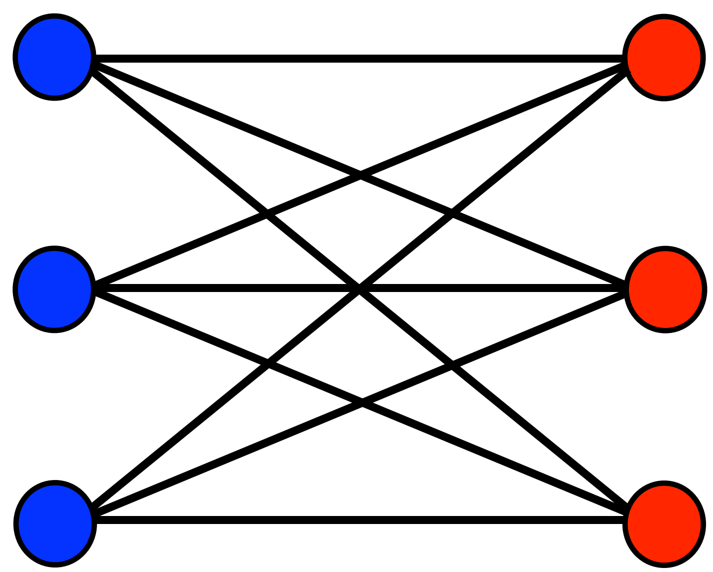
Extremal Graph:

spanning tree

Triangle-free graph

contains no  as subgraph

Example: bipartite graph



$|E|$ is maximized for
complete balanced bipartite graph

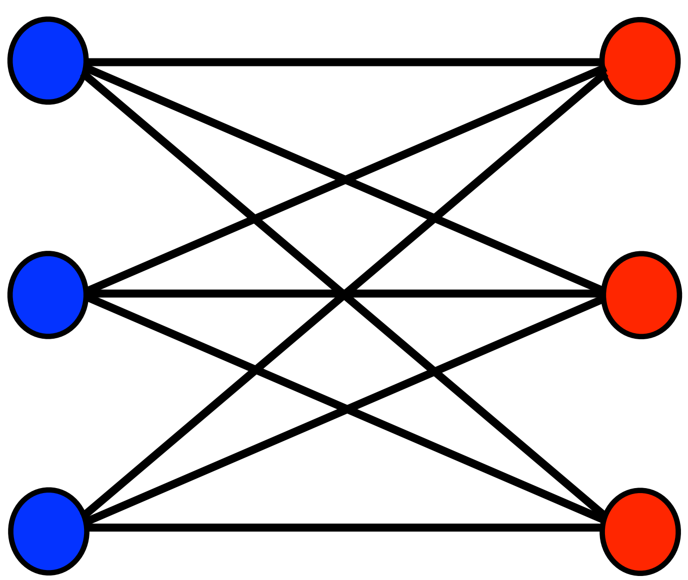
Extremal ?

Mantel's Theorem

Theorem (Mantel 1907)

If $G(V,E)$ has $|V|=n$ and is triangle-free, then

$$|E| \leq \frac{n^2}{4}.$$



For n is even,
extremal graph:

$$K_{\frac{n}{2},\frac{n}{2}}$$

$\triangle$-free $\Rightarrow$ $|E| \le n^2/4$

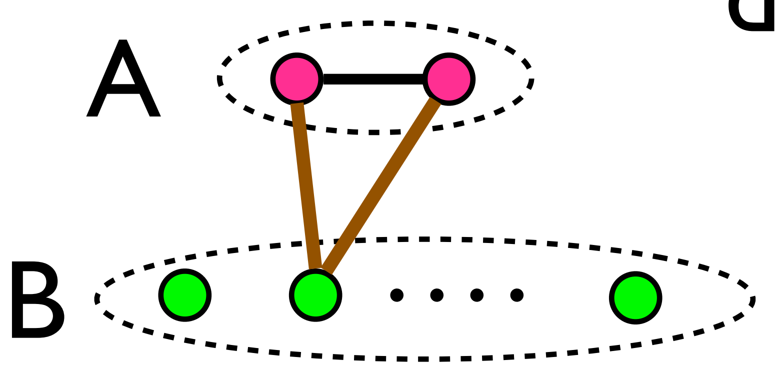
First Proof. Induction on n.

Basis: n=1,2. trivial

Induction Hypothesis: for any $n < N$

$$|E| > \frac{n^2}{4} \Rightarrow G \supseteq \triangle$$

Induction step: for $n = N$



due to I.H. $|E(B)| \le (n\text{-}2)^2/4$

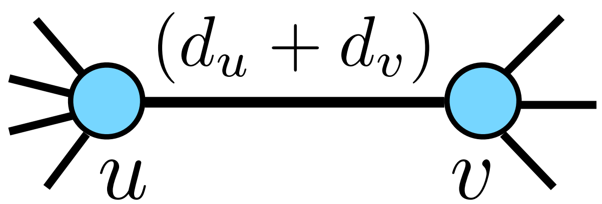
$$|E(A,B)| = |E| - |E(B)| - 1$$

$$> \frac{n^2}{4} - \frac{(n-2)^2}{4} - 1 \quad = n - 2$$

pigeonhole!

$$\triangle\text{-free} \Rightarrow |E| \leq n^2/4$$

Second Proof.



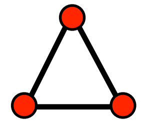
$$\sum_{uv \in E} (d_u + d_v) = \sum_{v \in V} d_v^2$$

$$\triangle\text{-free} \Rightarrow d_u + d_v \leq n \Rightarrow \sum_{uv \in E} (d_u + d_v) \leq n|E|$$

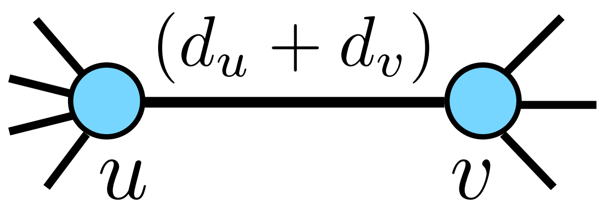
Cauchy-Schwarz

$$\sum_{v \in V} d_v^2 \geq \frac{1}{n} \left(\sum_{v \in V} d_v \right)^2 = \frac{4|E|^2}{n}$$ *(handshaking)*

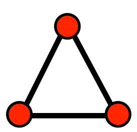
$$n|E| \geq \sum_{uv \in E} (d_u + d_v) = \sum_{v \in V} d_v^2 \geq \frac{\left(\sum_{v \in V} d_v\right)^2}{n} = \frac{4|E|^2}{n}$$

-free $\Rightarrow |E| \leq n^2/4$

Second Proof.



$$\sum_{uv \in E} (d_u + d_v) = \sum_{v \in V} d_v^2$$

-free $\Rightarrow d_u + d_v \leq n \Rightarrow \sum_{uv \in E} (d_u + d_v) \leq n|E|$

Cauchy-Schwarz

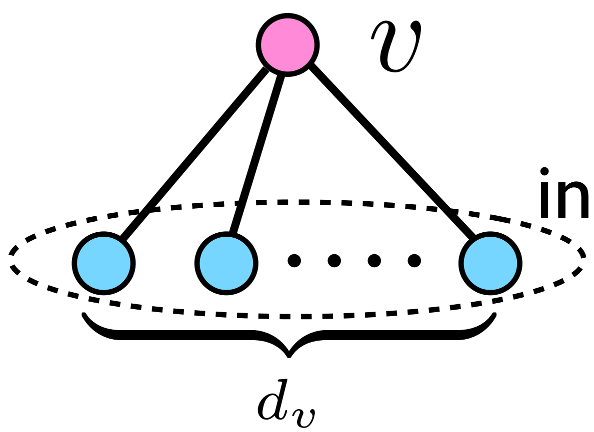
$$\sum_{v \in V} d_v^2 \geq \frac{1}{n} \left(\sum_{v \in V} d_v \right)^2 = \frac{4|E|^2}{n}$$ *(handshaking)*

$$n|E| \geq \frac{4|E|^2}{n} \Rightarrow |E| \leq \frac{n^2}{4}$$

$\triangle$-free $\Rightarrow$ $|E| \le n^2/4$

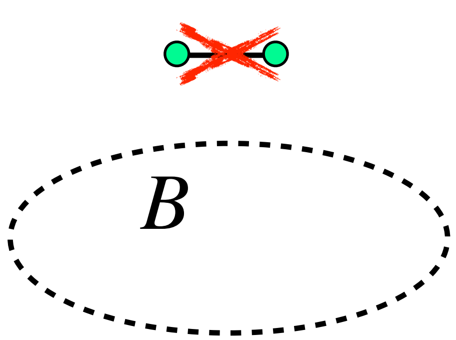
Third Proof.

A: maximum independent set $\qquad \alpha = |A|$



$$\forall v \in V, \quad d_v \le \alpha$$

$B = V \setminus A$ $\quad$ B incident to all edges $\quad$ $\beta = |B|$



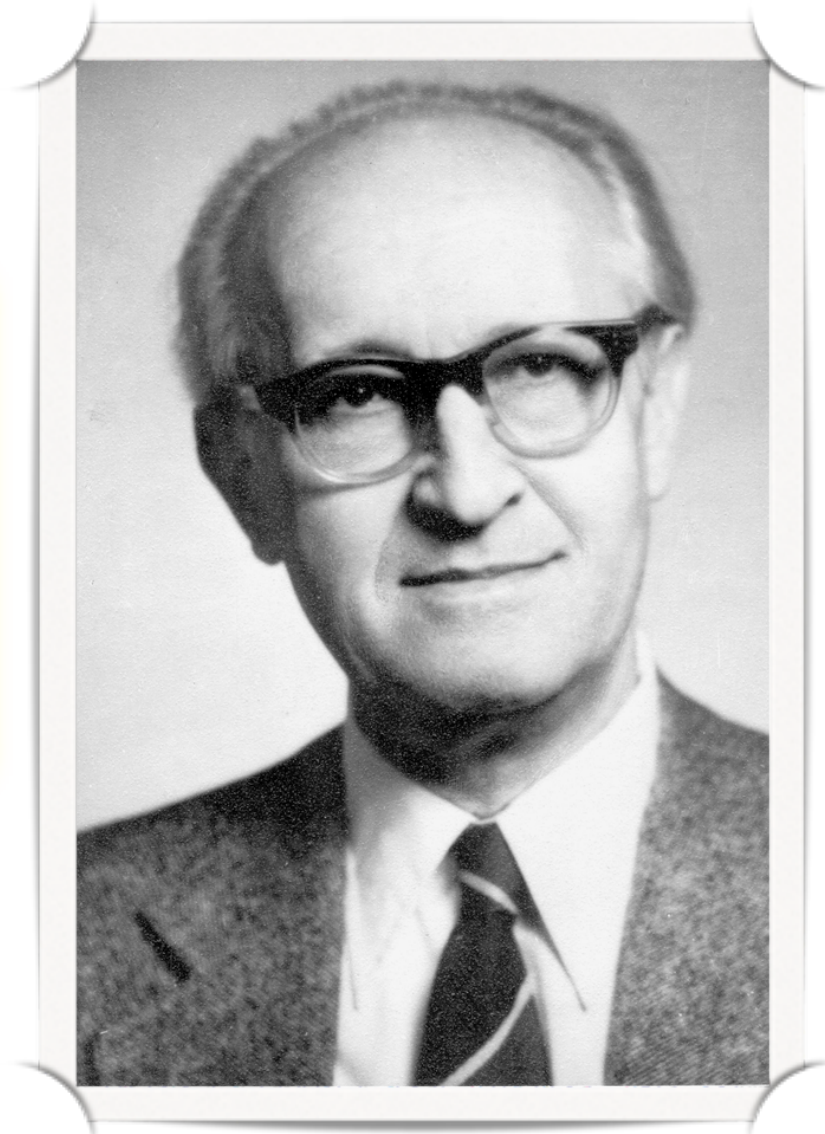
Inequality of the arithmetic and geometric mean

$$|E| \le \sum_{v \in B} d_v \le \alpha\beta \le \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{n^2}{4}$$

Turán's Theorem

"Suppose G is a K_r-free *graph.*

What is the largest number of edges that G can have?"



Paul Turán
(1910-1976)

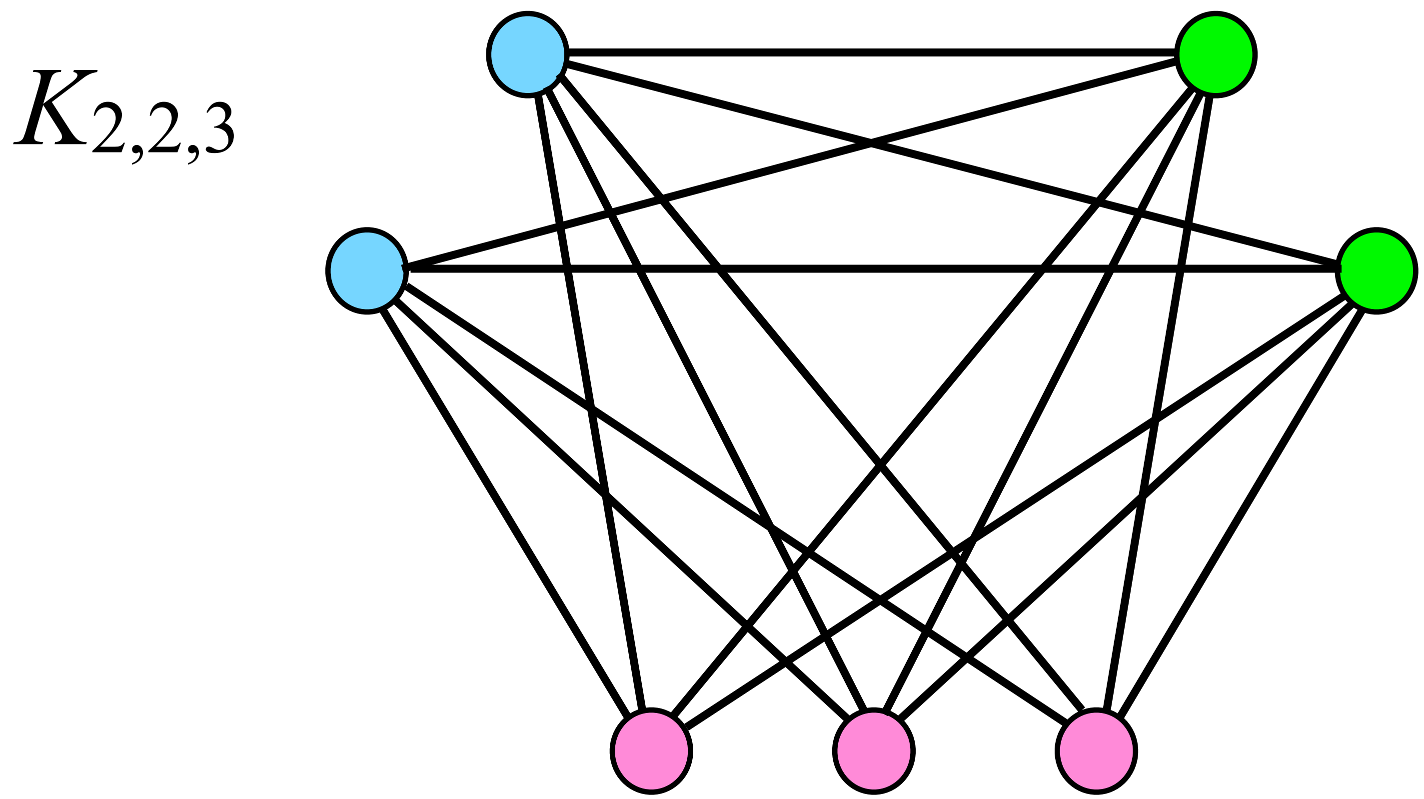
Turán's Theorem

Theorem (Turán 1941)

If $G(V,E)$ has $|V|=n$ and is K_r-free, then

$$|E| \leq \frac{r-2}{2(r-1)}n^2.$$

Complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$



Turán graph $T(n,r)$

$$T(n,r) = K_{n_1,n_2,\ldots,n_r}$$

$$n_1 + n_2 + \cdots + n_r = n \qquad n_i \in \left\{ \left\lfloor \frac{n}{r} \right\rfloor, \left\lceil \frac{n}{r} \right\rceil \right\}$$

Turán graph $T(n, r)$

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$T(n, r\text{-}1)$ has no K_r

$$|T(n, r-1)| \le \binom{r-1}{2} \left(\frac{n}{r-1} \right)^2$$

$$= \frac{r-2}{2(r-1)} n^2$$

$$K_r\text{-free} \Rightarrow |E| \leq \frac{r-2}{2(r-1)} n^2$$

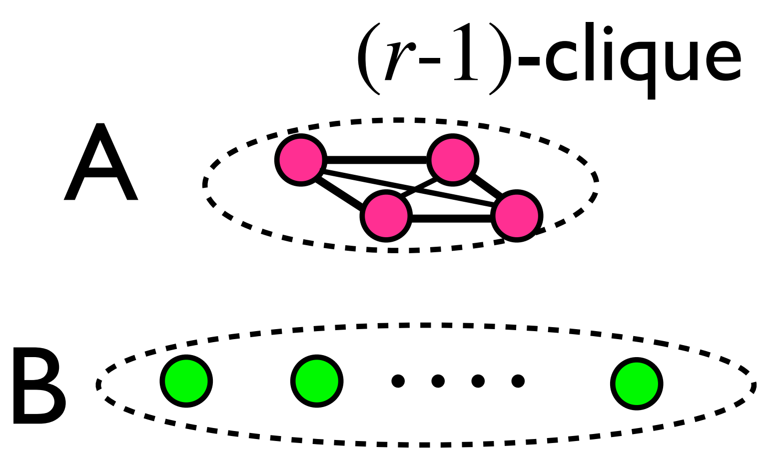
First Proof. Induction on n.

Basis: $n=1,\ldots,r-1$.

Induction Hypothesis: true for any $n < N$

Induction step: for $n = N$,

suppose G is maximum K_r-free



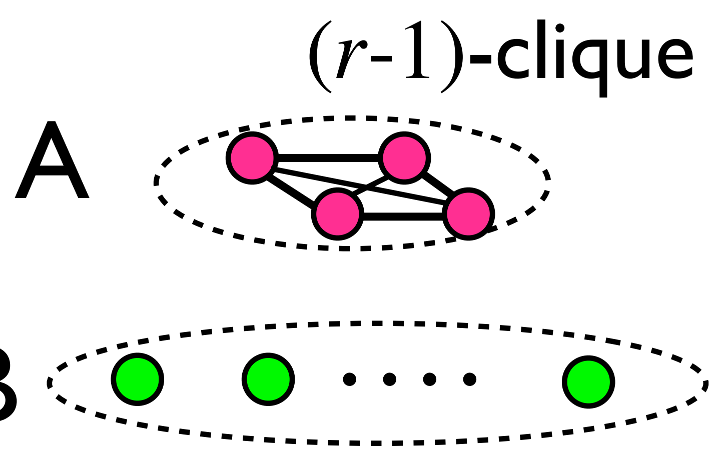
$\exists (r-1)$-clique

K_r-free $\Rightarrow$ $|E| \le \frac{r-2}{2(r-1)} n^2$

First Proof. Induction on n.

Induction step: for $n = N$,

suppose G is maximum K_r-free



due to I.H.

$$|E(B)| \le \frac{r-2}{2(r-1)}(n-r+1)^2$$

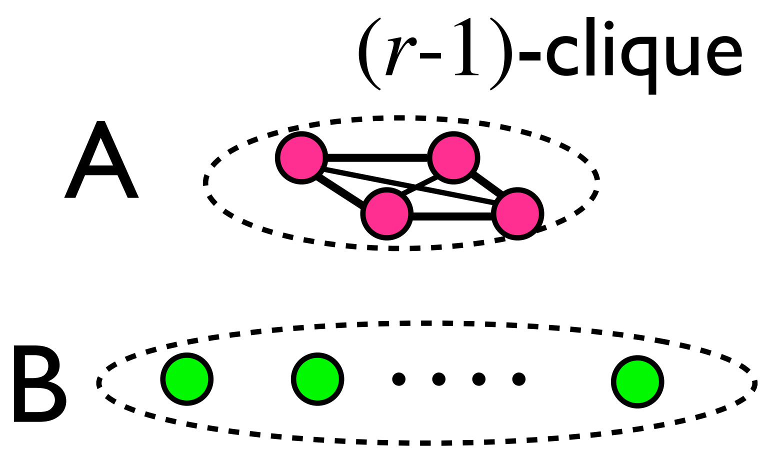
K_r-free $\Rightarrow$ no $u \in B$ adjacent to all $v \in A$

$$E(A,B) \le (r-2)(n-r+1)$$

$$K_r\text{-free} \Rightarrow |E| \le \frac{r-2}{2(r-1)} n^2$$

First Proof. Induction on n.

Induction step: for $n = N$,



$$|E(B)| \le \frac{r-2}{2(r-1)}(n-r+1)^2$$

$$E(A,B) \le (r-2)(n-r+1)$$

$$|E| = |E(A)| + |E(B)| + |E(A,B)|$$

$$= \binom{r-1}{2} + \frac{r-2}{2(r-1)}(n-r+1)^2 + (r-2)(n-r+1)$$

$$\le \frac{r-2}{2(r-1)} n^2$$

K_r-free $\Rightarrow$ $|E| \le \frac{r-2}{2(r-1)} n^2$

Second Proof. (weight shifting)

assign each vertex v a weight $w_v \ge 0$ with $\sum_{v \in V} w_v = 1$

evaluate $S = \sum_{uv \in E} w_u w_v$

let $W_u = \sum_{v: v \sim u} w_v$ For $u \not\sim v$ that $W_u \ge W_v$

$$(w_u + \epsilon) W_u + (w_v - \epsilon) W_v \ge w_u W_u + w_v W_v$$

shifting all weight of v to u $\Rightarrow$ S non-decreasing

S is maximized $\Rightarrow$ all weights on a clique

K_r-free ⇒ $|E| \le \frac{r-2}{2(r-1)} n^2$

Second Proof. (weight shifting)

assign each vertex v a weight $w_v \ge 0$ with $\sum_{v \in V} w_v = 1$

evaluate $S = \sum_{uv \in E} w_u w_v \le \binom{r-1}{2} \frac{1}{(r-1)^2}$

S is maximized ⇒ all weights on a clique

when all $w_i = \frac{1}{n}$

$$S = \sum_{uv \in E} w_u w_v = \frac{|E|}{n^2}$$

$$K_r\text{-free} \Rightarrow |E| \le \frac{r-2}{2(r-1)} n^2$$

Third Proof.

$$V = \{v_1, v_2, \ldots, v_n\} \qquad d_i = d(v_i)$$

clique number $\omega(G)$: size of the largest clique

$$\omega(G) \ge \sum_{i=1}^{n} \frac{1}{n - d_i}$$

(The probabilistic method)

random permutation π

$$S = \{i \mid \forall \pi_j < \pi_i, v_i \sim v_j\}$$

$i \in S$ iff v_i adjacent to all v_j that $\pi_j < \pi_i$

S is a clique

$$K_r\text{-free} \Rightarrow |E| \le \frac{r-2}{2(r-1)}n^2$$

Third Proof. (The probabilistic method)

$$\omega(G) \ge \sum_{i=1}^{n} \frac{1}{n-d_i}$$

random permutation π

$S = \{i \mid \forall \pi_j < \pi_i, v_i \sim v_j\}$

S is a clique

$i \in S$ iff v_i adjacent to all v_j that $\pi_j < \pi_i$

$$X_i = \begin{cases} 1 & v_i \in S \\ 0 & \text{otherwise} \end{cases} \qquad |S| = X = \sum_{i=1}^{n} X_i$$

$$v_i \in S \Leftarrow \forall v_j \not\sim v_i, \ \pi_i < \pi_j$$

$$\mathbf{E}[X_i] \ge \frac{1}{n-d_i} \qquad \mathbf{E}[|S|] \ge \sum_{i=1}^{n} \frac{1}{n-d_i}$$

$$K_r\text{-free} \Rightarrow |E| \leq \frac{r-2}{2(r-1)}n^2$$

Third Proof.

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n-d_i}$$

Cauchy-Schwarz $a_i = \sqrt{n-d_i},\ b = \frac{1}{\sqrt{n-d_i}}$

$$n^2 = \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$$= \left(\sum_{i=1}^{n}(n-d_i)\right)\left(\sum_{i=1}^{n}\frac{1}{n-d_i}\right) \leq \omega(G)\left(\sum_{i=1}^{n}(n-d_i)\right)$$

K_r-free $\Rightarrow$ $|E| \leq \frac{r-2}{2(r-1)} n^2$

Third Proof.

$$n^2 \leq \omega(G) \left(\sum_{i=1}^{n} (n - d_i) \right)$$

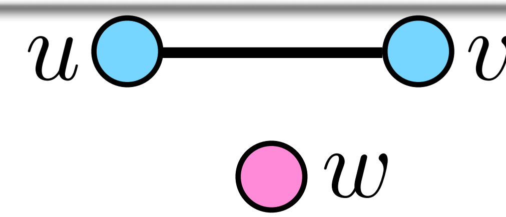
$$\leq (r-1)(n^2 - 2|E|)$$

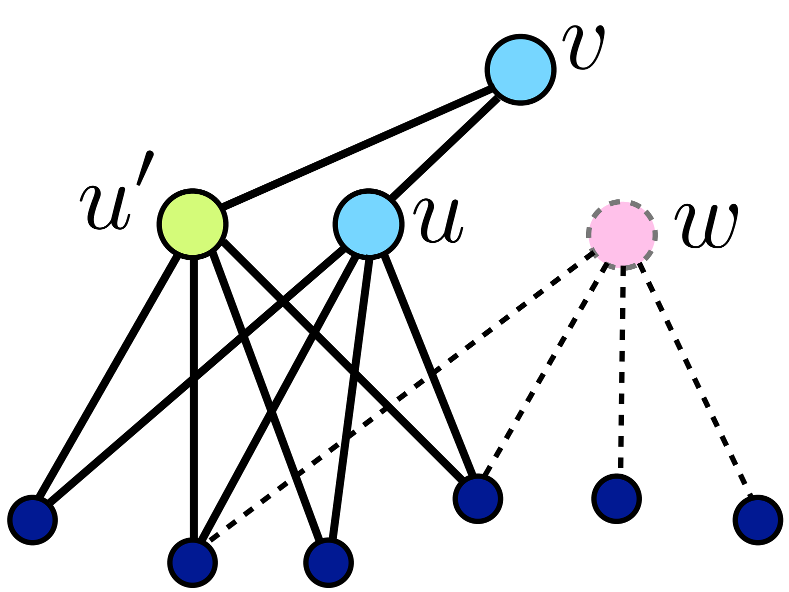
K_r-free $\Rightarrow$ $|E| \le \frac{r-2}{2(r-1)} n^2$

Fourth Proof.

Suppose G is K_r-free with maximum edges.

G does not have





By contradiction.

Case.1 $d_w < d_u$ or $d_w < d_v$

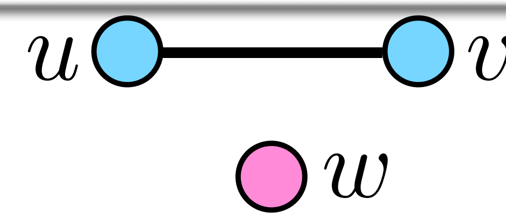
duplicate u, delete w, still K_r-free

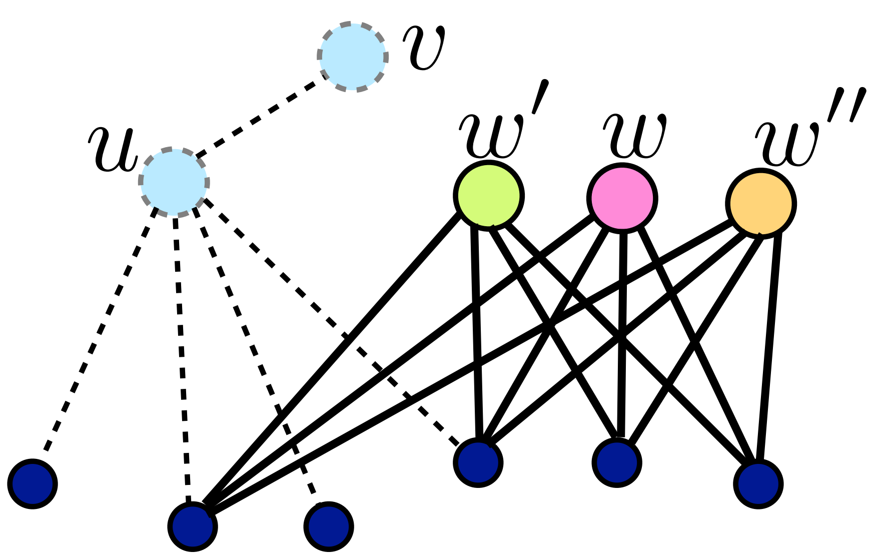
$$|E'| = |E| + d_u - d_w > |E|$$

K_r-free ⇒ $|E| \le \frac{r-2}{2(r-1)} n^2$

Fourth Proof.

Suppose G is K_r-free with maximum edges.

G does not have 



Case.2 $d_w \ge d_u \wedge d_w \ge d_v$

delete u,v, duplicate w, twice

still K_r-free

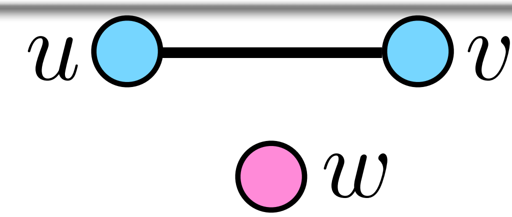
$$|E'| = |E| + 2d_w - (d_u + d_v - 1) > |E|$$

K_r-free ⇒ $|E| \leq \frac{r-2}{2(r-1)} n^2$

Fourth Proof.

Suppose G is K_r-free with maximum edges.

G does not have



$u \not\sim v$ is an equivalence relation

G is a complete multipartite graph

optimize $K_{n_1, n_2, \ldots, n_{r-1}}$

subject to $n_1 + n_2 + \ldots + n_{r-1} = n$

Turán's Theorem (clique)

If $G(V,E)$ has $|V|=n$ and is K_r-free, then

$$|E| \leq \frac{r-2}{2(r-1)} n^2.$$

Turán's Theorem (independent set)

If $G(V,E)$ has $|V|=n$ and $|E|=m$, then G has an independent set of size

$$\geq \frac{n^2}{2m+n}.$$

Parallel Max

- compute max of n distinct numbers
  - computation model: parallel, comparison-based
- 1-round algorithm: $\binom{n}{2}$ comparisons of all pairs
- lower bound for one-round:
  - $\binom{n}{2}$ comparisons are required in the worst case

adversary argument

Parallel Max

- 2-round algorithm:
  - divide n numbers into k groups of n/k each
  - 1st round: find max of each group;
    $k\binom{n/k}{2}$ comparisons
  - 2nd round: find the max of the k maxes
    $\binom{k}{2}$ comparisons
- total comparisons: $k\binom{n/k}{2} + \binom{k}{2} = O\left(n^{4/3}\right)$

for $k = n^{2/3}$

3-round?

optimal?

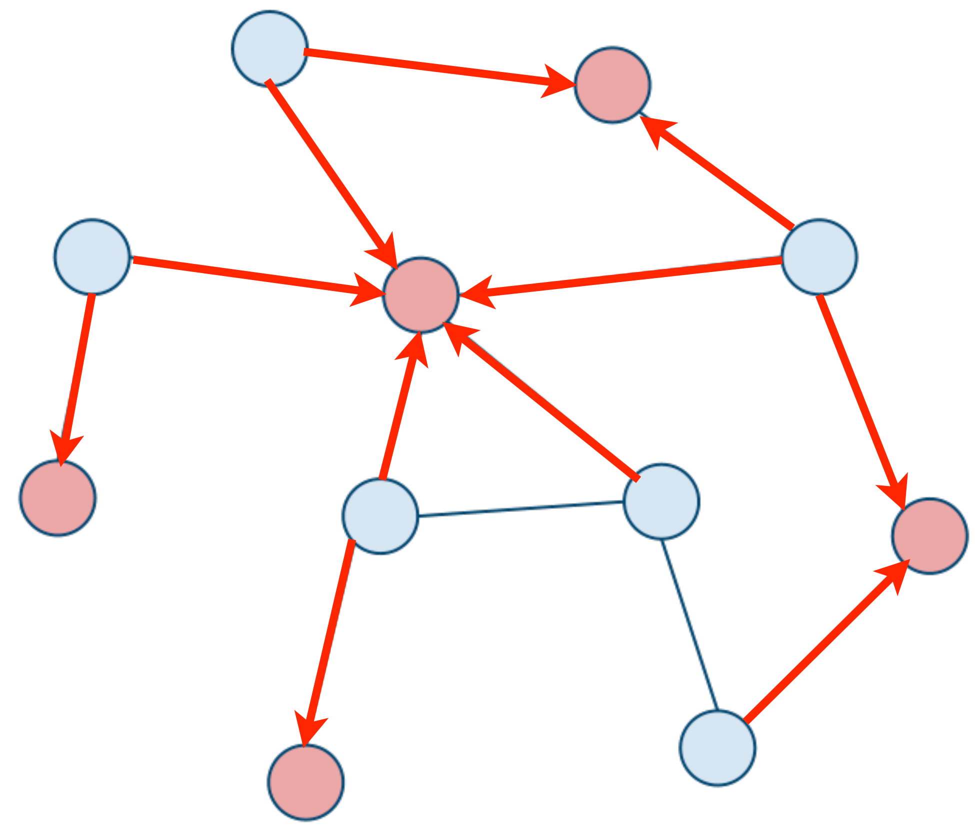
1st round:

Alg: m comparisons

choose an independent set

of size $\geq \dfrac{n^2}{2m+n}$ (Turán)

make them local maximal



2nd round:

a parallel max problem of size $\geq \dfrac{n^2}{2m+n}$

requires $\geq \dbinom{\frac{n^2}{2m+n}}{2}$ comparisons

total comparisons $\geq m + \dbinom{\frac{n^2}{2m+n}}{2} = \Omega(n^{4/3})$

Extremal Graph Theory

Fix a graph H.

$$\mathrm{ex}(n, H)$$

largest possible number of edges of $G \not\supseteq H$ on n vertices

$$\mathrm{ex}(n, H) = \max_{\substack{G \not\supseteq H \\ |V(G)|=n}} |E(G)|$$

Turán's Theorem

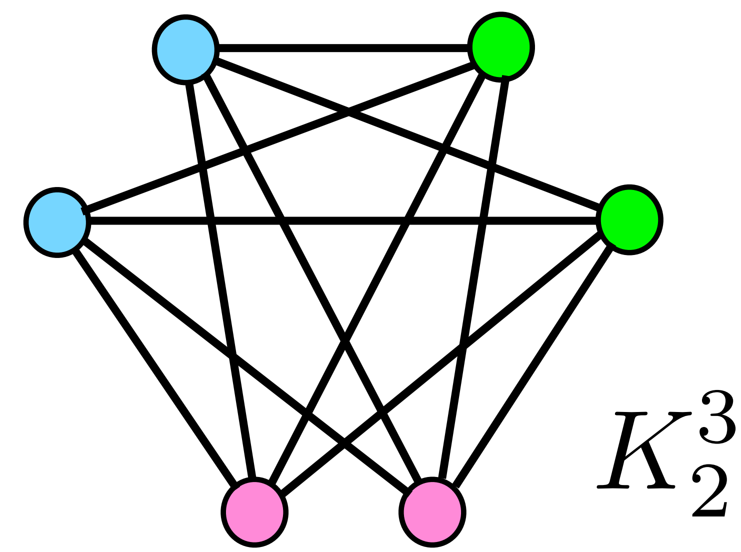
$$\mathrm{ex}(n, K_r) = |T(n, r-1)| \leq \frac{r-2}{2(r-1)} n^2$$

Erdős–Stone theorem

(Fundamental theorem of extremal graph theory)

$$K_s^r = K_{\underbrace{s, s, \cdots, s}_{r}} = T(rs, r)$$

complete r-partite graph
with s vertices in each part



Theorem (Erdős–Stone 1946)

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1) \right) n^2$$

Theorem (Erdős–Stone 1946)

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

$\mathrm{ex}(n, H)/\binom{n}{2}$ extremal density of subgraph H

Corollary

For any nonempty graph H

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

$\chi(H) = r$

$H \not\subseteq T(n, r-1)$ for any n

$$\mathrm{ex}(n,H) \geq |T(n, r-1)|$$

$H \subseteq K_s^r$ for sufficiently large s

$$\mathrm{ex}(n,H) \leq \mathrm{ex}(n, K_s^r)$$

$$= \left(\tfrac{r-2}{2(r-1)} + o(1)\right) n^2$$

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

$$\chi(H) = r$$

$$|T(n, r-1)| \le \mathrm{ex}(n,H) \le \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

$$\frac{r-2}{r-1} - o(1) \le \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \le \frac{r-2}{r-1} + o(1)$$