Combinatorics

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尹一通

Compositions by 1 and 2

of compositions of *n* with summands from {1,2}

of $(x_1, x_2, \ldots, x_k)$

for some $k \leq n$

$x_1 + \cdots + x_k = n$

$x_i \in \{1, 2\}$

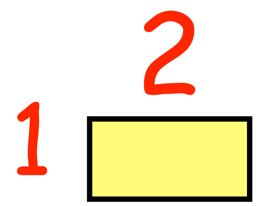
$$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 1 \quad a_1 = 1$$

Case.1 $x_k = 1$ $\quad x_1 + \cdots + x_{k-1} = n - 1$

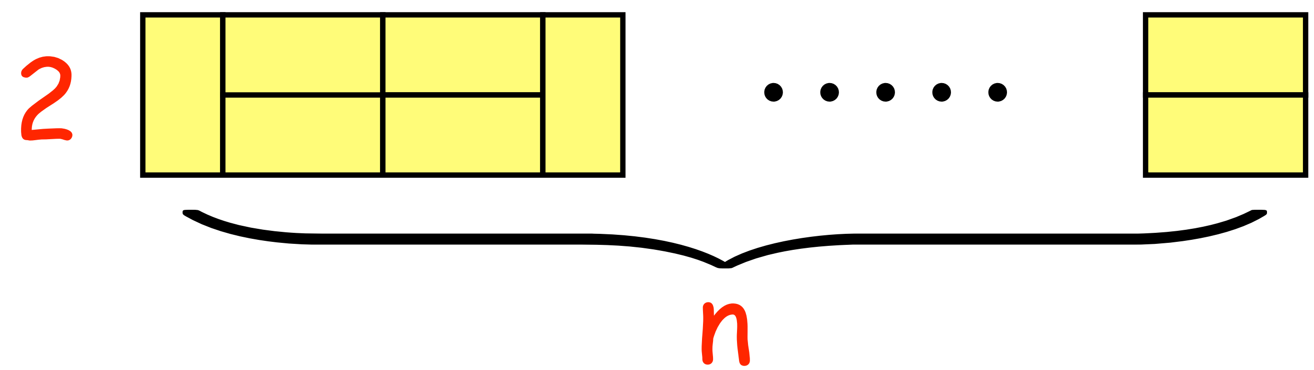
Case.2 $x_k = 2$ $\quad x_1 + \cdots + x_{k-1} = n - 2$

Dominos

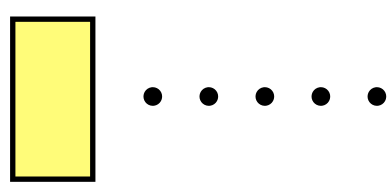
domino:

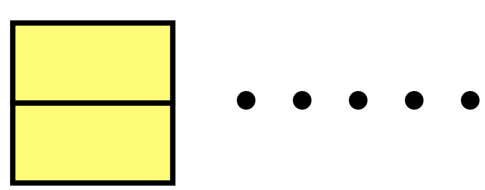


# of tilings



$$a_n = a_{n-1} + a_{n-2} \qquad a_0 = 1 \quad a_1 = 1$$

Case.1  2×(n-1)

Case.2  2×(n-2)

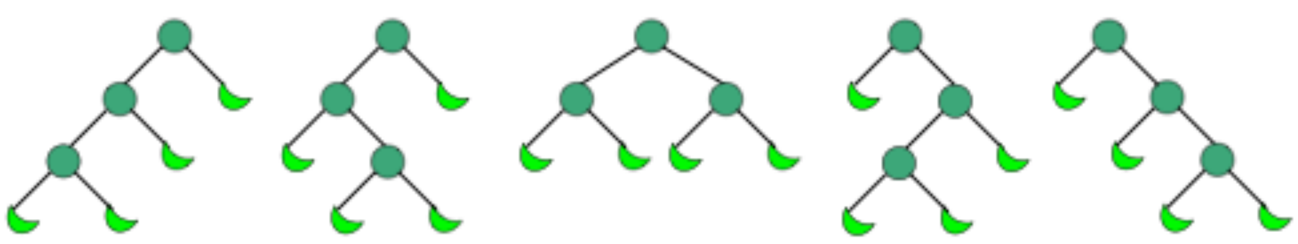
Fibonacci number

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

full parenthesization of $n+1$ factors

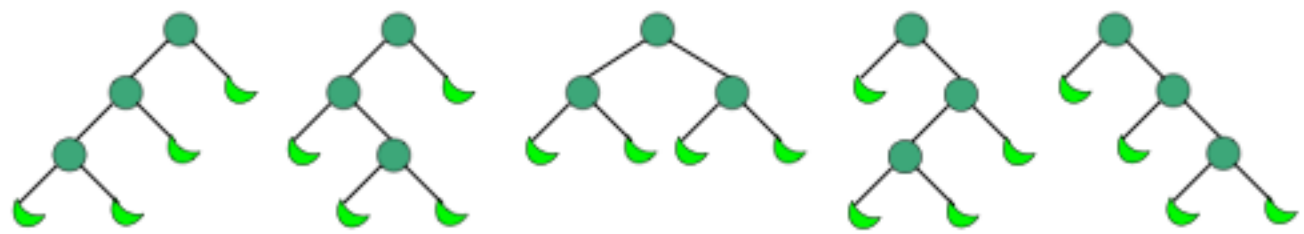
$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

full binary trees with $n+1$ leaves



Catalan Number

C_n : # of full binary trees with $n+1$ leaves



Recursion:

C_k C_{n-1-k}

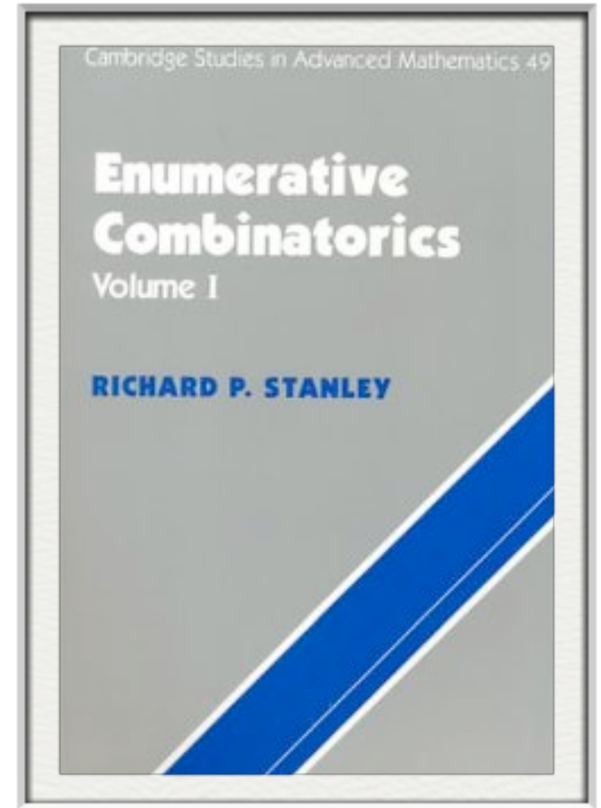
$$C_0 = 1 \quad \text{for } n \geq 1,$$

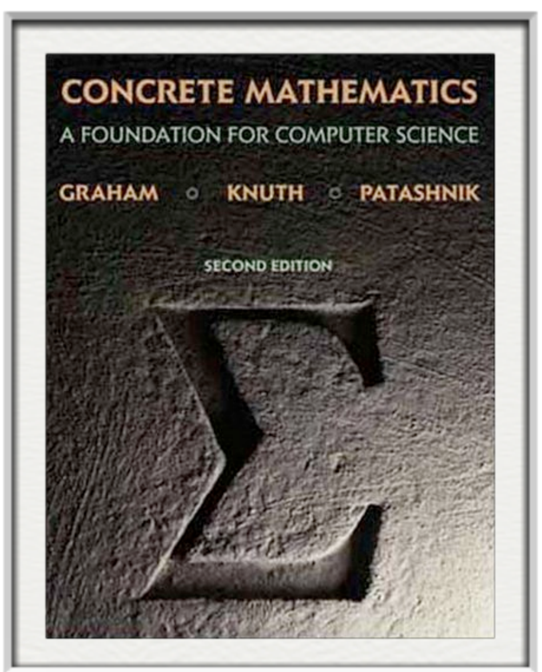
$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Generating Functions

Generating Functions:

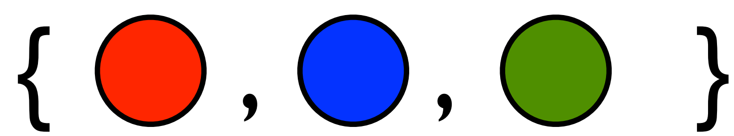
"the most *useful* but most *difficult to understand* method (for counting)"





"the *most powerful* way to deal with sequences of numbers, as far as anybody knows," generate

~~enumerate~~ all subsets of



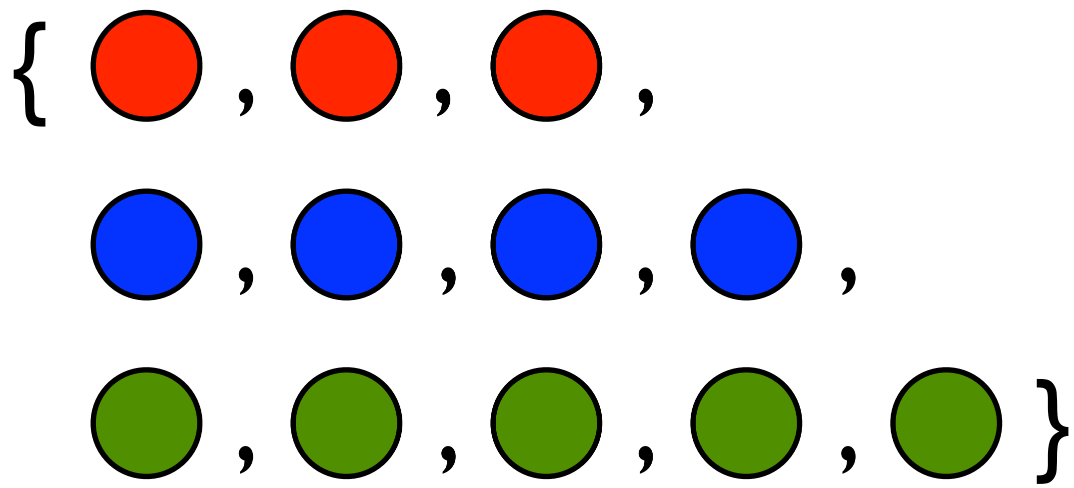
$$\color{red}{(x^0 + x^1)}\color{blue}{(x^0 + x^1)}\color{green}{(x^0 + x^1)}$$

$$= \color{red}{x^0}\color{blue}{x^0}\color{green}{x^0} + \color{red}{x^0}\color{blue}{x^0}\color{green}{x^1} + \color{red}{x^0}\color{blue}{x^1}\color{green}{x^0} + \color{red}{x^0}\color{blue}{x^1}\color{green}{x^1}$$

$$+ \color{red}{x^1}\color{blue}{x^0}\color{green}{x^0} + \color{red}{x^1}\color{blue}{x^0}\color{green}{x^1} + \color{red}{x^1}\color{blue}{x^1}\color{green}{x^0} + \color{red}{x^1}\color{blue}{x^1}\color{green}{x^1}$$

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

coefficient of x^k : # of *k*-subsets



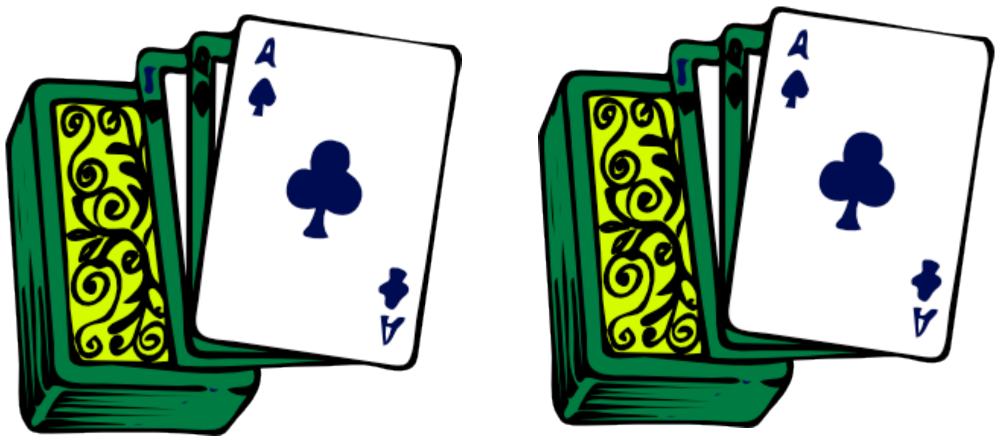
$$(1 + x + x^2 + x^3)$$

$$(1 + x + x^2 + x^3 + x^4)$$

$$(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$= \quad 1 + 3x + 6x^2 + 10x^3 + 14x^4 + 17x^5 + 18x^6$$
$$+17x^7 + 14x^8 + 10x^9 + 6x^{10} + 3x^{11} + x^{12}$$

Double Decks



choose *m* cards from

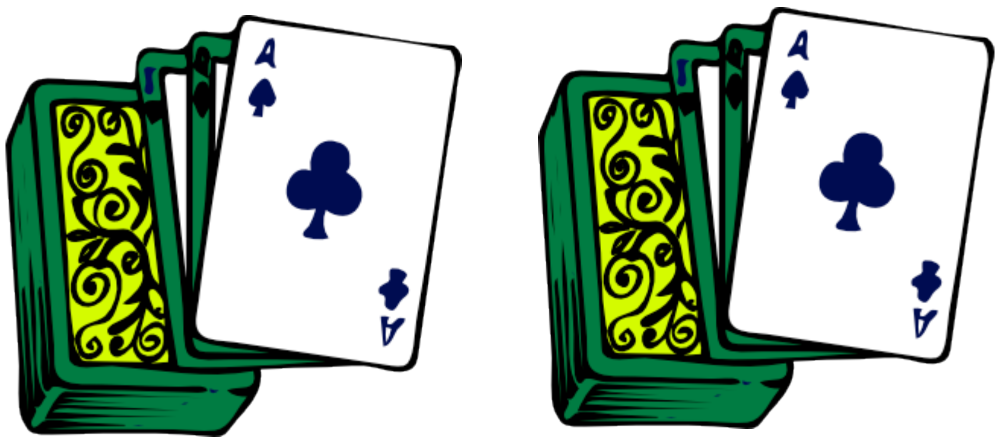
2 decks of *n* cards

$$(x_1^0 + x_1^1 + x_1^2)\,(x_2^0 + x_2^1 + x_2^2)\cdots(x_n^0 + x_n^1 + x_n^2)$$

of *m*-order terms

coefficient of x^m in $(1 + x + x^2)^n$

Double Decks



choose m cards from

2 decks of n cards

$$(1+(x+x^2))^n = \sum_k \binom{n}{k}(x+x^2)^k$$

$$= \sum_k \binom{n}{k} x^k \sum_{\ell \le k} \binom{k}{\ell} x^\ell = \sum_k \sum_{\ell \le k} \binom{n}{k}\binom{k}{\ell} x^{k+\ell}$$

$$= \sum_m \left(\sum_\ell \binom{n}{m-\ell}\binom{m-\ell}{\ell} \right) x^m$$

Multisets

multisets on $S = \{x_1, x_2, \ldots, x_n\}$

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots)\cdots(1 + x_n + x_n^2 + \cdots)$$

$$= \sum_{m:S\to\mathbb{N}} \prod_{x_i\in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \cdots)^n = \sum_{m\in\mathbb{N}^n} x^{m_1+\cdots+m_n} = \sum_{k\geq 0} \left(\!\binom{n}{k}\!\right) x^k$$

|| geometric

$$(1 - x)^{-n} = \sum_{k\geq 0} \frac{(-n)(-n-1)\cdots(-n-k+1)(-1)^k}{k!} x^k$$

Taylor

Multisets

multisets on $S = \{x_1, x_2, \dots, x_n\}$

$$(1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots)\cdots(1 + x_n + x_n^2 + \cdots)$$

$$= \sum_{m:S\to\mathbb{N}} \prod_{x_i \in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \cdots)^n = \sum_{m\in\mathbb{N}^n} x^{m_1+\cdots+m_n} = \sum_{k\geq 0} \left(\!\!\binom{n}{k}\!\!\right) x^k$$

$$\|$$

$$(1-x)^{-n} = \sum_{k\geq 0} \binom{n+k-1}{k} x^k$$

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

Ordinary Generating Function (OGF)

$\{a_n\}$ $\qquad a_0, a_1, a_2, \ldots$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$[x^n]G(x) = a_n$$

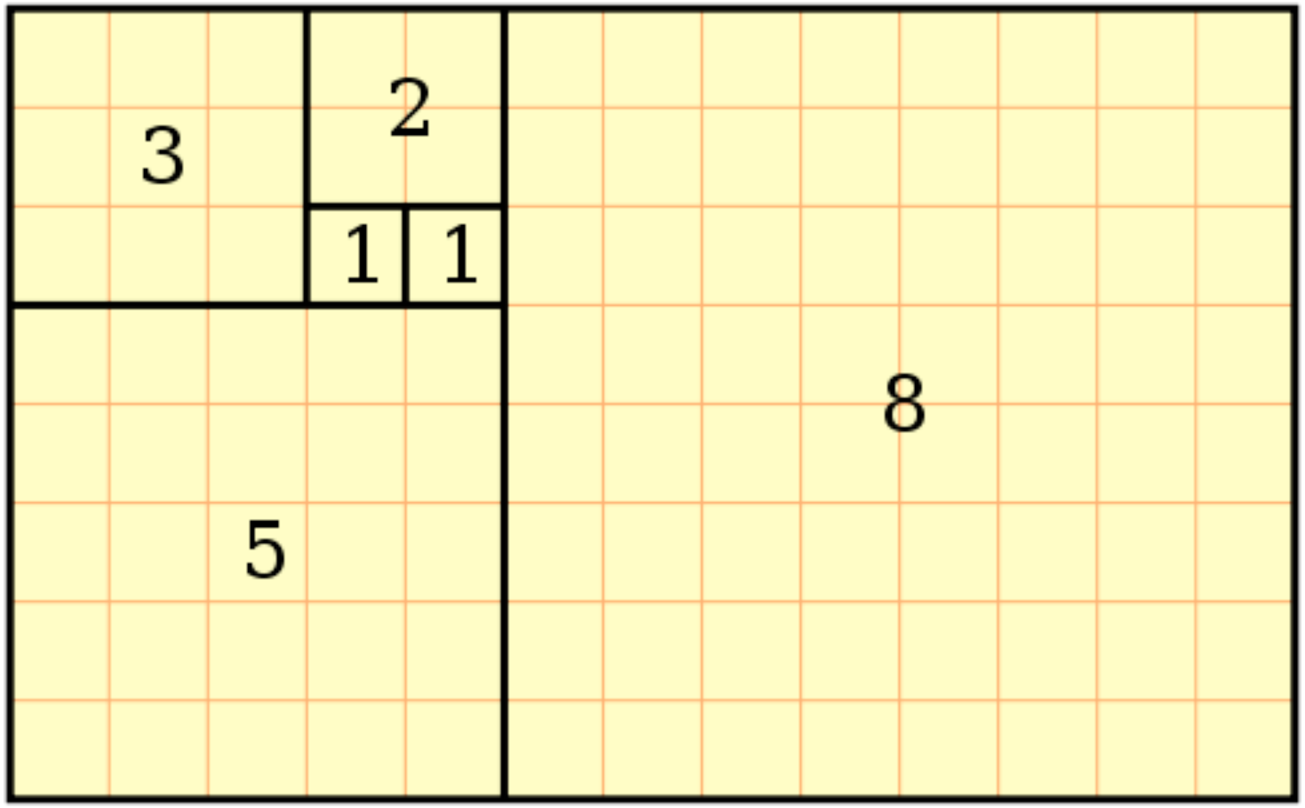
Fibonacci number

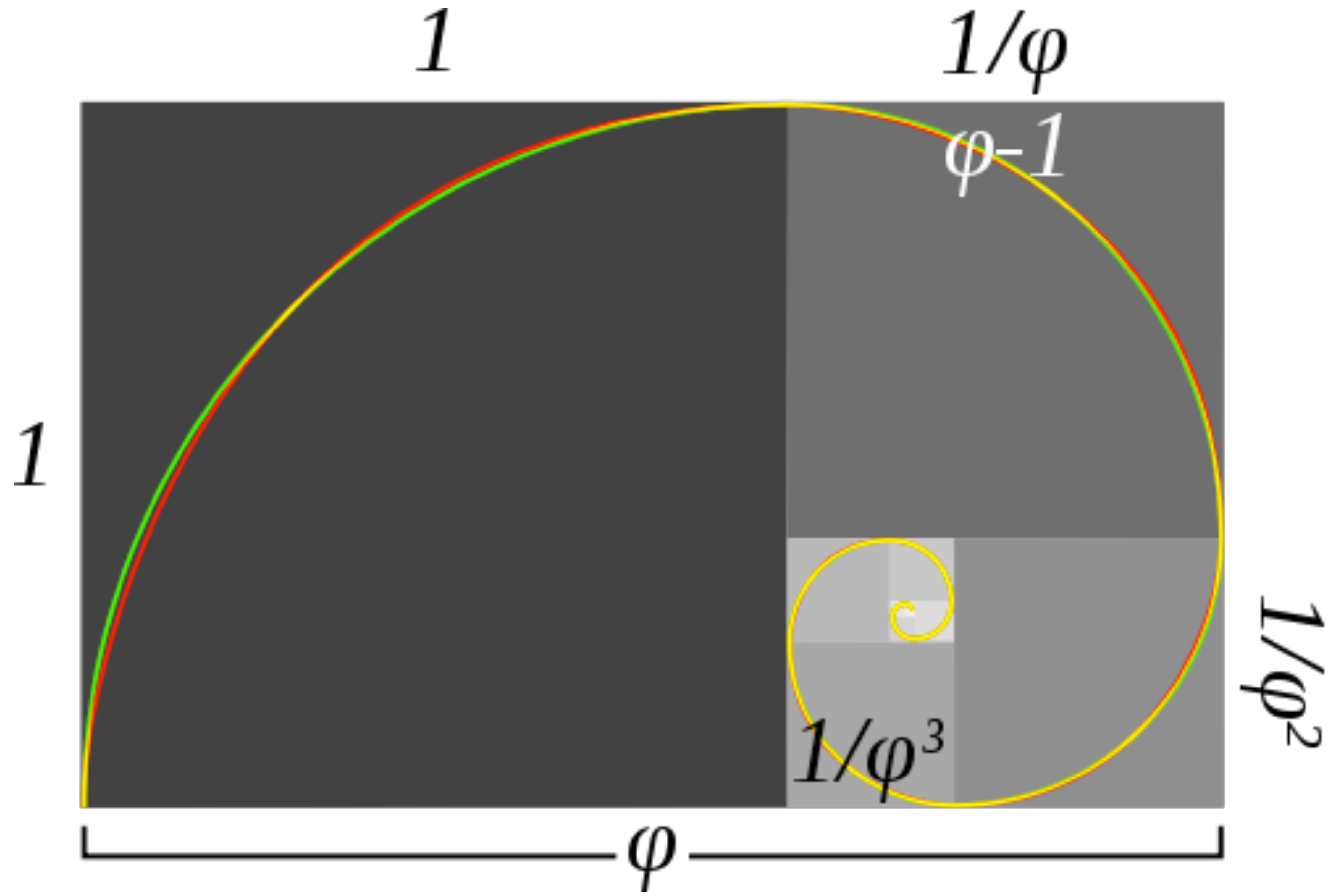
$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

Fibonacci number

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right)$$


3
2
1
1
5
8


1
1/φ
φ-1
1
1/φ²
1/φ³
φ

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n\geq 0} F_n x^n$$

recursion:

$$G(x) = F_0 + F_1 x + \sum_{n\geq 2} F_n x^n = x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n$$

$$\sum_{n\geq 2} F_{n-1} x^n = \sum_{n\geq 1} F_{n-1} x^n = \sum_{n\geq 0} F_n x^{n+1} = xG(x)$$

$$\sum_{n\geq 2} F_{n-2} x^n = \sum_{n\geq 0} F_n x^{n+2} = x^2 G(x)$$

identity: $G(x) = x + (x + x^2)G(x)$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n\geq 0} F_n x^n$$

identity: $G(x) = x + (x + x^2)G(x)$

solution: $G(x) = \dfrac{x}{1 - x - x^2}$ =? Taylor ?

denote $\phi = \dfrac{1+\sqrt{5}}{2}$ $\hat{\phi} = \dfrac{1-\sqrt{5}}{2}$

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n \geq 0} F_n x^n$$

identity:

$$G(x) = x + (x + x^2)G(x)$$

solution:

$$G(x) = \frac{x}{1 - x - x^2}$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\phi x)^n - \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\hat{\phi} x)^n$$

$$= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n$$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right)$$

Ordinary Generating Function (OGF)

$\{a_n\}$ $\qquad a_0, a_1, a_2, \ldots$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\frac{1}{1 - \alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

Formal Power Series

formal power series: $$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$\mathbb{C}[[x]]$: ring of formal power series with complex coefficient

inverse

$$F(x)G(x) = 1 \Rightarrow F(x) = G(x)^{-1} = \frac{1}{G(x)}$$

$$(1-\alpha x)\left(\sum_{n=0}^{\infty} \alpha^n x^n\right) = 1 \Rightarrow \frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

“Generatingfunctionology”

1. Recurrence:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2}$$

2. Manipulation:

$$\begin{aligned} G(x) &= \sum_{n\geq 0} a_n x^n = x + \sum_{n\geq 1} a_{n-1}x^n + \sum_{n\geq 2} a_{n-2}x^n \\ &= x + (x + x^2)G(x) \end{aligned}$$

3. Solving:

$$G(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n\right) x^n$$

Generating function manipulations

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution: $$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation: $$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

“Generatingfunctionology”

1. Recurrence:

$$a_0 = 0 \qquad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2}$$

2. Manipulation:

$$\begin{aligned} G(x) &= \sum_{n\geq 0} a_n x^n = x + \sum_{n\geq 1} a_{n-1} x^n + \sum_{n\geq 2} a_{n-2} x^n \\ &= x + (x + x^2) G(x) \end{aligned}$$

3. Solving: expending!

$$G(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$= \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n$$

Expanding generating functions

Taylor's expansion:

$$G(x) = \sum_{n \geq 0} \frac{G^{(n)}(0)}{n!} x^n$$

Geometric sequence:

$$\frac{a}{1 - bx} = a \sum_{n \geq 0} (bx)^n$$

$$G(x) = \frac{a_1}{1 - b_1 x} + \frac{a_2}{1 - b_2 x} + \cdots + \frac{a_k}{1 - b_k x}$$

$$[x^n]G(x) = a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n$$

Expanding generating functions

Binomial theorem: Newton's formula

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

$$((1+x)^\alpha)^{(n)} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$$

generalized binomial coefficient:

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

Changing Money

$$(壹, 伍)_n = \sum_{k=0}^{n} 壹_k 伍_{n-k}$$

$壹_n$: # of ways to change n yuan using $\boxed{壹圆}$

$$\sum_{n\geq 0} 壹_n x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$伍_n$: # of ways to change n yuan using $\boxed{伍圆}$

$$\sum_{n\geq 0} 伍_n x^n = 1 + x^5 + x^{10} + \cdots = \frac{1}{1-x^5}$$

Changing Money

$$(壹,伍)_n = \sum_{k=0}^{n} 壹_k 伍_{n-k}$$

$$\sum_{n\geq 0} 壹_n x^n = \frac{1}{1-x} \qquad \sum_{n\geq 0} 伍_n x^n = \frac{1}{1-x^5}$$

$$\sum_{n\geq 0} (壹,伍)_n x^n = \sum_{n\geq 0}\sum_{k=0}^{n} 壹_k 伍_{n-k}\, x^n$$

convolution! $\quad = \dfrac{1}{(1-x)(1-x^5)}$

Changing Money

$$\sum_{n\geq 0}(\text{壹}, \text{伍}, \text{拾}, \text{贰拾}, \text{伍拾}, \text{壹佰})_n x^n$$

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$\sum_{n\geq 0}(\text{壹}, \text{伍}, \text{拾}, \text{贰拾}, \text{伍拾}, \text{壹佰})_n x^n$$

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$= (1+x+x^2+x^3+\cdots+x^{99})\cdot(1+x^5+x^{10}+x^{15}+\cdots+x^{95})$$

$$\cdot(1+x^{10}+x^{20}+x^{30}+\cdots+x^{90})\cdot(1+x^{20}+x^{40}+x^{60}+x^{80})$$

$$\cdot(1+x^{50})\cdot\boxed{\frac{1}{(1-x^{100})^6}}$$

Newton's formula

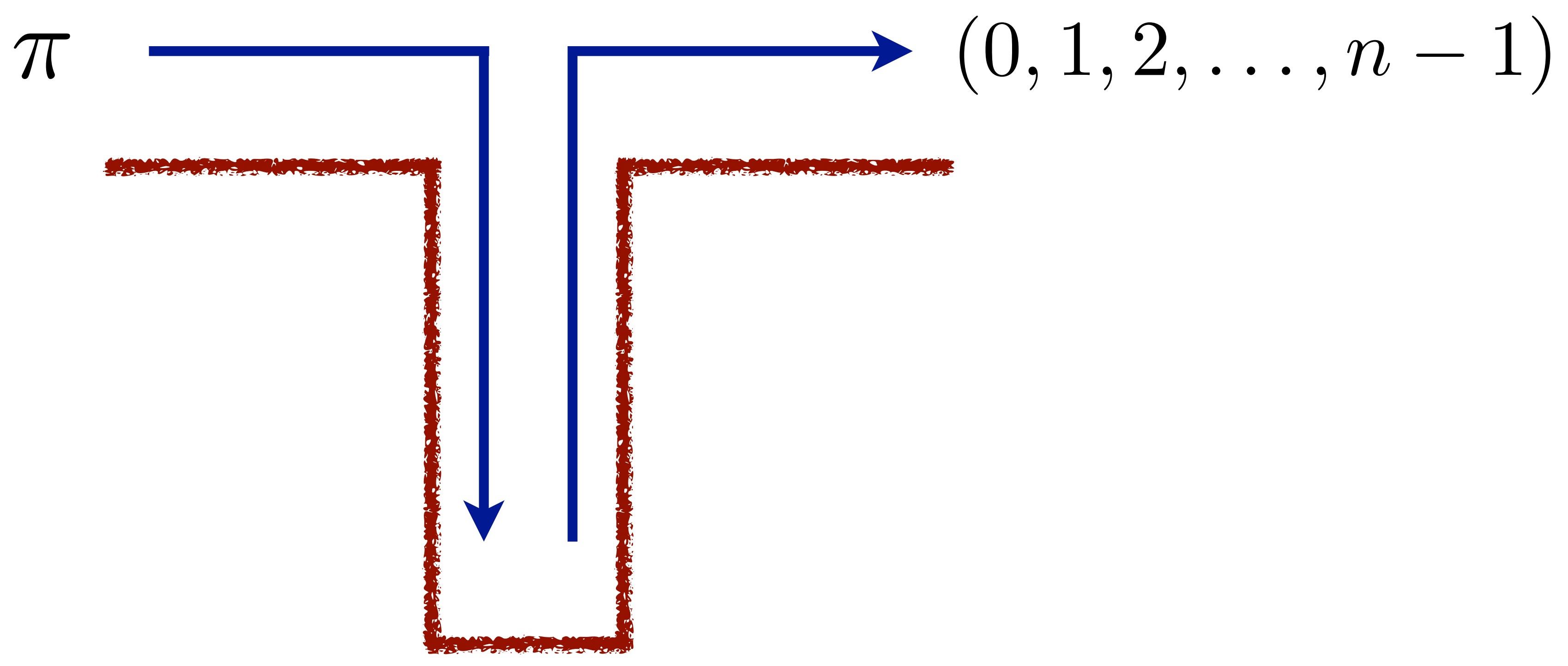
$$\| $$

$$\sum_{n\geq 0}\binom{-6}{n}\left(-x^{100}\right)^n$$

n pairs of matching parenthesis

((()))   ()(())   ()()()   (())()   (()())

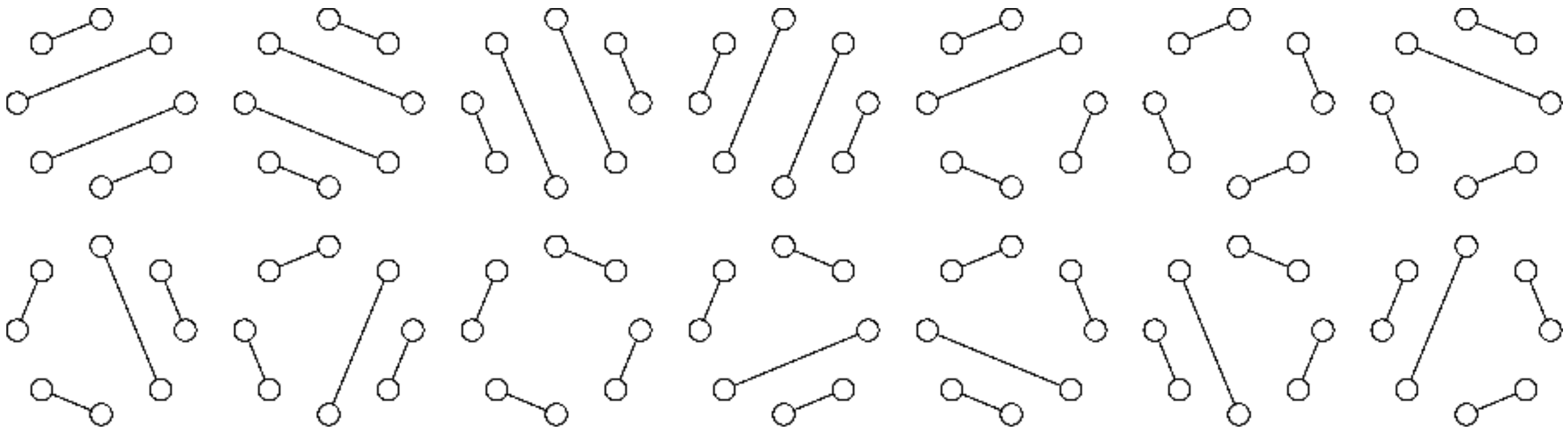
stack-sortable permutations of $[n]$



n pairs of matching parenthesis

((()))   ()(())   ()()()   (())()   (()())

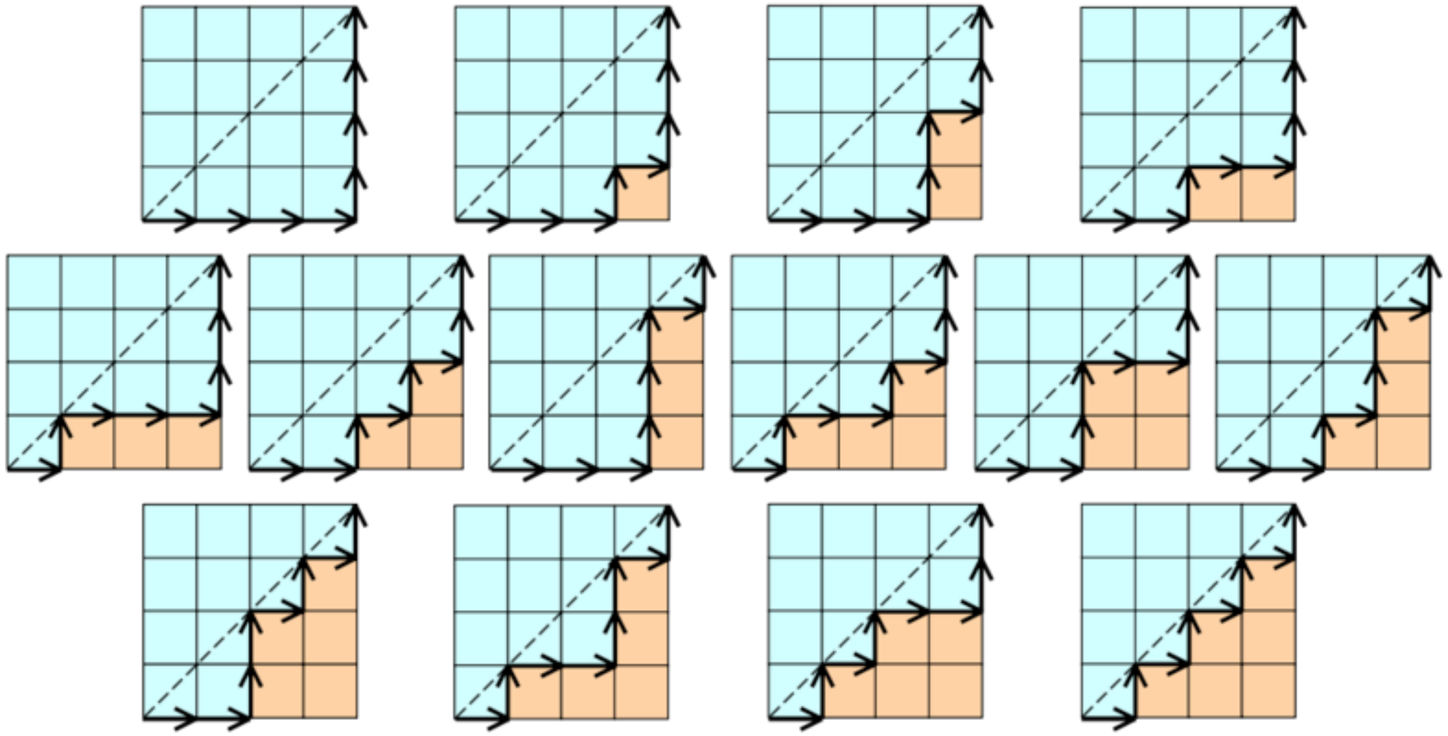
$2n$ people around a circular table shake hands, no arms cross each other



n pairs of matching parenthesis

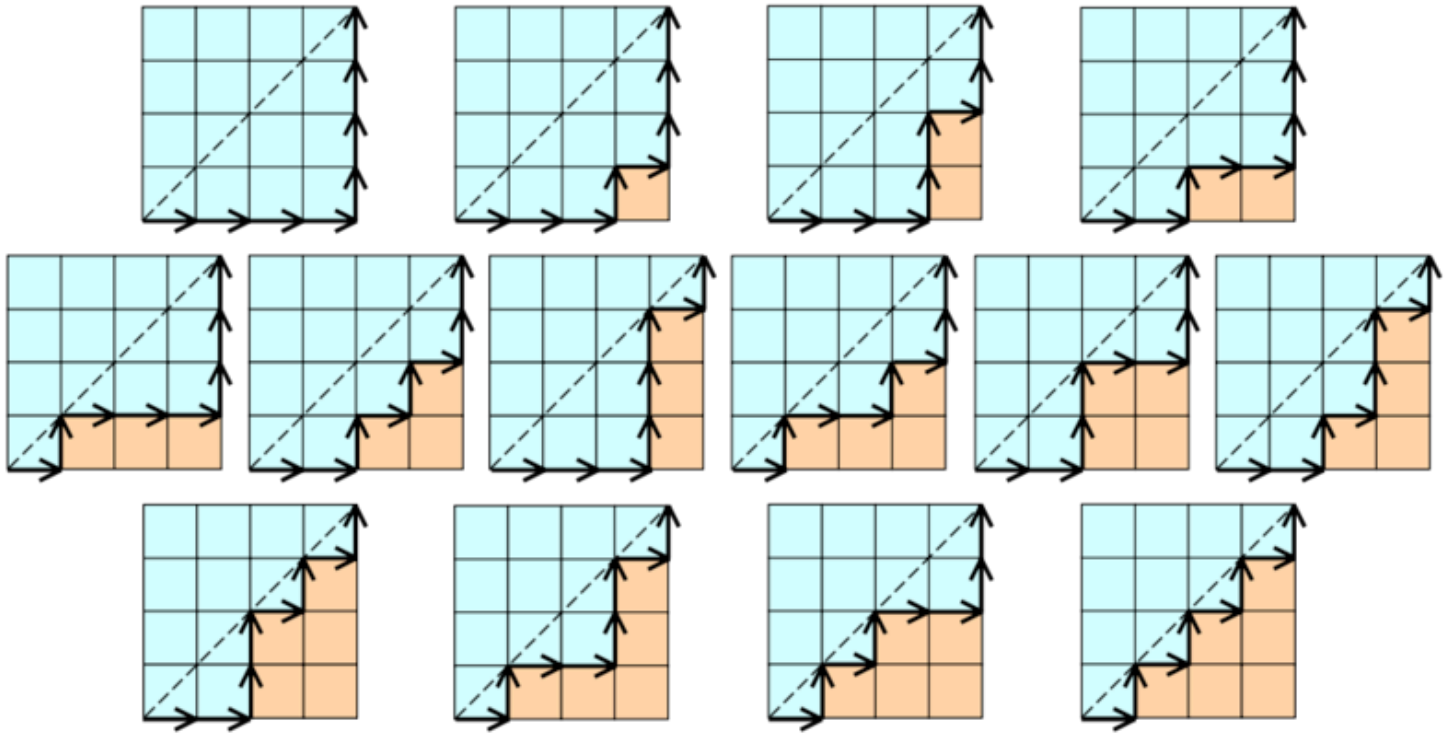
((())) ()(()) ()()() (())() (()())

monotone paths along $n \times n$ grids below diagonal



monotone paths along $n \times n$ grids

below diagonal

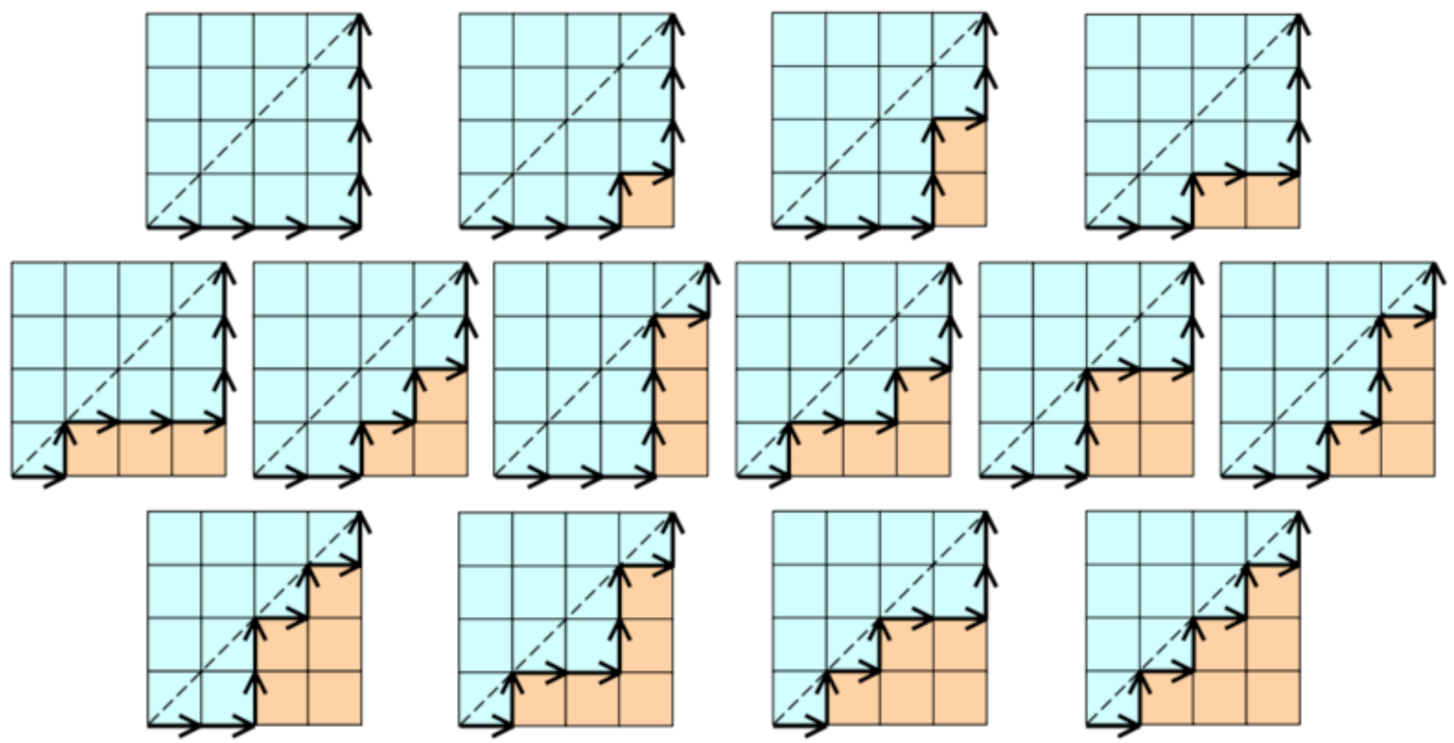


monotonically non-decreasing function:

$$f : [n] \to [n]$$

satisfying $f(i) \leq i$ for all $i \in [n]$

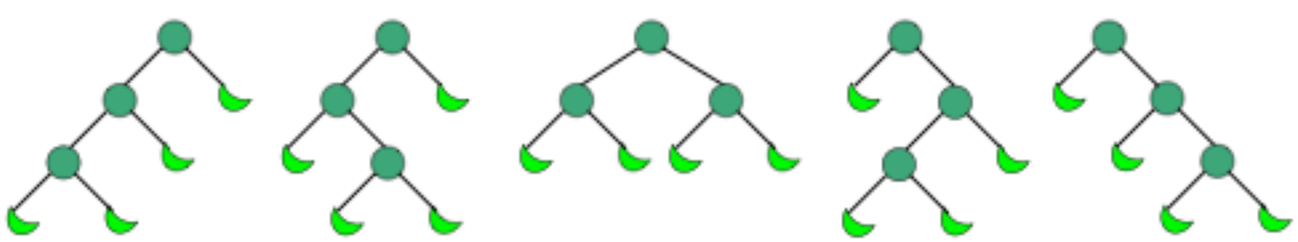
monotone paths along $n \times n$ grids
below diagonal



sequence of n $(+1)$s and n (-1)s,
every prefix-sum is nonnegative

sequence of n (+1)s and n (-1)s,
every prefix-sum is nonnegative

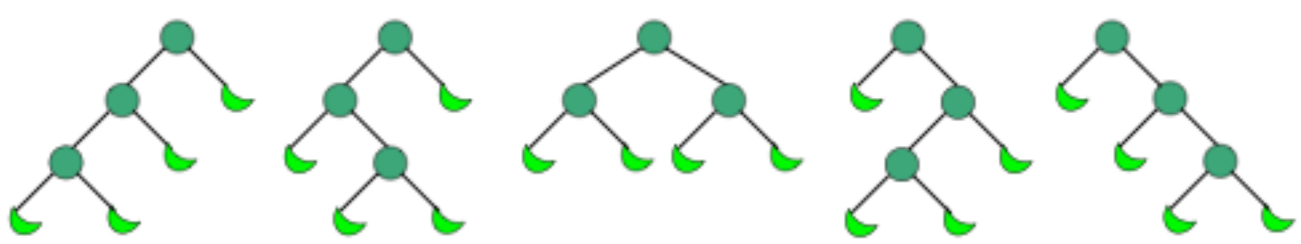
full binary trees with $n+1$ leaves



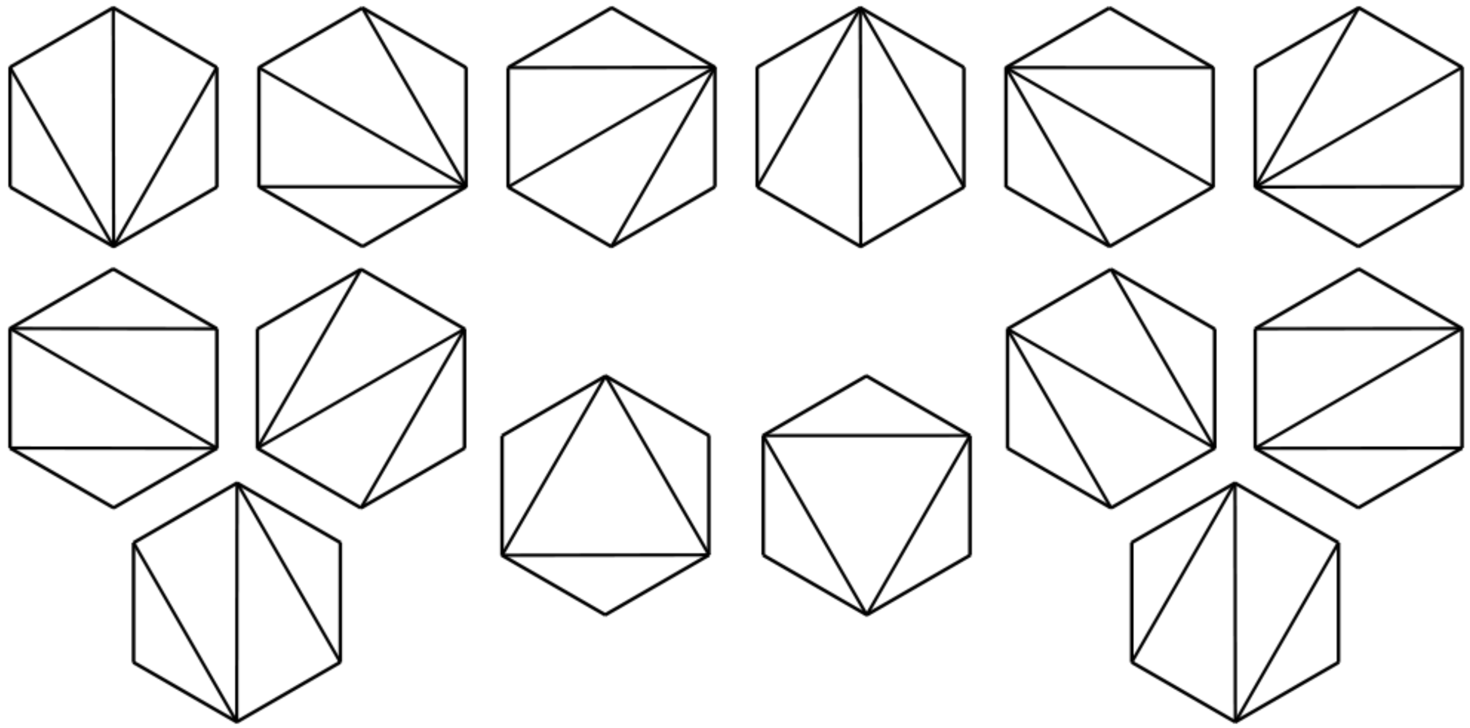
full parenthesization of $n+1$ factors

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))$$

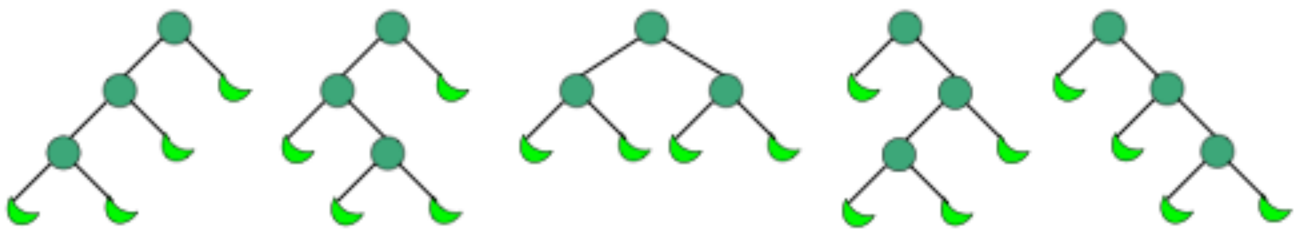
full binary trees with $n+1$ leaves



triangulations of a convex $(n+2)$-gon:

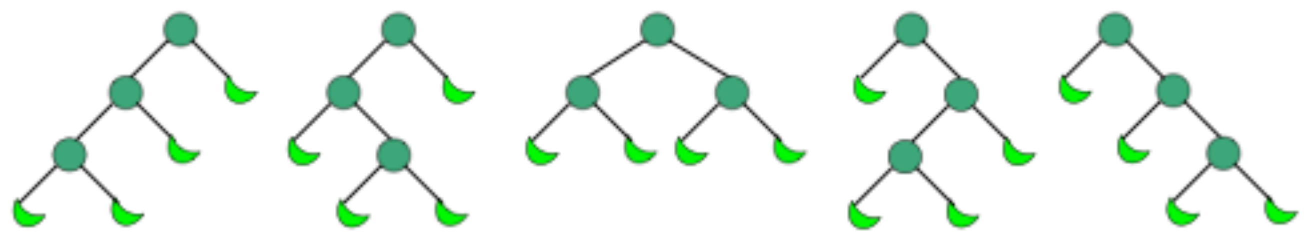


full binary trees with $n+1$ leaves



Catalan Number

C_n : # of full binary trees with $n+1$ leaves



Recursion:

C_k $\quad\quad$ C_{n-1-k}

$$C_0 = 1 \quad \text{for } n \geq 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Generating function manipulations

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution: $$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation: $$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$G(x)^2 = \sum_{n \geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Generating function manipulations

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

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Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

$$G(x)^2 = \sum_{n\geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^n$$

$$xG(x)^2 = \sum_{n\geq 0} \sum_{k=0}^{n} C_k C_{n-k} x^{n+1}$$

$$= \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Manipulation:

$$G(x) = \sum_{n\geq 0} C_n x^n = C_0 + \sum_{n\geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n = 1 + xG(x)^2$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}$$

$$G(x) = \sum_{n \geq 0} C_n x^n \quad \Rightarrow \quad \lim_{x \to 0} G(x) = C_0 = 1$$

$$\Rightarrow \quad G(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1-(1-4x)^{1/2}}{2x}$$

Expanding: Newton's formula

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n\geq 1} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n\geq 0} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n\geq 0} \binom{1/2}{n+1} (-4x)^n$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1-(1-4x)^{1/2}}{2x}$$

Expanding:

$$(1-4x)^{1/2} = 1 - 4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{1-(1-4x)^{1/2}}{2x} = 2 \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$= \sum_{n \geq 0} 2 \binom{1/2}{n+1} (-4)^n x^n$$

$$G(x) = \sum_{n\geq 0} 2\binom{1/2}{n+1}(-4)^n x^n = \sum_{n\geq 0} C_n x^n$$

$$C_n = 2\binom{1/2}{n+1}(-4)^n$$

$$= 2 \cdot \left(\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2}\right) \cdot \frac{1}{(n+1)!} \cdot (-4)^n$$

$$= \frac{2^n}{(n+1)!} \prod_{k=1}^{n} (2k-1) \quad = \frac{2^n}{(n+1)!} \prod_{k=1}^{n} \frac{(2k-1)2k}{2k}$$

$$= \frac{1}{n!(n+1)!} \prod_{k=1}^{n} (2k-1)2k \quad = \frac{(2n)!}{n!(n+1)!}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Quicksort

input: an array A of n numbers

Qsort(A):
choose a pivot $x = A[1]$; partition A into L with all $L[i] < x$, R with all $R[i] > x$; Qsort(L) and Qsort(R);

Complexity: number of comparisons

worst-case: $\Theta(n^2)$

average-case: ?

Qsort(A):

choose a pivot $x = A[1]$;
partition A into L with all $L[i] < x$,
R with all $R[i] > x$;
Qsort(L) and Qsort(R);

T_n :

average # of comparisons used by Qsort

taken over all $n!$ total orders of A

pivot: the k-th smallest number in A

$|L| = k\text{-}1 \qquad |R| = n\text{-}k$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$nT_n = \sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0}\left(\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})\right)x^n$$

$$= \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= x^2 \sum_{n\geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

$$= 2x\sum_{n\geq 0}\left(\sum_{k=0}^{n} T_k\right)x^n$$

Generating function manipulations

$$G(x) = \sum_{n \geq 0} g_n x^n \qquad F(x) = \sum_{n \geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution: $$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^{n} f_k g_{n-k} x^n$$

differentiation: $$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \underbrace{\sum_{n\geq 0} n(n-1)x^n}_{\text{blue}} + \underbrace{2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right) x^n}_{\text{orange}}$$

(blue) $$= x^2 \sum_{n\geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

(orange) $$= 2x\sum_{n\geq 0}\left(\sum_{k=0}^{n} T_k\right) x^n = 2x\sum_{n\geq 0} x^n \sum_{n\geq 0} T_n x^n = \frac{2xG(x)}{1-x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n - 1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

 $= \dfrac{2x^2}{(1-x)^3}$

 $= \dfrac{2xG(x)}{1-x}$

Generating function manipulations

$$G(x) = \sum_{n\geq 0} g_n x^n \qquad F(x) = \sum_{n\geq 0} f_n x^n$$

shift: $$x^k G(x) = \sum_{n\geq k} g_{n-k} x^n$$

addition: $$F(x) + G(x) = \sum_{n\geq 0} (f_n + g_n) x^n$$

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differentiation: $$G'(x) = \sum_{n\geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= \frac{2x^2}{(1-x)^3}$$

$$= \frac{2xG(x)}{1-x}$$

$$= x\sum_{n\geq 0}(n+1)T_{n+1}x^n = xG'(x)$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n - 1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$\sum_{n\geq 0} nT_n x^n = \sum_{n\geq 0} n(n-1)x^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} T_k\right)x^n$$

$$= xG'(x)$$

$$= \frac{2x^2}{(1-x)^3}$$

$$= \frac{2xG(x)}{1-x}$$

$$xG'(x) = \frac{2x^2}{(1-x)^3} + \frac{2x}{1-x}G(x)$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: first-order linear differential equation

$$y' + P(x)y = Q(x)$$

$$y(x) = \frac{1}{u(x)}\int u(x)Q(x)\,\mathrm{d}x \quad \textbf{with} \quad u(x) = \mathrm{e}^{\int P(x)\,\mathrm{d}x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: first-order linear differential equation

$$G(x) = e^{\int \frac{2}{1-x}\,\mathrm{d}x}\int \frac{2x}{(1-x)^3}e^{-\int\frac{2}{1-x}\,\mathrm{d}x}\,\mathrm{d}x$$

$$= \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving: $$G(x) = \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Expanding:

Taylor

$$\frac{2}{(1-x)^2} = 2\sum_{n\geq 0}(n+1)x^n \qquad \ln\frac{1}{1-x} = \sum_{n\geq 1}\frac{x^n}{n}$$

$$G(x) = 2\sum_{n\geq 0}(n+1)x^n \sum_{n\geq 1}\frac{x^n}{n} = 2\sum_{n\geq 1}\left(\sum_{k=1}^{n}(n-k+1)\frac{1}{k}\right)x^n$$

Recursion:

$$T_n = \frac{1}{n}\sum_{k=1}^{n}(n-1+T_{k-1}+T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n\geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x}G(x)$$

Solving:

$$G(x) = \frac{2}{(1-x)^2}\ln\frac{1}{1-x}$$

Expanding:

$$G(x) = 2\sum_{n\geq 1}\left(\sum_{k=1}^{n}(n-k+1)\frac{1}{k}\right)x^n$$

$$T_n = 2\sum_{k=1}^{n}(n-k+1)\frac{1}{k} = 2(n+1)\sum_{k=1}^{n}\frac{1}{k} - 2\sum_{k=1}^{n}k\cdot\frac{1}{k}$$

$$= 2(n+1)H(n) - 2n \quad = 2n\ln n + O(n)$$