

Combinatorics

南京大学
尹一通

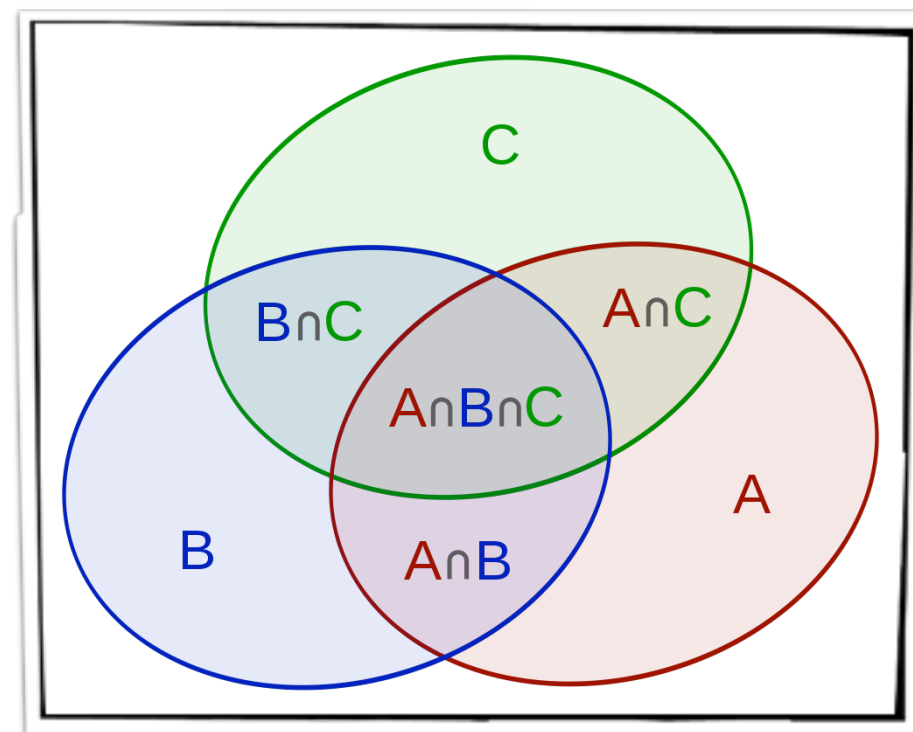
The Sieve Methods

PIE

(**P**rinciple of **I**nclusion-**E**xclusion)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$\begin{aligned} |A \cup B \cup C| = & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C| \end{aligned}$$



PIE

(Principle of Inclusion-Exclusion)

$$\begin{aligned} \left| \bigcup_{i=1}^n A_i \right| &= \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \\ &\quad \dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n| \\ &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right| \end{aligned}$$

PIE

(**P**rinciple of **I**nclusion-**E**xclusion)

$$A_1, A_2, \dots, A_n \subseteq U \quad \leftarrow \text{universe}$$

$$|\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_n}| = \left| U - \bigcup_{i=1}^n A_i \right|$$

$$= |U| - \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|$$

$$A_I = \bigcap_{i \in I} A_i \qquad A_{\emptyset} = U$$

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PIE

(Principle of Inclusion-Exclusion)

$$A_1, A_2, \dots, A_n \subseteq U \quad \leftarrow \text{universe}$$

$$|\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_n}| = S_0 - S_1 + S_2 - \dots + (-1)^n S_n$$

$$A_I = \bigcap_{i \in I} A_i \quad A_\emptyset = U$$

$$S_k = \sum_{|I|=k} |A_I| \quad S_0 = |A_\emptyset| = |U|$$

Surjections

of

$$f : [n] \xrightarrow{\text{onto}} [m]$$

$$U = [n] \rightarrow [m] \qquad A_i = [n] \rightarrow ([m] \setminus \{i\})$$

$$\left| \bigcap_{i \in [m]} \overline{A_i} \right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I|$$

$$A_I = \bigcap_{i \in I} A_i \qquad A_\emptyset = U$$

Surjections

$$U = [n] \rightarrow [m] \quad A_i = [n] \rightarrow ([m] \setminus \{i\})$$

$$A_\emptyset = U \quad A_I = \bigcap_{i \in I} A_i = [n] \rightarrow ([m] \setminus I)$$

$$|A_I| = (m - |I|)^n$$

$$\left| \bigcap_{i \in [m]} \overline{A_i} \right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I|$$

$$= \sum_{I \subseteq [m]} (-1)^{|I|} (m - |I|)^n = \sum_{k=0}^m (-1)^k \binom{m}{k} (m - k)^n$$

$$= \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} k^n$$

Surjections

$$\left| [n] \xrightarrow{\text{onto}} [m] \right| = \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} k^n$$

$$(f^{-1}(0), f^{-1}(1), \dots, f^{-1}(m-1))$$

ordered m -partition of $[n]$

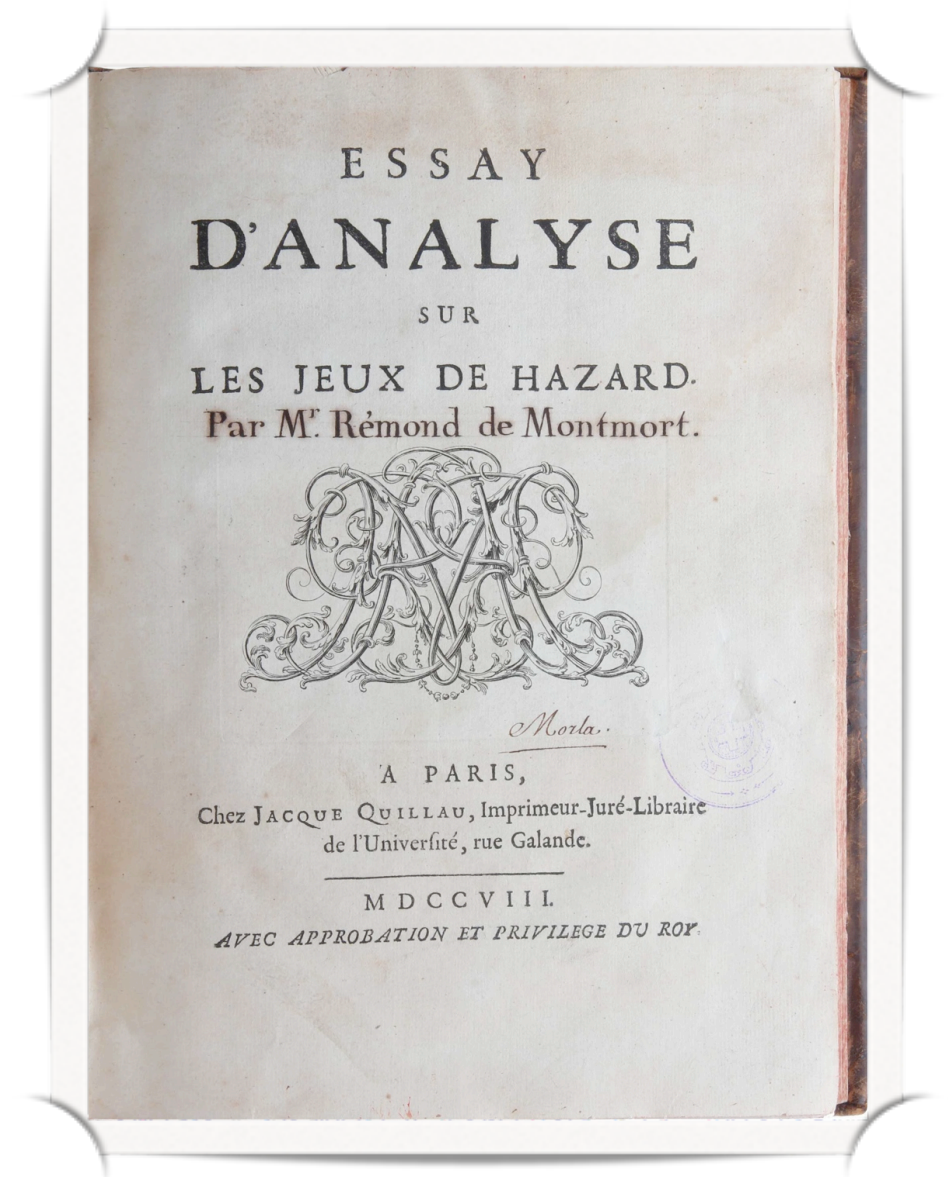
$$\left| [n] \xrightarrow{\text{onto}} [m] \right| = m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}$$
$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \frac{1}{m!} \sum_{k=1}^m (-1)^{m-k} \binom{m}{k} k^n$$

Derangement

les problèmes des rencontres :

Two decks, A and B , of cards:
The cards of A are laid out in a row,
and those of B are placed at random,
one at the top on each card of A .

What is the probability that
no 2 cards are the same in each pair?



Derangement

permutation π of $[n]$

$$\forall i \in [n], \quad \pi(i) \neq i$$

“permutations with no fixed point” $!n$

U : permutations of $[n]$

Derangement

permutation π of $[n]$

$$\forall i \in [n], \quad \pi(i) \neq i$$

“permutations with no fixed point” $!n$

$U = S_n$ symmetric group $A_i = \{\pi \mid \pi(i) = i\}$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

$$A_I = \{\pi \mid \forall i \in I, \pi(i) = i\} \quad |A_I| = (n - |I|)!$$

Derangement

$$U = S_n \quad A_i = \{\pi \mid \pi(i) = i\}$$

$$A_I = \{\pi \mid \forall i \in I, \pi(i) = i\} \quad |A_I| = (n - |I|)!$$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^n (-1)^k \binom{n}{k} (n - k)!$$

$$= n! \sum_{k=0}^n \frac{(-1)^k}{k!} \approx \frac{n!}{e}$$

Permutations with restricted positions

permutation π of $[n]$

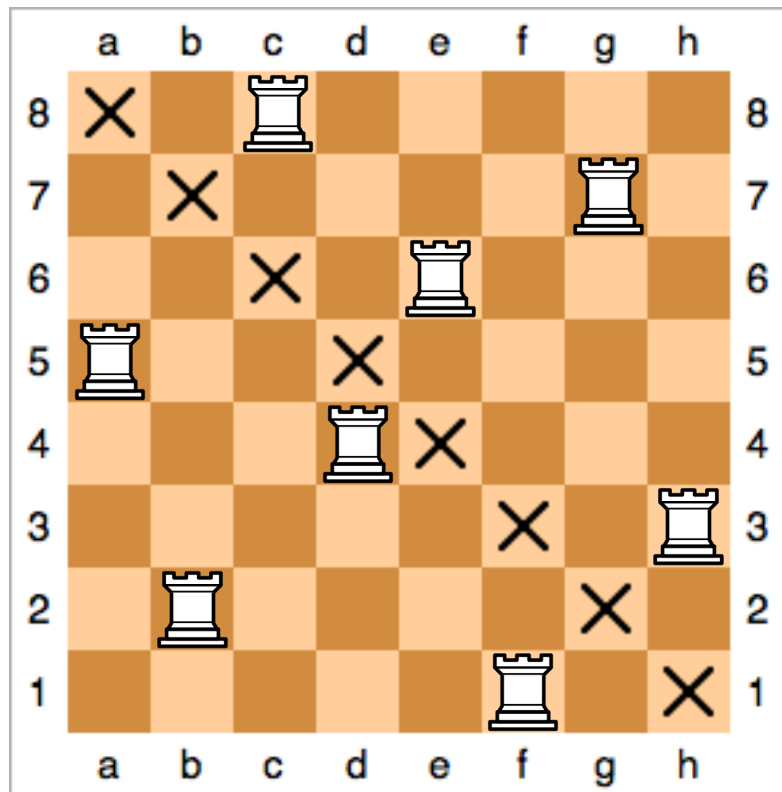
derangement: $\forall i \in [n], \pi(i) \neq i$

generally: $\pi(i_1) \neq j_1, \pi(i_2) \neq j_2, \dots$

forbidden positions $B \subseteq [n] \times [n]$

$$\forall i \in [n], (i, \pi(i)) \notin B$$

Chess board



permutation π of $[n]$

$$\{(i, \pi(i)) \mid i \in [n]\}$$

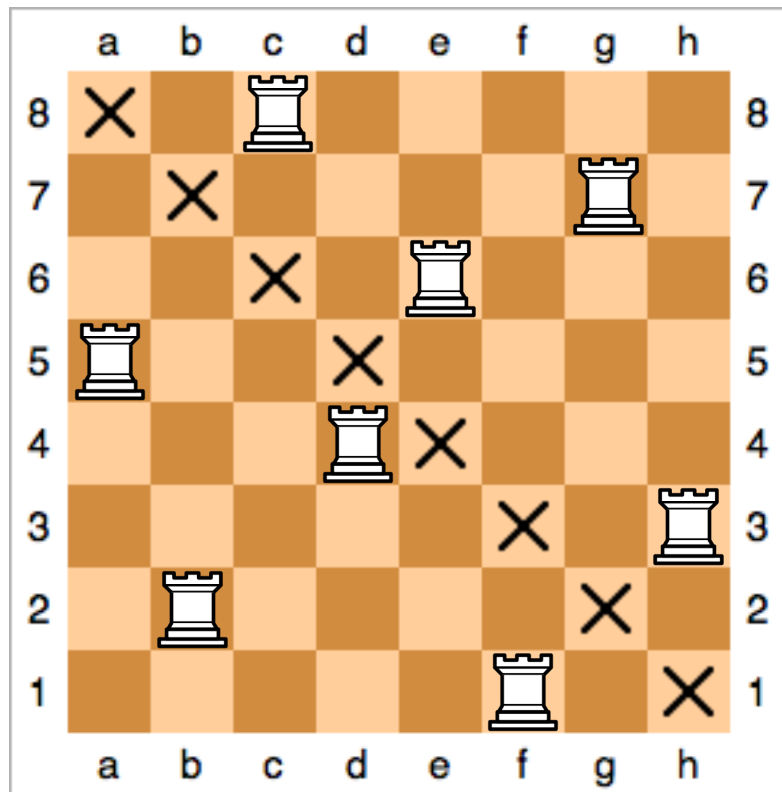
“A placement of
non-attacking rooks”

forbidden positions $B \subseteq [n] \times [n]$

derangement:

$$B = \{(i, i) \mid i \in [n]\}$$

Chess board



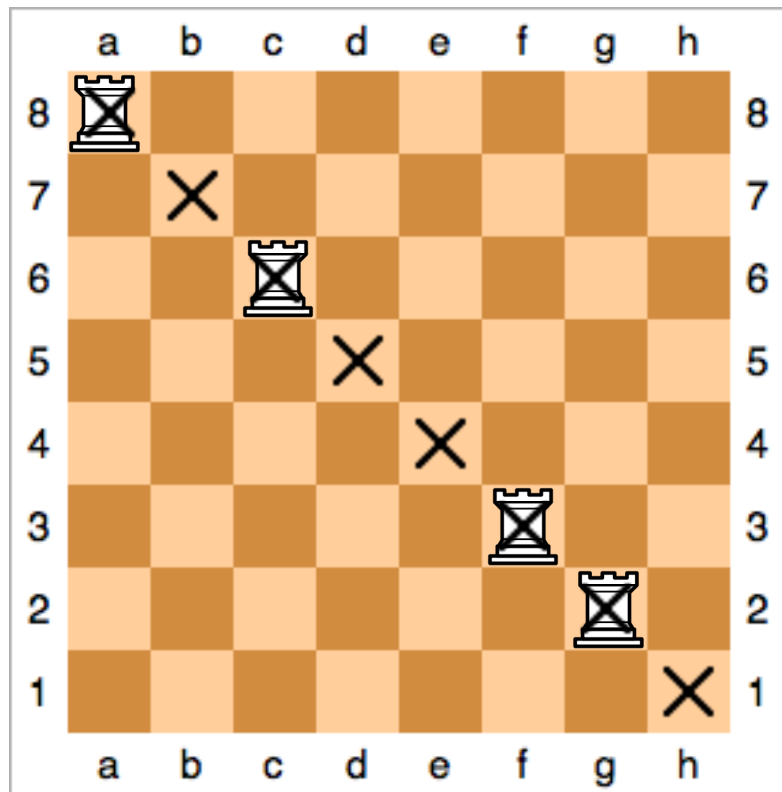
For a particular set of forbidden positions

$$B \subseteq [n] \times [n]$$

N_0 :

the # of placements of n non-attacking rooks?

Chess board



For a particular set of forbidden positions

$$B \subseteq [n] \times [n]$$

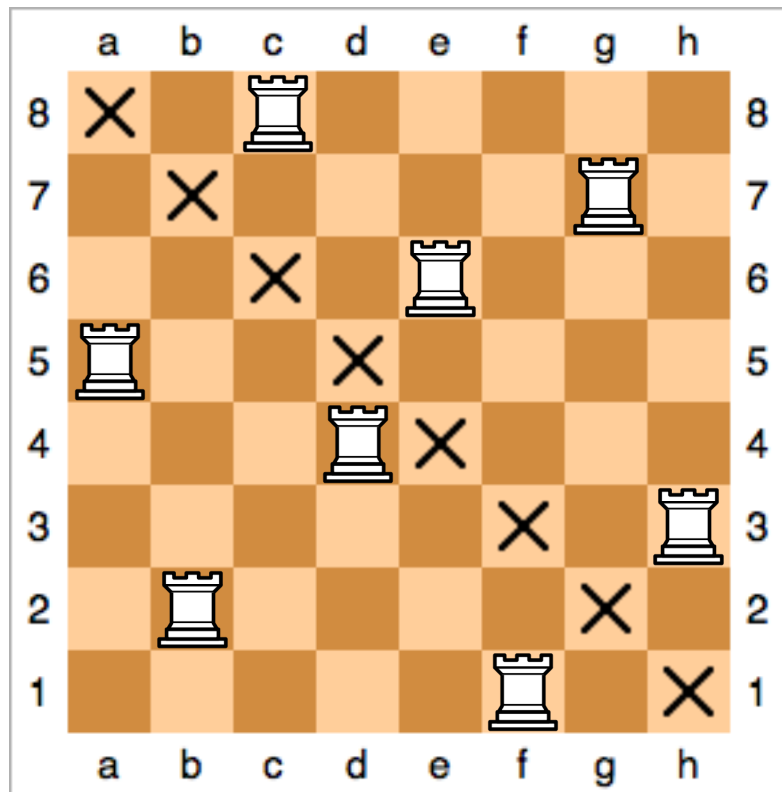
$r_k :$

of ways of placing k non-attacking rooks in B

$N_0 :$

the # of placements of n non-attacking rooks?

Chess board



$$B \subseteq [n] \times [n]$$

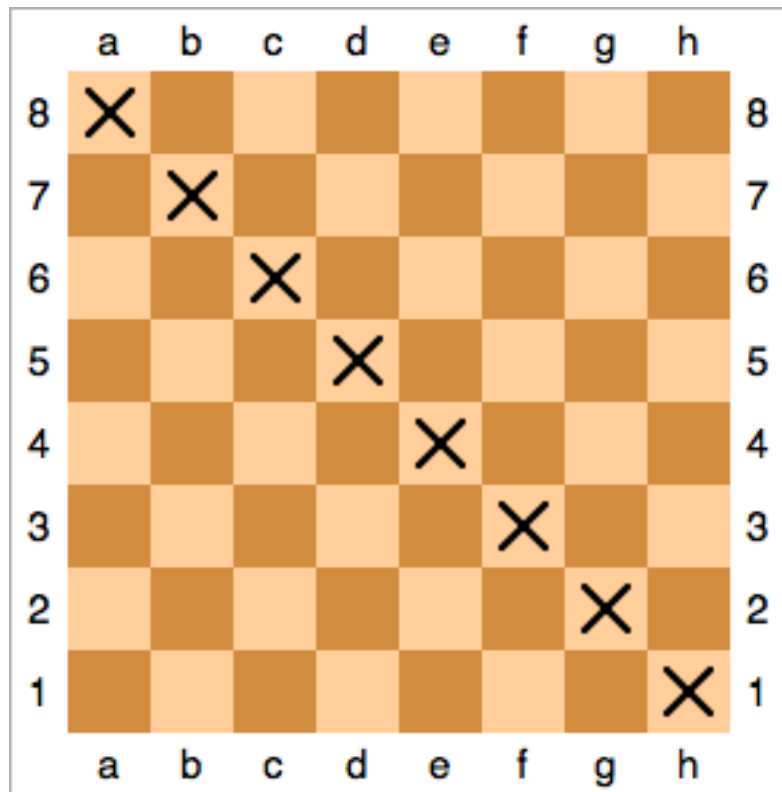
$r_k :$

of ways of placing k
non-attacking rooks in B

$N_0 :$ # of placements of n non-attacking rooks

$$N_0 = \sum_{k=0}^n (-1)^k r_k (n - k)!$$

Derangement again



$$B = \{(i, i) \mid i \in [n]\}$$

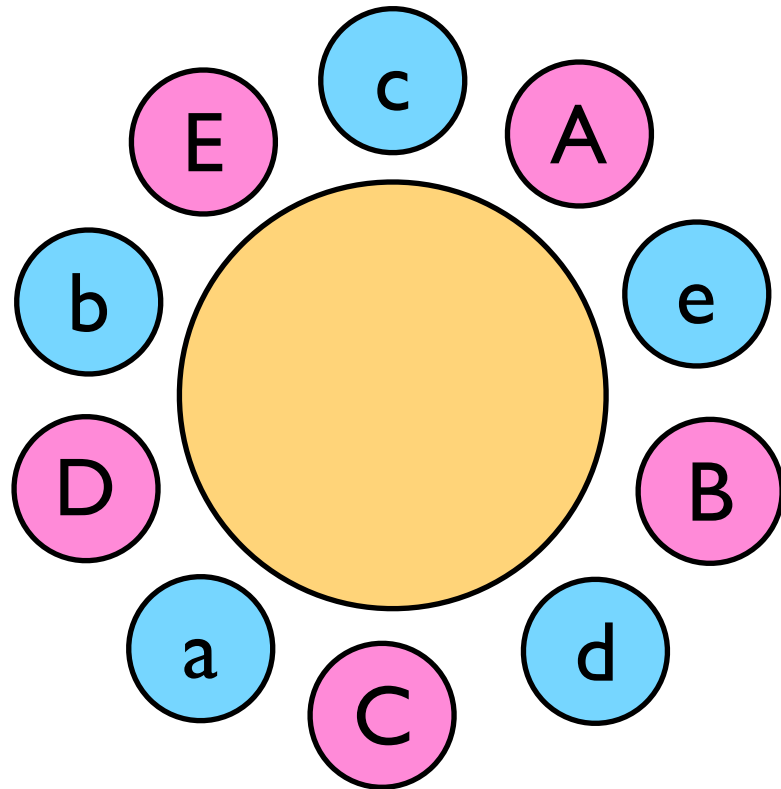
r_k : # of ways of placing k non-attacking rooks in B

$$\binom{n}{k}$$

$$N_0 = \sum_{k=0}^n (-1)^k r_k (n-k)! = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum_{k=0}^n (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^n (-1)^k \frac{1}{k!} \approx \frac{n!}{e}$$

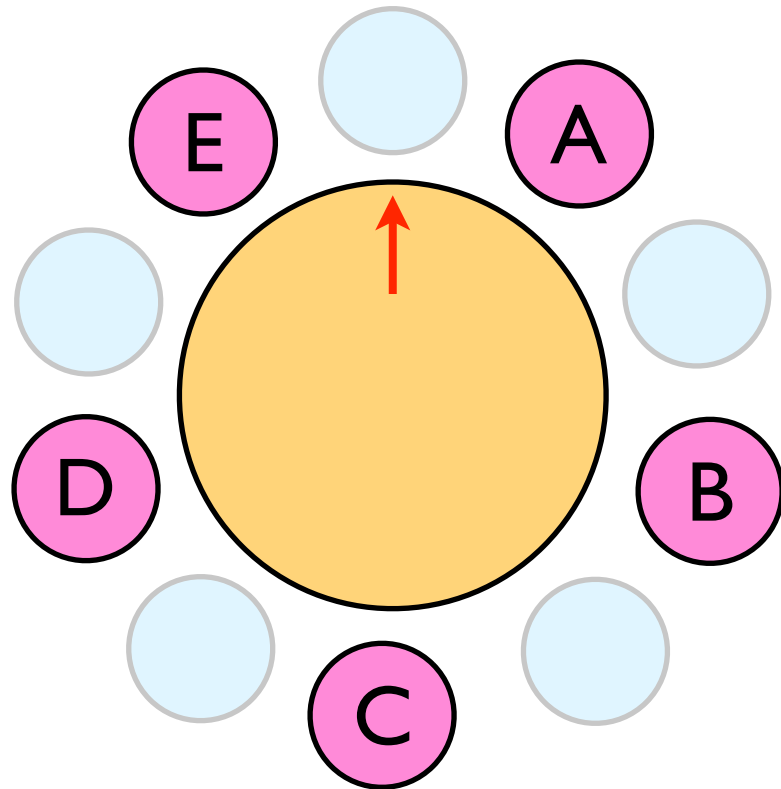
Problème des ménages



n couples sit around a table

- male-female alternative
- no one sit next to spouse

Problème des ménages



“Lady first!”

$2(n!)$ ways

“Gentlemen, please sit.”

permutation π of $[n]$

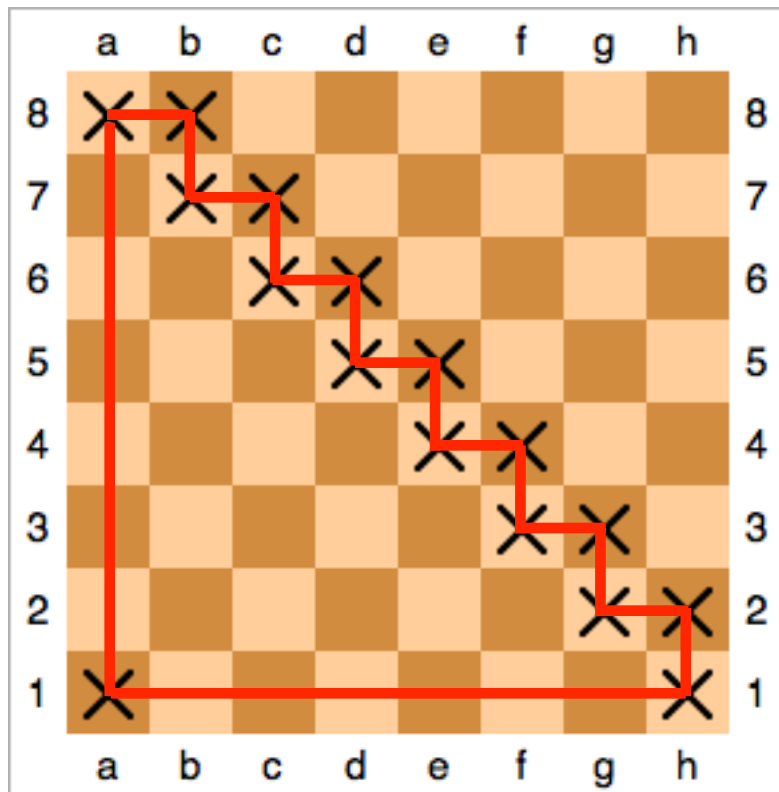
i : husband of the lady at the i -th position

$\pi(i)$: his seat

$$\pi(i) \neq i$$

$$\pi(i) \not\equiv i + 1 \pmod{n}$$

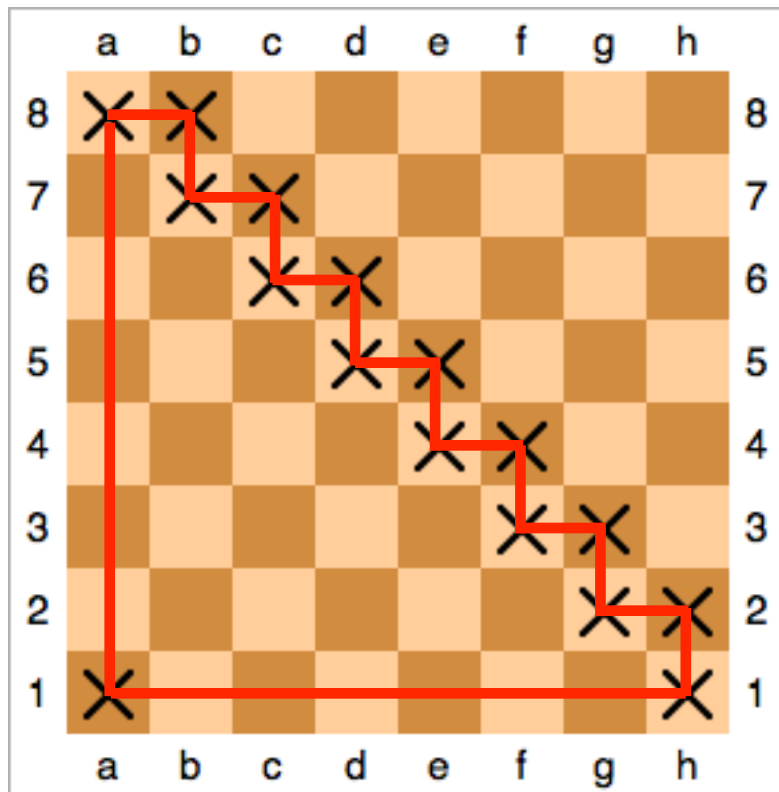
Problème des ménages



$$B = \{(i, i), (i, (i + 1) \bmod n)\}$$

r_k : # of ways of placing k
non-attacking rooks in B

Problème des ménages

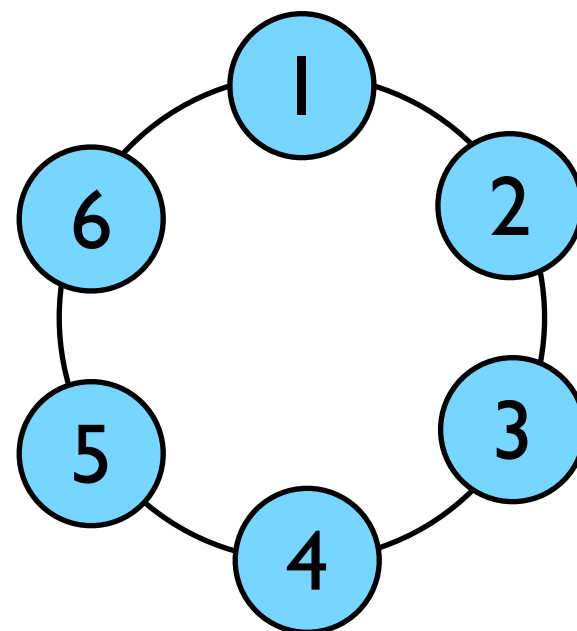


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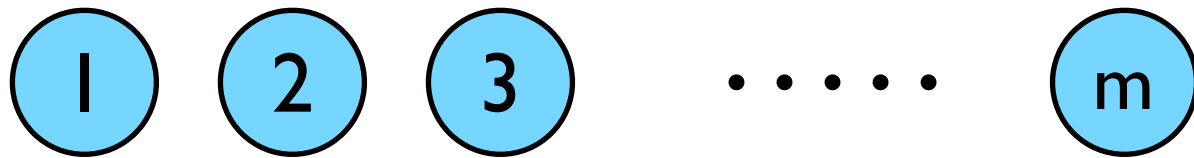
of ways of choosing k
 r_k : non-consecutive points
 from a circle of $2n$ points

$2n$ objects in a **circle**

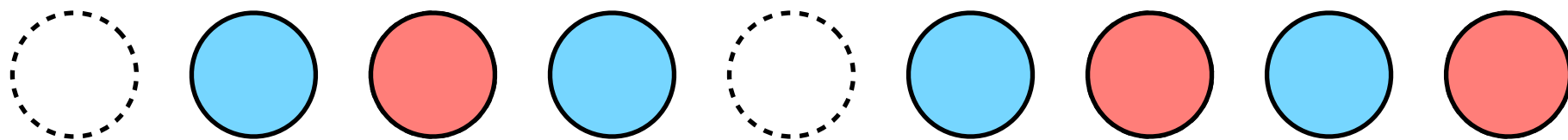
choose k
non-consecutive objects



m objects in a line



$L(m,k)$: choose k non-consecutive objects



$m-k$ objects, $m-k+1$ space

choose k from $m-k+1$ space

$$L(m,k) = \binom{m-k+1}{k}$$

m objects in a **circle**

$C(m,k)$: choose k non-consecutive objects

$(m-k)C(m,k)$:

||

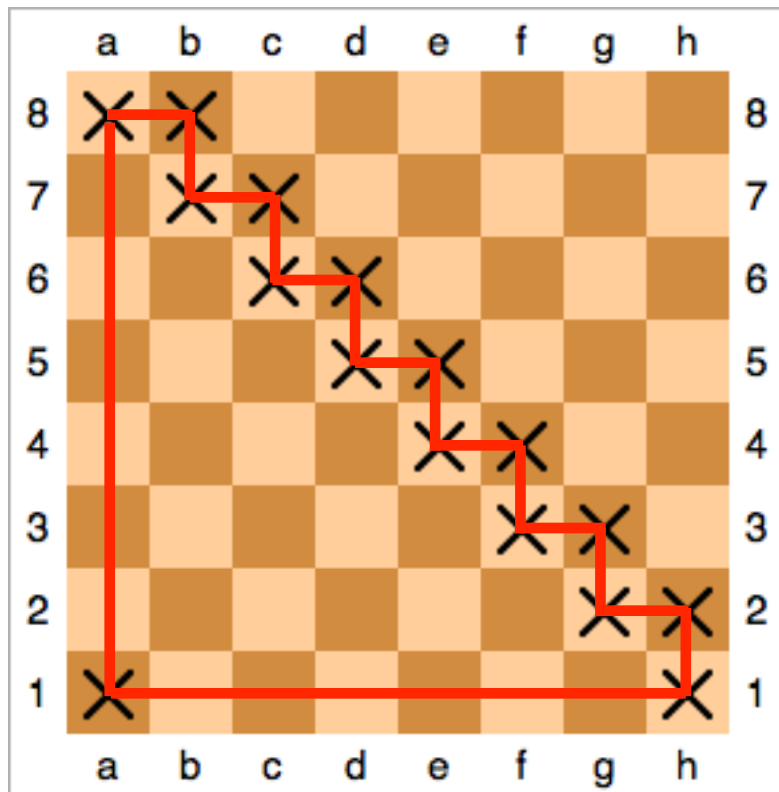
1. choose k non-consecutive objects from a circle
2. mark one of the remaining objects

$m L(m-1,k)$:

1. mark one object in the circle, cut the circle by removing the object
2. choose k non-consecutive objects from the $m-1$ objects in a line

$$C(m,k) = \frac{m}{m-k} \binom{m-k}{k}$$

Problème des ménages



$$B = \{(i, i), (i, (i + 1) \bmod n)\}$$

r_k : # of ways of choosing k
non-consecutive points
from a circle of $2n$ points

$$\frac{2n}{2n - k} \binom{2n - k}{k}$$

$$N_0 = \sum_{k=0}^n (-1)^k r_k (n - k)!$$

$$= \sum_{k=0}^n (-1)^k \frac{2n}{2n - k} \binom{2n - k}{k} (n - k)!$$

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$$A_I = \bigcap_{i \in I} A_i \qquad A_\emptyset = U$$

Inversion

V : 2^n -dimensional vector space of all mappings

$$f : 2^{[n]} \rightarrow \mathbb{N}$$

linear transformation $\phi : V \rightarrow V$

$$\forall S \subseteq [n], \quad \phi f(S) \triangleq \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} f(T)$$

then its inverse:

$$\forall S \subseteq [n], \quad \phi^{-1} f(S) = \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} (-1)^{|T \setminus S|} f(T)$$

$$\phi f(S) \triangleq \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} f(T) \qquad \phi^{-1} f(S) = \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} (-1)^{|T \setminus S|} f(T)$$

$$A_1, A_2, \dots, A_n \subseteq U \qquad I \subseteq [n]$$

$$f_{=}(I) = |\{x \in U \mid \forall i \in I, x \in A_i, \forall j \notin I, x \notin A_j\}|$$

$$= \left| \left(\bigcap_{i \in I} A_i \right) \setminus \left(\bigcup_{j \notin I} A_j \right) \right|$$

$$f_{\geq}(I) = \sum_{\substack{J \supseteq I \\ J \subseteq [n]}} f_{=}(J) = \left| \bigcap_{i \in I} A_i \right| = |A_I|$$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = f_{=}(\emptyset) = \sum_{\substack{I \supseteq \emptyset \\ I \subseteq [n]}} (-1)^{|I \setminus \emptyset|} f_{\geq}(I) = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

PIE

(**P**rinciple of **I**nclusion-**E**xclusion)

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $T \subseteq S$

$$\sum_{T \subseteq I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = T \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$A_1 = A_2 = \cdots = A_n = \{1\}$$

$$1 = \left| \bigcup_{i=1}^n A_i \right| = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1} |A_I|$$

$$A_I = \bigcap_{i \in I} A_i = \{1\}$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \dots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1, \dots, n\}} (-1)^{n-|I|} = 0$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \dots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1, \dots, n\}} (-1)^{-|I|} = 0$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \dots, n\} \neq \emptyset$

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PIE

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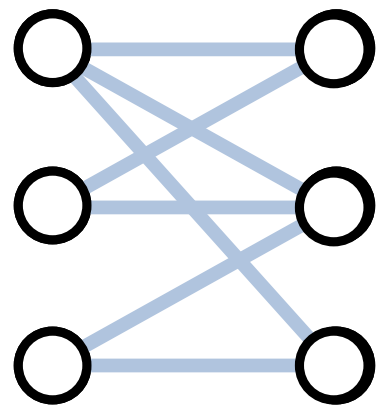
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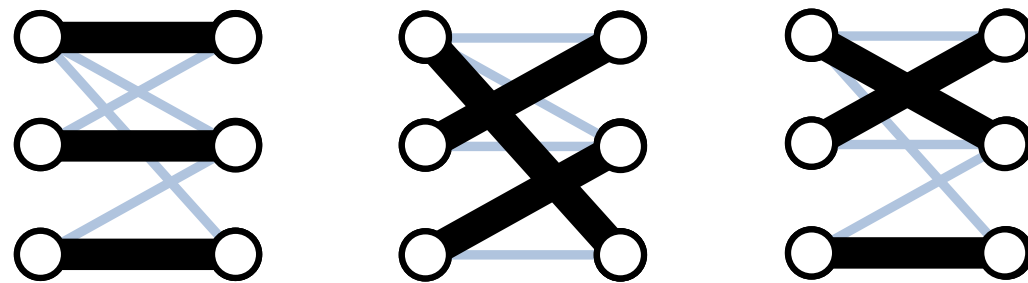
Bipartite Perfect Matching

bipartite graph



$$G([n], [n], E)$$

perfect matchings



permutation π of $[n]$ **s.t.** $(i, \pi(i)) \in E$

$n \times n$ matrix A :

$$A_{i,j} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

of P.M. in G

$$= \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)}$$

Permanent

$n \times n$ matrix A :

$$\text{perm}(A) = \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)}$$

#P-hard

determinant:

$$\det(A) = \sum_{\pi \in S_n} (-1)^{r(\pi)} \prod_{i \in [n]} A_{i, \pi(i)}$$

poly-time by Gaussian elimination

Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

term in $\prod_{i \in [n]} A_{i, f(i)}$ for some $f : [n] \rightarrow [n]$

$$T = f([n]) \subseteq I$$

coefficient of $\prod_{i \in [n]} A_{i, f(i)}$ in $\sum_{T \subseteq I \subseteq [n]} (-1)^{n-|I|}$:

$$\sum_{T \subseteq I \subseteq [n]} (-1)^{n-|I|} = \begin{cases} 1 & T = [n] \leftarrow f \text{ is a permutation} \\ 0 & o.w. \end{cases}$$

Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i, \pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

$O(n!)$ time

$O(n2^n)$ time

Sieve of Eratosthenes

	2	3	4	5	6	7	8	9	10	Prime numbers			
11	12	13	14	15	16	17	18	19	20	2	3	5	7
21	22	23	24	25	26	27	28	29	30	11	13	17	19
31	32	33	34	35	36	37	38	39	40	23	29	31	37
41	42	43	44	45	46	47	48	49	50	41	43	47	53
51	52	53	54	55	56	57	58	59	60	59	61	67	71
61	62	63	64	65	66	67	68	69	70	73	79	83	89
71	72	73	74	75	76	77	78	79	80	97	101	103	107
81	82	83	84	85	86	87	88	89	90	109	113		
91	92	93	94	95	96	97	98	99	100				
101	102	103	104	105	106	107	108	109	110				
111	112	113	114	115	116	117	118	119	120				

Euler Totient Function

$\phi(n)$: ~~# of $a \in \{1, 2, \dots, n\}$ relative prime to n~~
~~# of $a \in \{1, 2, \dots, n\}$ relative prime to n~~

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$\phi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$

Euler Totient Function

$$\phi(n) = |\{1 \leq a \leq n \mid \gcd(a, n) = 1\}|$$

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Universe: $U = \{1, 2, \dots, n\}$

$$i = 1, 2, \dots, r \quad A_i = \{1 \leq a \leq n \mid p_i \mid a\}$$

$$I \subseteq \{1, 2, \dots, r\} \quad A_I = \{1 \leq a \leq n \mid \forall i \in I, p_i \mid a\}$$

$$|A_i| = \frac{n}{p_i} \quad |A_I| = \frac{n}{\prod_{i \in I} p_i}$$

$$\phi(n) = \left| \bigcap_{i \in \{1, \dots, r\}} \overline{A_i} \right| = \sum_{I \subseteq \{1, \dots, r\}} (-1)^{|I|} |A_I|$$

Euler Totient Function

$$\phi(n) = |\{1 \leq a \leq n \mid \gcd(a, n) = 1\}|$$

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$I \subseteq \{1, 2, \dots, r\} \quad A_I = \{1 \leq a \leq n \mid \forall i \in I, p_i \mid a\}$$

$$|A_I| = \frac{n}{\prod_{i \in I} p_i}$$

$$\begin{aligned} \phi(n) &= \sum_{I \subseteq \{1, \dots, r\}} (-1)^{|I|} |A_I| \\ &= n \sum_{k=0}^r \sum_{I \in \binom{\{1, \dots, r\}}{k}} \frac{(-1)^{|I|}}{\prod_{i \in I} p_i} = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right) \end{aligned}$$

Euler Totient Function

$$\phi(n) = |\{1 \leq a \leq n \mid \gcd(a, n) = 1\}|$$

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$\phi(n) = n \prod_{i=1}^r \left(1 - \frac{1}{p_i}\right)$$