Combinatorics

Ramsey Theory

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Ramsey's Theorem

Frank P. Ramsey
(1903-1930)

"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

Color edges of K_6 with 2 colors.

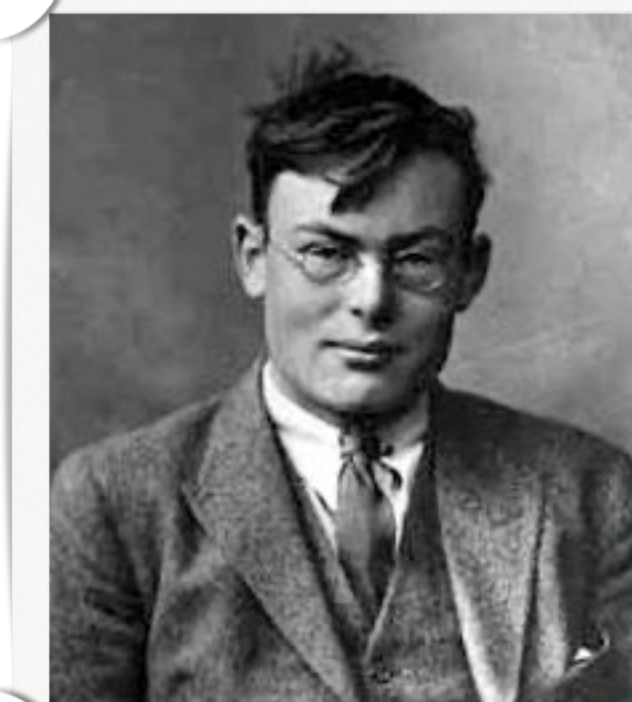
There must be a monochromatic K_3.

ON A PROBLEM OF FORMAL LOGIC

By F. P. Ramsey.

[Received 28 November, 1928.—Read 13 December, 1928.]

This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.



Frank P. Ramsey
(1903-1930)

$R(k,l) \triangleq$ the smallest integer satisfying:

if $n \geq R(k,l)$, then no matter how to color edges of K_n with ■ and ■, there must exist a red K_k or a blue K_l.

2-coloring of K_n

$$f : \binom{[n]}{2} \to \{\texttt{red}, \texttt{blue}\}$$

Ramsey Theorem

$R(k, l)$ is finite.

$R(3,3) = 6$

Frank P. Ramsey
(1903-1930)

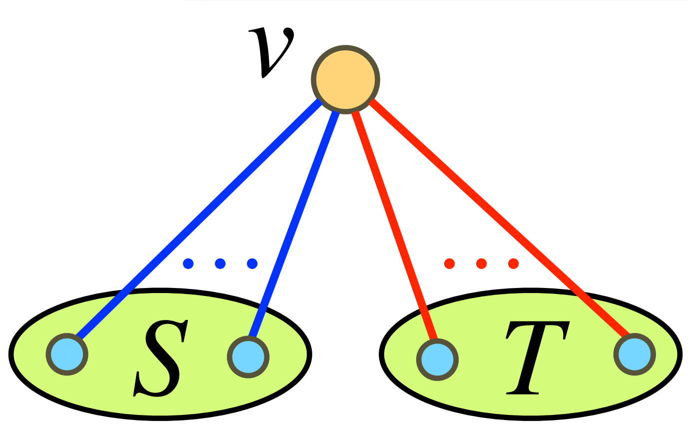
if $n \geq R(k,l)$, then no matter how to color edges of K_n with ■ and ■, there must exist a red K_k or a blue K_l.

$$R(k,2) = k \;; \quad R(2,l) = l \;;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

if $n \geq R(k,l)$, then no matter how to color edges of K_n with red and blue, there must exist a red K_k or a blue K_l.

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$



take $n = R(k,l\text{-}1) + R(k\text{-}1,l)$

arbitrary vertex v

$$|S| + |T| + 1 = n = R(k,l\text{-}1) + R(k\text{-}1,l)$$

or

$|S| \geq R(k,l\text{-}1)$ → or: K_k in S; $K_{l\text{-}1}$ in S $\xrightarrow{+v}$ K_l

$|T| \geq R(k\text{-}1,l)$ → or: $K_{k\text{-}1}$ in T $\xrightarrow{+v}$ K_k; K_l in T

if $n \geq R(k,l)$, then no matter how to color edges of K_n with ■ and ■, there must exist a red K_k or a blue K_l.

$$R(k,2) = k\ ; \quad R(2,l) = l\ ;$$

$$R(k,l) \leq R(k,l-1) + R(k-1,l)$$

Ramsey Theorem $R(k,l)$ is finite.

By induction:

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

$$R(k,k) \geq n$$

"$\exists$ a 2-coloring of K_n with no monochromatic K_k."

a random 2-coloring of K_n:

$\forall \{u, v\} \in K_n$, *uniformly* and *independently* $\begin{cases} uv \\ uv \end{cases}$

$\forall S \in \binom{[n]}{k}$ event A_S: S is a monochromatic K_k

$$\Pr[A_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent $\Leftrightarrow$ $|S \cap T| \geq 2$

max degree of dependency graph $d \leq \binom{k}{2}\binom{n}{k-2}$

To prove: $\Pr\left[\bigwedge_{S \in \binom{[n]}{k}} \overline{A_S}\right] > 0$

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \le p$
- $ep(d+1) \le 1$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

$$\Pr[A_S] = 2^{1-\binom{k}{2}}$$

$$d \le \binom{k}{2}\binom{n}{k-2}$$

$\Rightarrow$ for some $n = ck2^{k/2}$ with constant c

$$e2^{1-\binom{k}{2}}(d+1) \le 1$$

To prove: $\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$

$$R(k,k) \ge n = \Omega(k2^{k/2})$$

Ramsey Number

$$\Omega\left(k2^{k/2}\right) \le R(k,k) \le \binom{2k-2}{k-1} = \mathrm{O}\left(\frac{4^k}{\sqrt{k}}\right)$$

R(k,l)	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	[40,42]
4				**18**	25	[36,40]	[49,58]	[59,79]	[73,106]	[92,136]
5					**[43,48]**	[58,85]	[80,133]	[101,194]	[133,282]	[149,381]
6						**[102,161]**	[115,273]	[134,427]	[183,656]	[204,949]
7							**[205,497]**	[219,840]	[252,1379]	[292,2134]
8								**[282,1532]**	[329,2683]	[343,4432]
9									**[565,6588]**	[581,12677]
10										**[798,23556]**

Multicolor

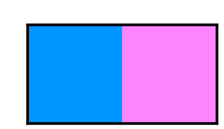
> if $n \geq R(k,l)$, for any 2-coloring of edges of K_n,
> there exists a red K_k or a blue K_l.

$$R(r; k_1, k_2, \ldots, k_r)$$

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,
for any r-coloring of edges of K_n, for some $i \in [r]$
there exists a k_i-clique **monochromatic** with color i.

$$R(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R(r-1; k_1, \ldots, k_{r-2}, R(2; k_{r-1}, k_r))$$

the mixing color trick:

color 

Multicolor

if $n \geq R(k,l)$, for any 2-coloring of edges of K_n,
there exists a red K_k or a blue K_l.

$R(r; k_1, k_2, \ldots, k_r)$

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,
for any r-coloring of edges of K_n, for some $i \in [r]$
there exists a k_i-clique **monochromatic** with color i.

Ramsey Theorem

$R(r; k_1, k_2, \ldots, k_r)$ is finite.

Hypergraph

if $n \geq R(r; k_1, k_2, \ldots, k_r)$,

for any r-coloring of edges of K_n, for some $i \in [r]$

there exists a k_i-clique **monochromatic** with color i.

complete t-uniform **hypergraph** $\binom{[n]}{t}$

r-coloring $\quad f : \binom{[n]}{t} \to \{1, 2, \ldots, r\}$

Hypergraph

if $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

for any r-coloring of $\binom{[n]}{t}$, there exists a monochromatic $\binom{S}{t}$ with color i and $|S| = k_i$ for some $i \in \{1,2,\ldots,r\}$

complete t-uniform **hypergraph** $\binom{[n]}{t}$

r-coloring $\quad f : \binom{[n]}{t} \to \{1, 2, \ldots, r\}$

Partition of Set Family

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$, there exists an $S \subseteq [n]$ such that $|S| = k_i$ and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \ldots, r\}$.

Erdős-Rado partition arrow

$$n \longrightarrow (k_1, k_2, \ldots, k_r)^t$$

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

for any r-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$, there exists an $S \subseteq [n]$ such that $|S| = k_i$ and $\binom{S}{t} \subseteq C_i$ for some $i \in \{1, 2, \ldots, r\}$.

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k, l) \leq R_{t-1}(R_t(k-1, l), R_t(k, l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1, l), R_t(k, l-1)) + 1$$

goal: $\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$

$\exists \binom{X}{t}, |X| = k$ or $\binom{Y}{t}, |Y| = l$

$$R_t(k,l) \le R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\mathtt{red}, \mathtt{blue}\}$$

remove n from $[n]$, consider $\binom{[n-1]}{t-1}$

($[n] = \{1, 2, ..., n\}$)

define $f' : \binom{[n-1]}{t-1} \to \{\mathtt{red}, \mathtt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

$$n\text{-}1 = R_{t-1}(R_t(k-1,l), R_t(k,l-1))$$

or $\begin{cases} \exists S \subseteq [n-1], |S| = R_t(k-1,l), \binom{S}{t-1} \text{ by } f' \\ \exists T \subseteq [n-1], |T| = R_t(k,l-1), \binom{T}{t-1} \text{ by } f' \end{cases}$

$$R_t(k,l) \le R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$n = R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

$$\forall f : \binom{[n]}{t} \to \{\texttt{red}, \texttt{blue}\}$$

define $f' : \binom{[n-1]}{t-1} \to \{\texttt{red}, \texttt{blue}\}$

$$\forall A \in \binom{[n-1]}{t-1}, \quad f'(A) = f(A \cup \{n\})$$

by symmetry

$$\exists S \subseteq [n-1], |S| = R_t(k-1,l), \binom{S}{t-1} \text{ by } f'$$

or

$$\exists X \subseteq S, |X| = k-1, \binom{X}{t} \text{ by } f$$

$$\exists Y \subseteq S, |Y| = l, \binom{Y}{t} \text{ by } f \quad ✓$$

$$✓ \quad \binom{X \cup \{n\}}{t} \text{ by } f$$

If $n \geq R_t(r; k_1, k_2, \ldots, k_r)$,

$\forall r$-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

such that $\binom{S}{t} \subseteq C_i$

mixing color:

$$R_t(r; k_1, \ldots, k_{r-2}, k_{r-1}, k_r) \leq R_t(r-1; k_1, \ldots, k_{r-2}, R_t(2; k_{r-1}, k_r))$$

$$R_t(k,l) \leq R_{t-1}(R_t(k-1,l), R_t(k,l-1)) + 1$$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

$\forall r$-partition of $\binom{[n]}{t} = C_1 \cup \cdots \cup C_r$

$\exists i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

such that $\binom{S}{t} \subseteq C_i$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem

If $n \geq R_t\left(r; k_1, k_2, \ldots, k_r\right)$,

$\forall\, r$-coloring $f : \binom{[n]}{t} \to [r]$

$\exists i \in [r]$ and $S \subseteq [n]$ with $|S| = k_i$

such that $f\left(\binom{S}{t}\right) = \{i\}$

Theorem (Ramsey 1930)

$R_t(r; k_1, k_2, \ldots, k_r)$ is finite.

Ramsey Theorem (diagonal)

If $n \geq R_t(r; k) \triangleq R_t(r; \underbrace{k, k, \ldots, k}_{r})$,

$\forall\, r$-coloring $f : \binom{[n]}{t} \to [r]$

$\exists$ a monochromatic $\binom{S}{t}$ with $S \in \binom{[n]}{k}$

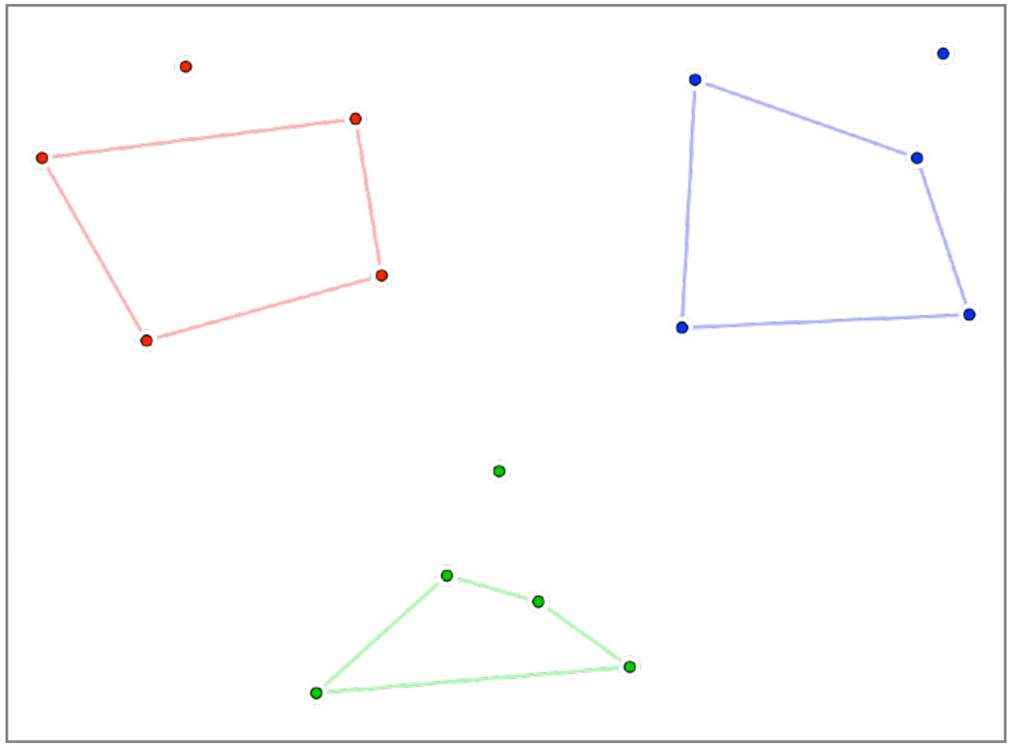
Theorem (Ramsey 1930)

$R_t(r; k)$ is finite.

Applications of Ramsey's Theorem

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.

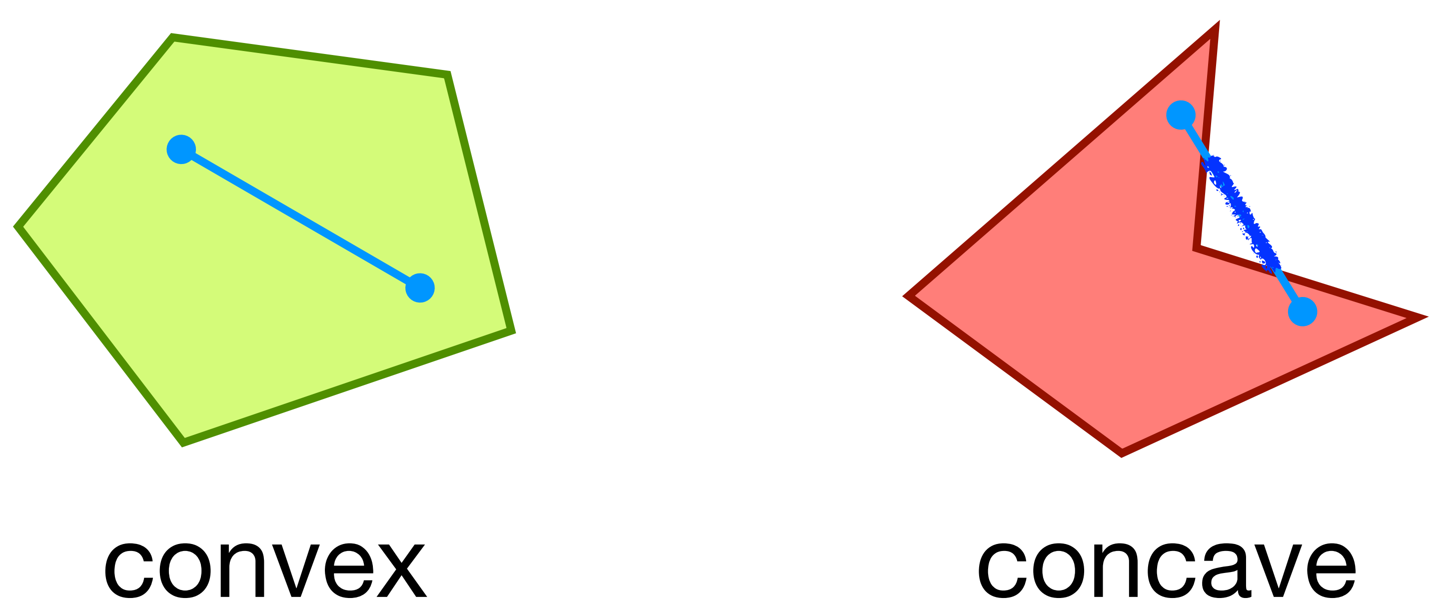


Theorem (Erdős-Szekeres 1935)

$\forall m \geq 3$, $\exists N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

general positioned

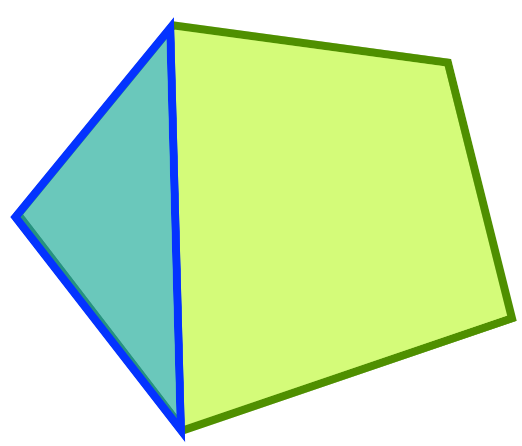
Polygon:



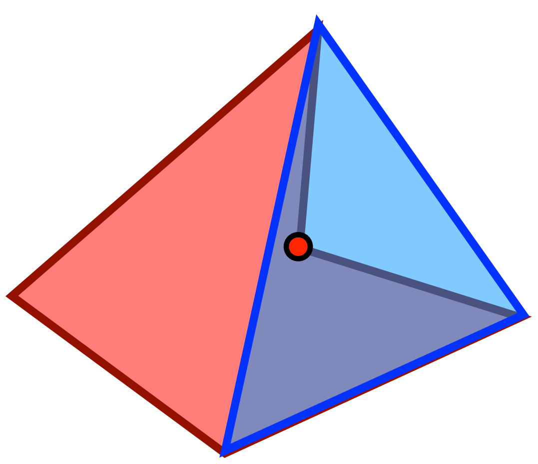
Theorem (Erdős-Szekeres 1935)

$\forall m \geq 3$, $\exists N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains *m* points that are the vertices of a convex m-gon.

Polygon:



convex



concave

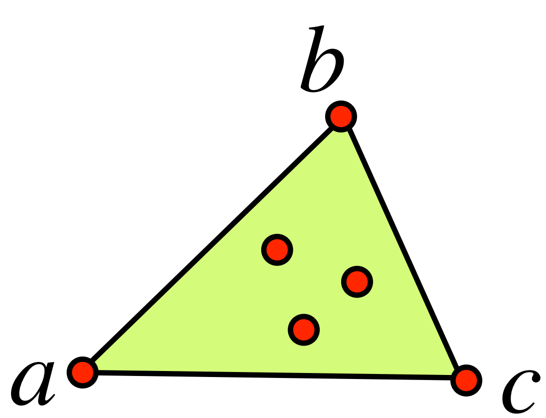
Theorem (Erdős-Szekeres 1935)
$\forall m \geq 3$, $\exists N(m)$ such that any set of $n \geq N(m)$ points in the plane, no three on a line, contains m points that are the vertices of a convex m-gon.

$N(m) = R_3(2; m, m)$ $|X| \geq N(m)$

$\forall f : \binom{X}{3} \to \{0, 1\}$ $\exists S \subseteq X, |S| = m$

monochromatic $\binom{S}{3}$

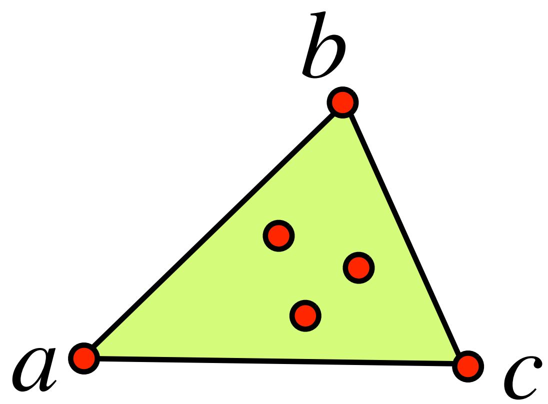
X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X$, $\triangle_{abc}$: points in triangle abc

$$f(\{a, b, c\}) = |\triangle_{abc}| \bmod 2$$

X: set of points in the plane, no 3 on a line



$\forall a, b, c \in X$, $\triangle_{abc}$: points in triangle abc

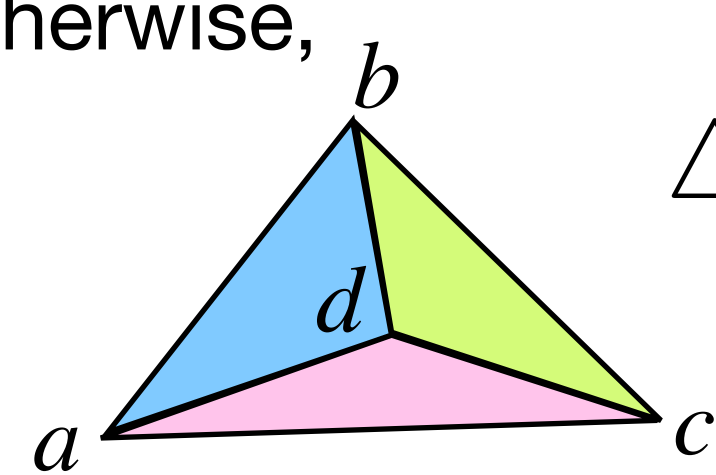
$$f(\{a,b,c\}) = |\triangle_{abc}| \bmod 2$$

$|X| \geq R_3(2; m, m)$ $\forall f : \binom{X}{3} \to \{0, 1\}$

$\exists S \subseteq X, |S| = m$ monochromatic $\binom{S}{3}$

S is a convex m-gon

Otherwise,



disjoint union:

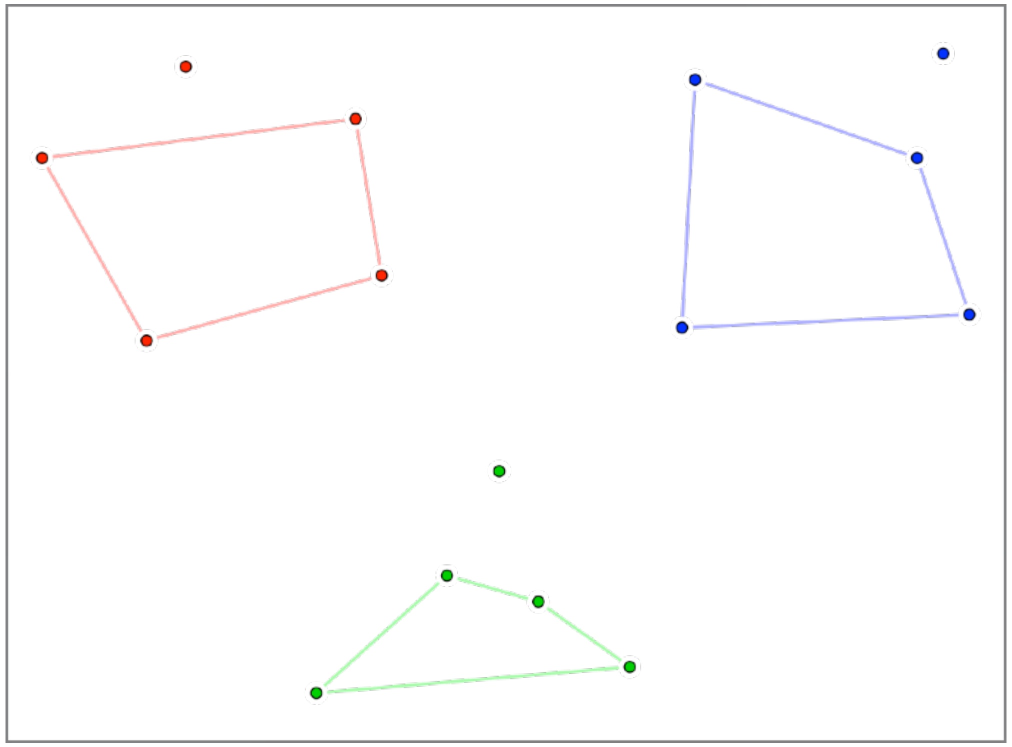
Contradiction!

$$\triangle_{abc} = \triangle_{abd} \cup \triangle_{acd} \cup \triangle_{bcd} \cup \{d\}$$

$$f(abc) = f(abd) + f(acd) + f(bcd) + 1$$

Happy Ending Problem

Any 5 points in the plane, no three on a line, has a subset of 4 points that form a convex quadrilateral.



Data Structures

Problem: "Is $x \in S$?"

data set $S \in \binom{[N]}{n}$ key $x \in [N]$ data universe $[N]$

Solution:

Data structure:

sorted table

Search Alg:

~~binary search~~

Complexity:

$\geq \log_2 n$

memory accesses

in the worst-case

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

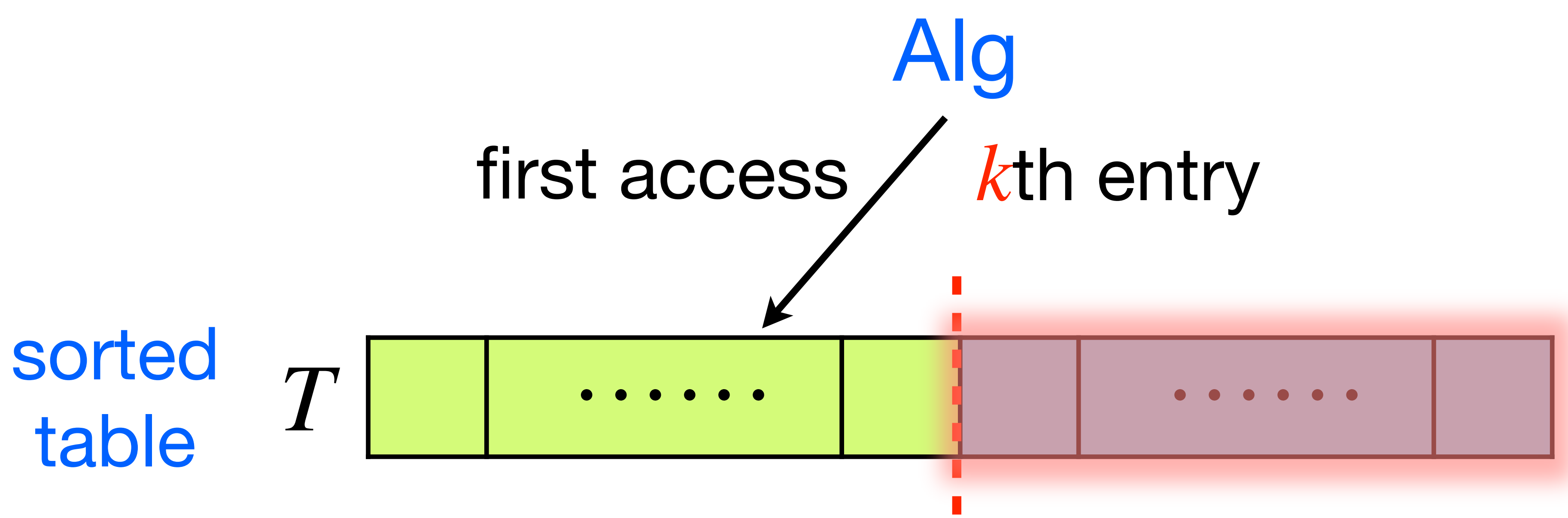
Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

Induction on n, $n = 2$, trivial

Suppose it is true for any smaller n.

adversarial argument + self-reduction

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

$$\binom{\{\frac{n}{2}, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{\frac{n}{2}, \ldots, N\}}{\frac{n}{2}} \subseteq$$ possible $\{\, T[\frac{n}{2}+1], \ldots, T[n] \,\}$

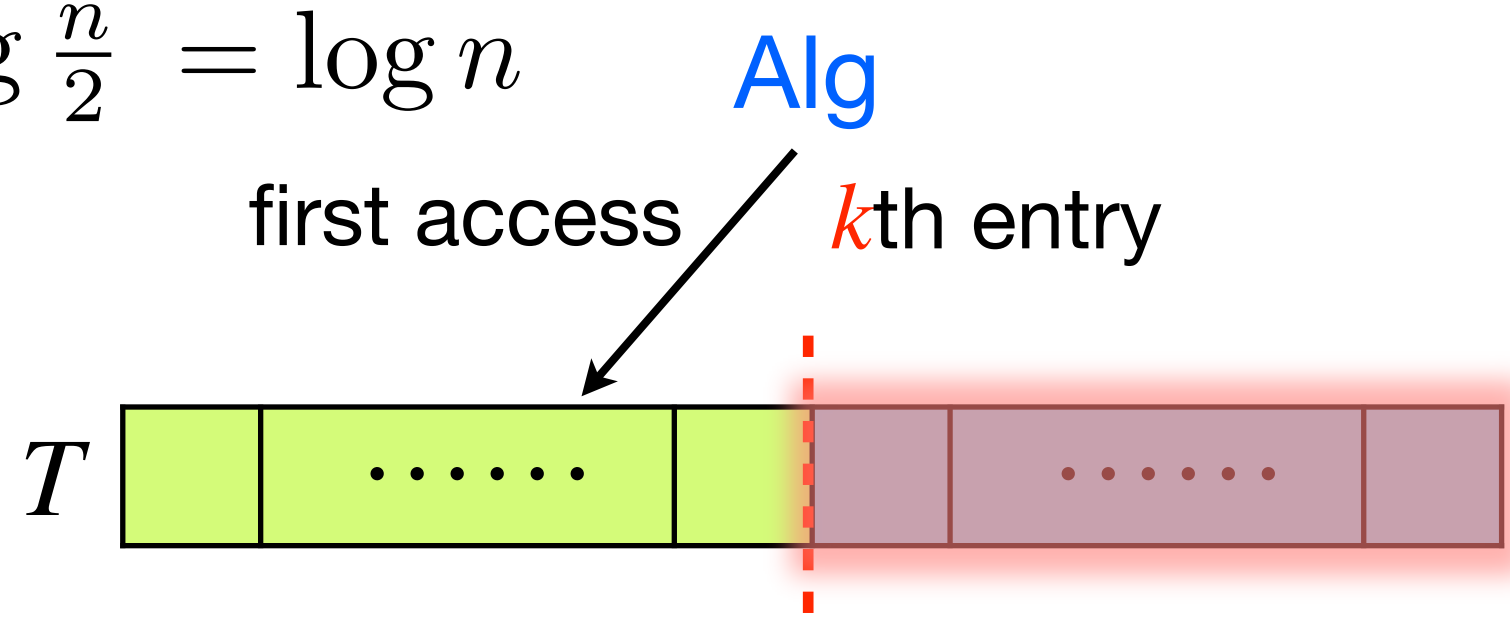
$$n' = \frac{n}{2} \qquad N' = |\{\tfrac{n}{2}, \ldots, N - \tfrac{n}{2}\}| \geq n \geq 2n'$$

relative key in $[N']$: $n - \frac{n}{2} = n'$

Problem: "Is $n \in S$?" $\forall S \in \binom{[N]}{n}$ $N \geq 2n$

$\geq 1 + \log \frac{n}{2} = \log n$

sorted table

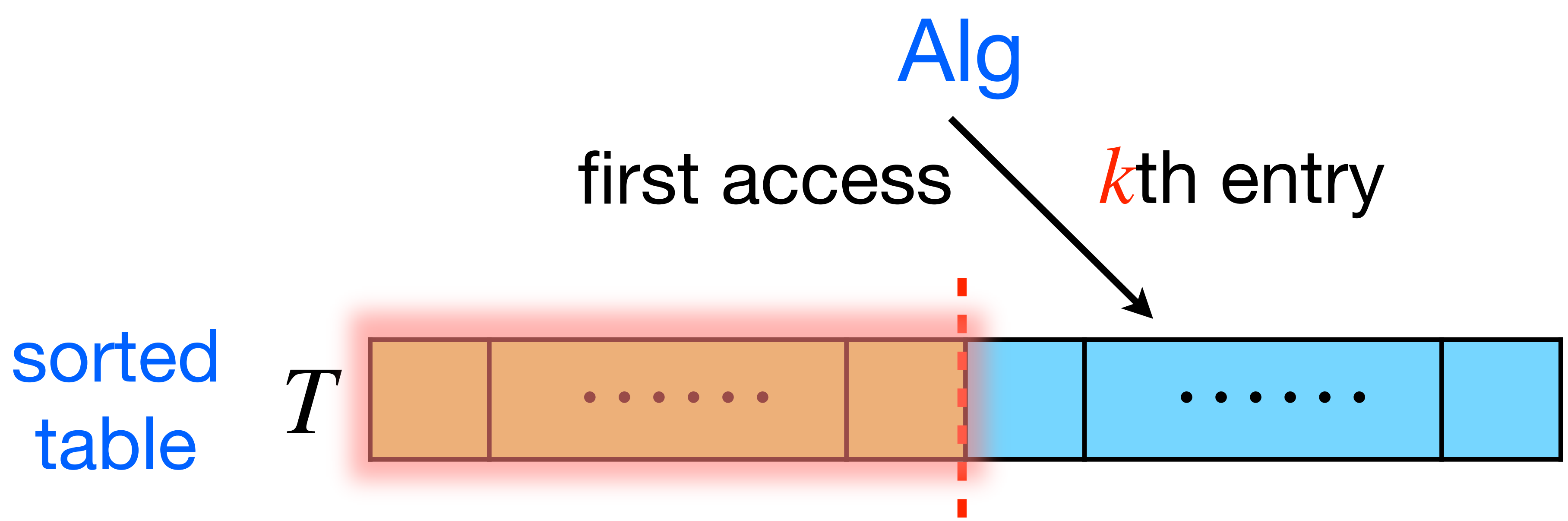


$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n - k) & \text{if } k > \frac{n}{2} \end{cases}$$

$n' = \frac{n}{2}$ "Is $n' \in S'$?" $\forall S' \in \binom{[N']}{n'}$ $N' \geq 2n'$

I.H. require $\log \frac{n}{2}$ memory accesses

Problem: “Is $n \in S$?” $\quad \forall S \in \binom{[N]}{n} \quad N \geq 2n$



$$T[k] = \begin{cases} k & \text{if } k \leq \frac{n}{2} \\ N - (n-k) & \text{if } k > \frac{n}{2} \end{cases}$$

possible

$$\binom{\{\frac{n}{2}, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \binom{\{1, \ldots, N - \frac{n}{2}\}}{\frac{n}{2}} \subseteq \{\, T[1], \ldots, T[\tfrac{n}{2}] \,\}$$

$$n' = \frac{n}{2} \qquad N' = |\{\tfrac{n}{2}, \ldots, N - \tfrac{n}{2}\}| \geq 2n'$$

relative key in [N’]: $\quad n - \frac{n}{2} = n'$

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
If $N \geq 2n$, on sorted table, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

implicit data structure:

each $S \in \binom{[N]}{n}$ is stored as a permutation of S

$$f : \binom{[N]}{n} \to [n!]$$

$f(S) = \pi$
dataset S: $x_1 < \cdots < x_n$
table: $(x_{\pi(1)}, \ldots, x_{\pi(n)})$

$\binom{[N]}{n}$ is mapped to the same π ⇒ same as sorted

$\exists X \subseteq [N], |X| \geq 2n$, $\binom{X}{n}$ monochromatic ⇒ same as sorted

"Is $x \in S$?" $x \in [N]$ $S \in \binom{[N]}{n}$

Theorem (Yao 1981)
For sufficiently large N, on any implicit data structure, any search Alg requires $\Omega(\log n)$ accesses in the worst-case.

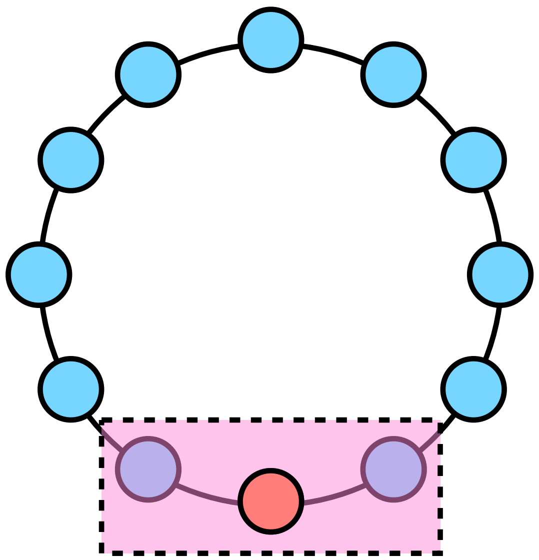
implicit data structure:

$$f : \binom{[N]}{n} \to [n!]$$

$N \geq R_n(n!; \underbrace{2n, \ldots, 2n}_{n!})$ or equivalently $N \to (\underbrace{2n, \ldots, 2n}_{n!})^n$

$\exists X \subseteq [N], |X| \geq 2n,$ $\binom{X}{n}$ monochromatic ⇒ $\geq \log n$ accesses

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

for any input ring,
locally compute the MIS

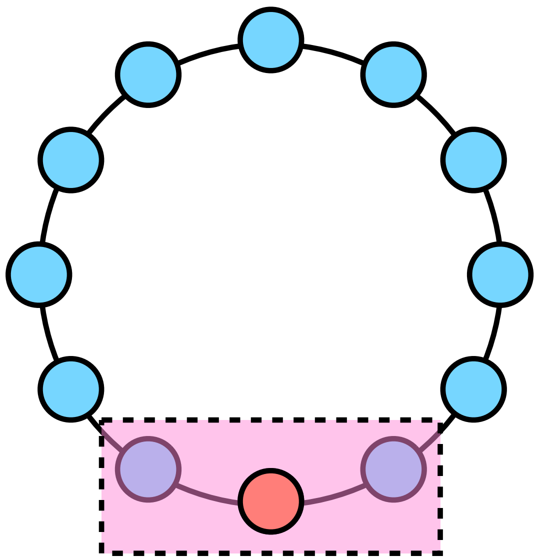
t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$f : \binom{[n]}{t} \to \{0, 1\}$$

$$f(\{a_1, \ldots, a_t\}) = \mathcal{L}(a_1, \ldots, a_t)$$
$$a_1 < a_2 < \ldots < a_t$$

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

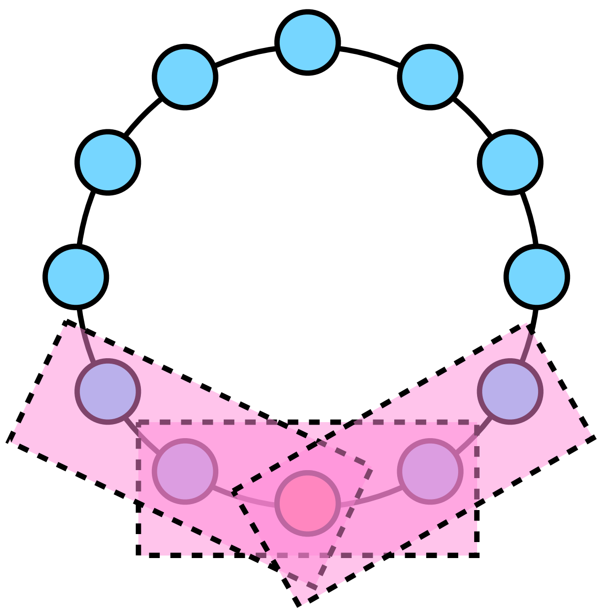
$$f : \binom{[n]}{t} \to \{0, 1\}$$

$$n \geq R_t(2; t+2, t+2)$$

∃ a monochromatic $\binom{S}{t}$ $\quad |S| = t + 2$

$S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\}$ $\quad a_1 < \ldots < a_t < a_{t+1} < a_{t+2}$

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$n \geq R_t(2; t+2, t+2)$ ⇒ $\exists S = \{a_1, \ldots, a_t, a_{t+1}, a_{t+2}\}$

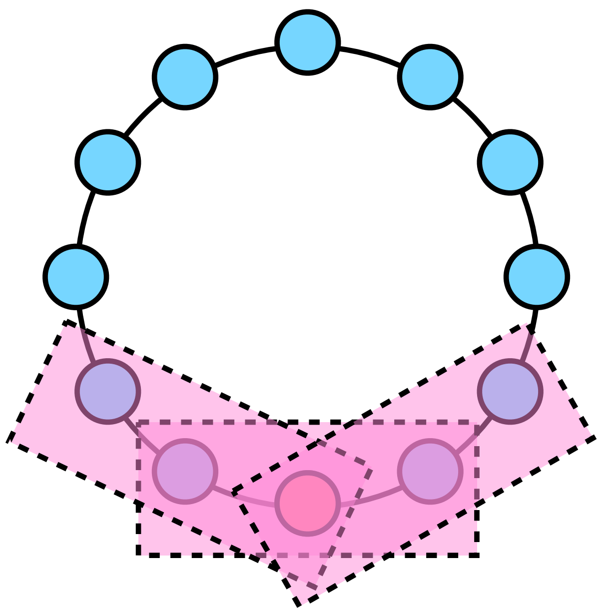
$$\mathcal{L}(a_1, \ldots, a_t) = \mathcal{L}(a_2, \ldots, a_{t+1}) = \mathcal{L}(a_3, \ldots, a_{t+2})$$

construct a bad ring starting with

$$(a_1, a_2, \ldots, a_t, a_{t+1}, a_{t+2})$$

Contradiction!

Local Computation



distributed computing in a ring

n nodes, ID from $[n]$

maximal independent set (MIS)

t-local algorithm:

$$\mathcal{L} : \{t\text{-permutations of } [n]\} \to \{0, 1\}$$

$$n < R_t(2; t+2, t+2) \le \underbrace{2^{2^{\cdot^{\cdot^{2^{ct}}}}}}_{t}$$

$$\Rightarrow \quad t = \Omega(\log^* n)$$

Ramsey Theory

Arithmetic Progression

2-coloring of 1 to 12

1 2 3 4 5 6 7 8 9 10 11 12

monochromatic arithmetic progression

Can you give a 2-coloring of $\mathbb{N}$, no infinite monochromatic arithmetic progression?

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 $\cdots$

Van der Waerden Theorem

$W(r,k) \triangleq$ the smallest integer satisfying:

if $n \geq W(r,k)$, for any r-coloring of $[n]$, there exists a monochromatic arithmetic progression of length k

VdW Theorem
(Van der Waerden 1927)

$W(r,k)$ is finite.

Bartel Leendert van der Waerden
(1903-1996)

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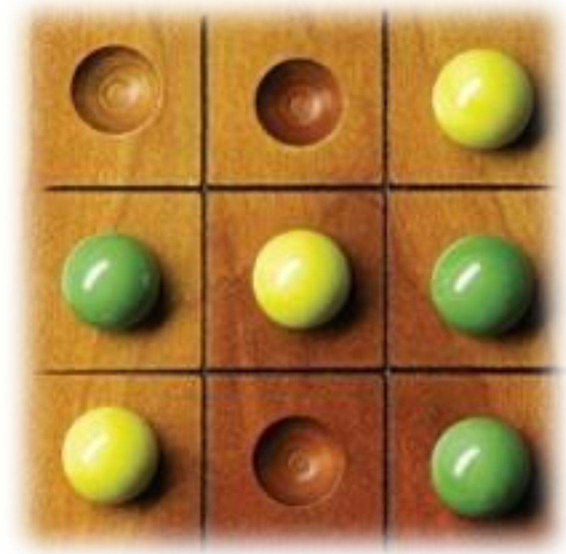
VdW Theorem $W(r,k)$ is finite.

$$\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$$

⇒ some C_i contains arbitrarily long arithmetic progression.

Hales-Jewett Theorem



Hales-Jewett Theorem

$[k] = \{1,\ldots,k\}$: an alphabet of k symbols

$[k]^n$: n-dimensional discrete cube

combinatorial line: $L_\tau = \{\tau(1), \tau(2), \ldots, \tau(k)\}$

$\tau \in ([k] \cup \{\star\})^n$ τ contains "$\star$"

$\forall i \in [k], \quad \tau(i) = \text{replacing } \star \text{ by } i \text{ in } \tau$

$\tau = 12\star3\star2$

$k = 4$

$$L_\tau = \{121312, 122322, 123332, 124342\}$$

Hales-Jewett Theorem

$[k] = \{1, \ldots, k\}$: an alphabet of k symbols

$[k]^n$: n-dimensional discrete cube

combinatorial line:

$HJ(r,k) \triangleq$ the smallest integer satisfying:

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

Hales-Jewett Theorem $HJ(r,k)$ is finite.

HJT $\Longrightarrow$ VdW

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

reduction ϕ: $[k]^n \rightarrow [N]$

$$\forall x \in [k]^n \qquad \phi(x) = x_1 + x_2 + \ldots + x_n$$

combinatorial line $L_\tau = \{\tau(1), \ldots, \tau(k)\}$

arithmetic progression $\{\phi(\tau(1)), \ldots, \phi(\tau(k))\}$

$\tau = 12\star3\star2$ $\qquad L_\tau =$ { 121312, 122322
123332, 124342 }

if $N \geq W(r,k)$, for any r-coloring of $[N]$, there exists a monochromatic arithmetic progression of length k

HJT $\Longrightarrow$ VdW

If $n \geq HJ(r,k)$, for every r-coloring of the cube $[k]^n$, there exists a monochromatic combinatorial line.

reduction $\phi: [k]^n \to [N]$

$$\forall x \in [k]^n \qquad \phi(x) = x_1 + x_2 + \ldots + x_n$$

combinatorial line $\qquad L_\tau = \{\tau(1), \ldots, \tau(k)\}$

arithmetic progression $\qquad \{\phi(\tau(1)), \ldots, \phi(\tau(k))\}$

$f : [N] \to [r]$ $\Rightarrow$ $f' : [k]^n \to [r]$

$$f'(x) = f(\phi(x))$$

if $N \geq W(r,k)$, for any r-coloring of $[N]$, there exists a monochromatic arithmetic progression of length k

Szemerédi's Regularity Lemma

History

- Ramsey theory
- Van der Waerden's theorem (1927)
- Erdős-Turán conjecture (1936)
- Szemerédi's theorem (1969,1975)
  - Szemerédi regularity lemma (1975, 1978)

Arithmetic Progression

2-coloring of 1 to 12

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12$

monochromatic arithmetic progression

Can you give a 2-coloring of $\mathbb{N}$, no infinite monochromatic arithmetic progression?

$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ \cdots$

Van der Waerden Theorem

$W(r,k) \triangleq$ the smallest integer satisfying:

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VdW Theorem
(Van der Waerden 1927)

$W(r,k)$ is finite.

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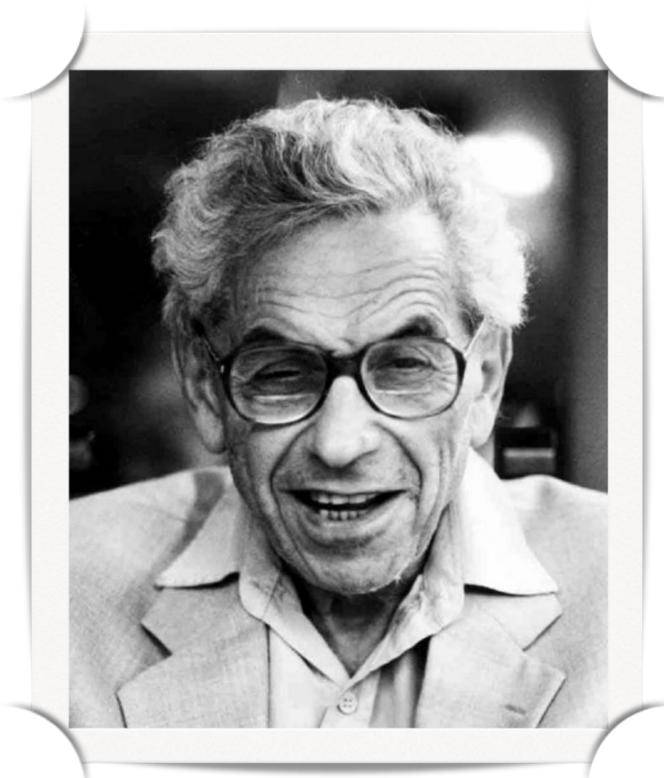
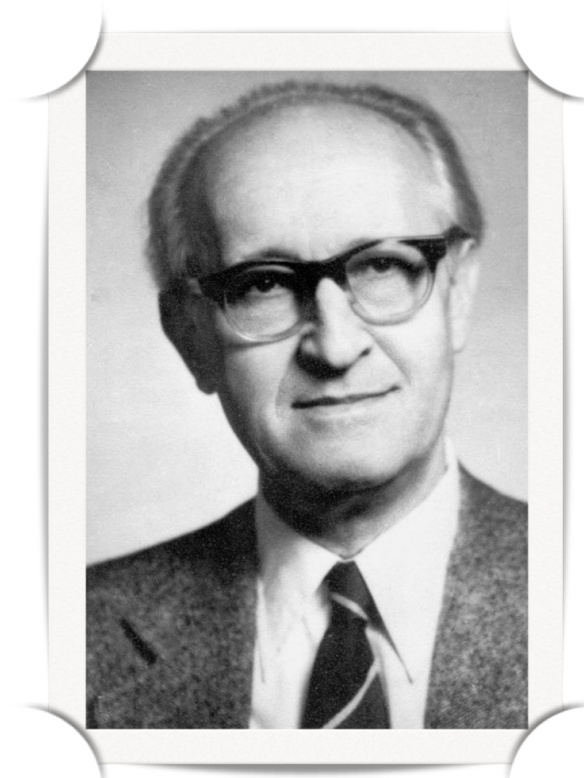
(1903-1996)

Erdős-Turán Conjecture

if $n \geq W(r,k)$, for any r-coloring of $[n]$, there exists a monochromatic arithmetic progression of length k





Erdős-Turán (1936): Coloring is not essential!

Every dense enough integer set has long arithmetic progression.

Erdős-Turán Conjecture

if $n \geq W(r,k)$, for any r-coloring of $[n]$, there exists a monochromatic arithmetic progression of length k

$\forall$ density $0 < \delta < 1$, $\forall$ integer k, $\exists N(\delta, k)$

$n \geq N(\delta, k)$ ⇒ every δn-subset S of $[n]$ contains a length-k arithmetic progression

Erdős-Turán (1936): Coloring is not essential!

Every dense enough integer set has long arithmetic progression.

Szemerédi's Theorem

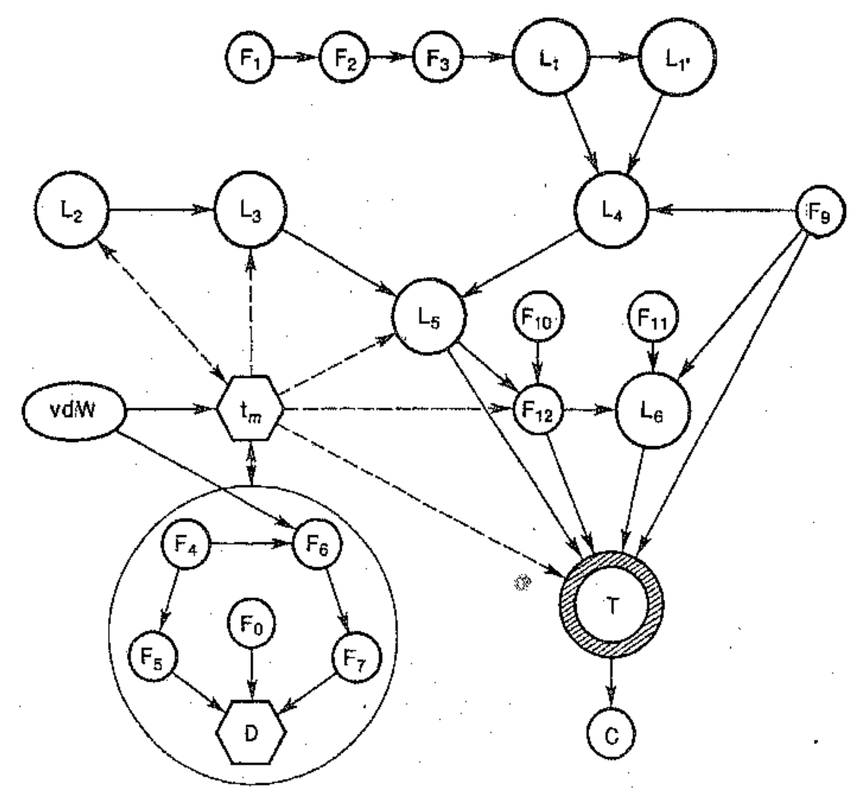
Theorem (Szemerédi 1975)

$\forall$ density $0 < \delta < 1$, $\forall$ integer k, $\exists\, N(\delta, k)$

$n \geq N(\delta, k)$ ⇨ every δn-subset S of $[n]$ contains a length-k arithmetic progression

key ingredient of the proof:

Szemerédi Regularity Lemma

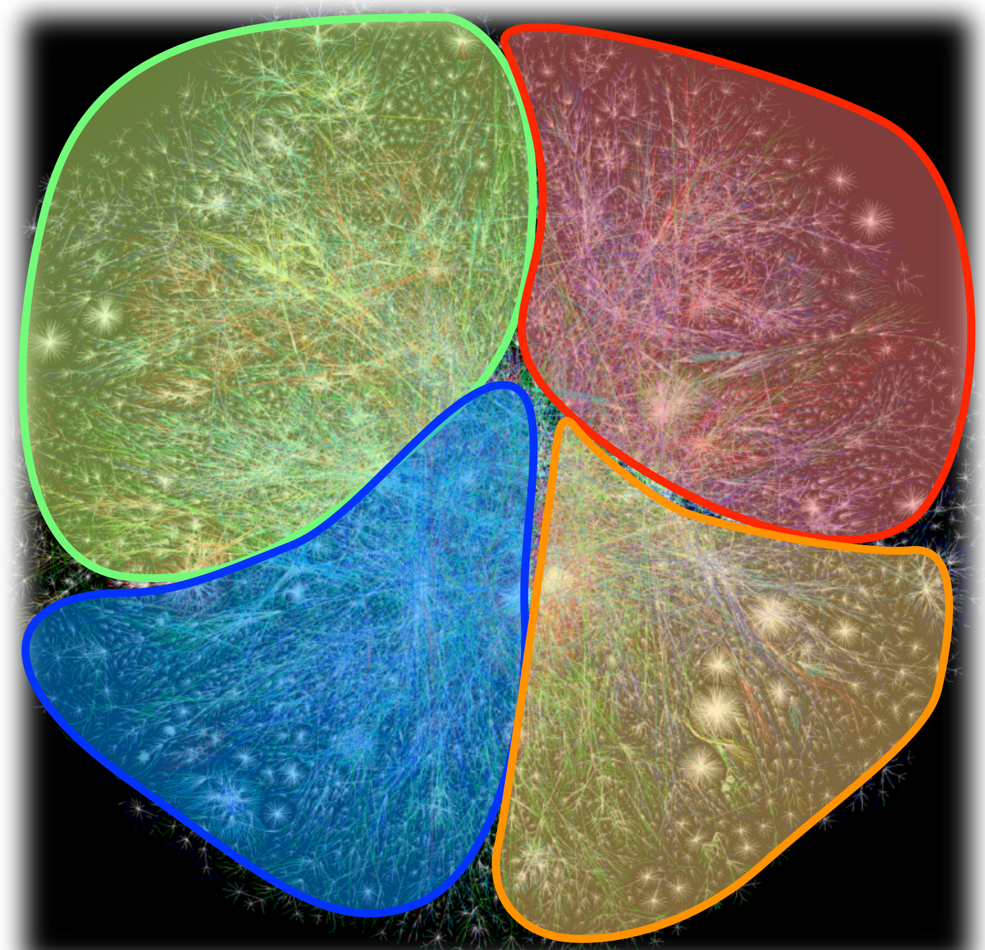


Szemerédi Regularity Lemma

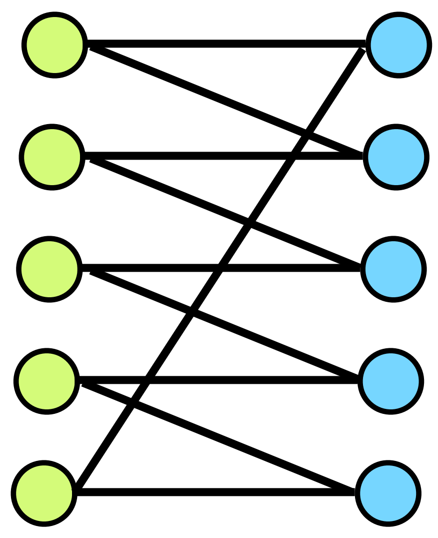
Every large enough graph can be approximated by a simple object of constant complexity.

$\forall$ sufficient large graph can be partitioned into constant equal-sized parts $V_1, V_2, \ldots, V_k$

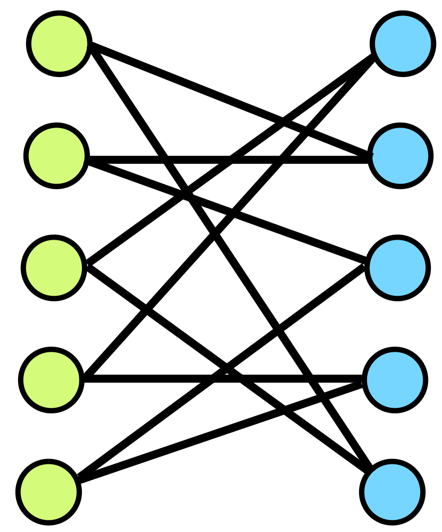
most (V_i, V_j) look like random bipartite graphs



Regularity

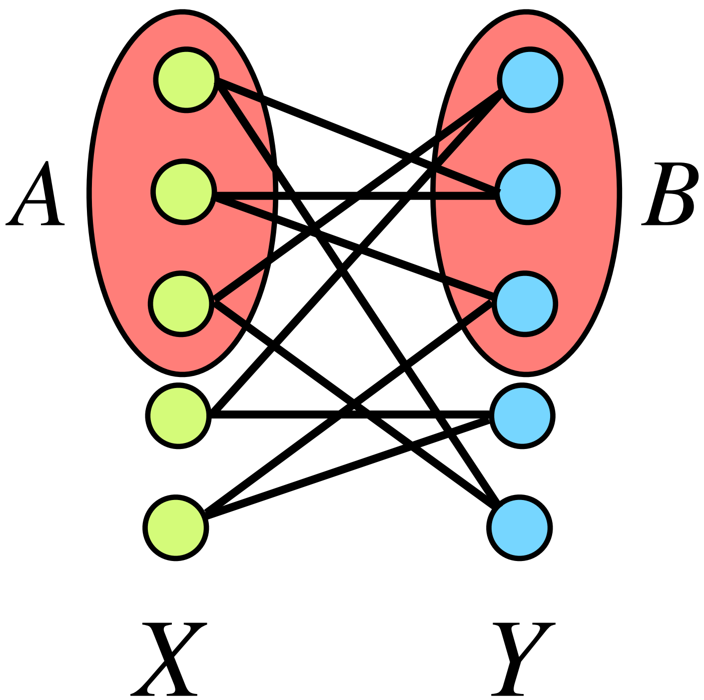


not random



random or is it?

Regularity



$$E(X,Y) = \{uv \mid u \in X, v \in Y\}$$

density:

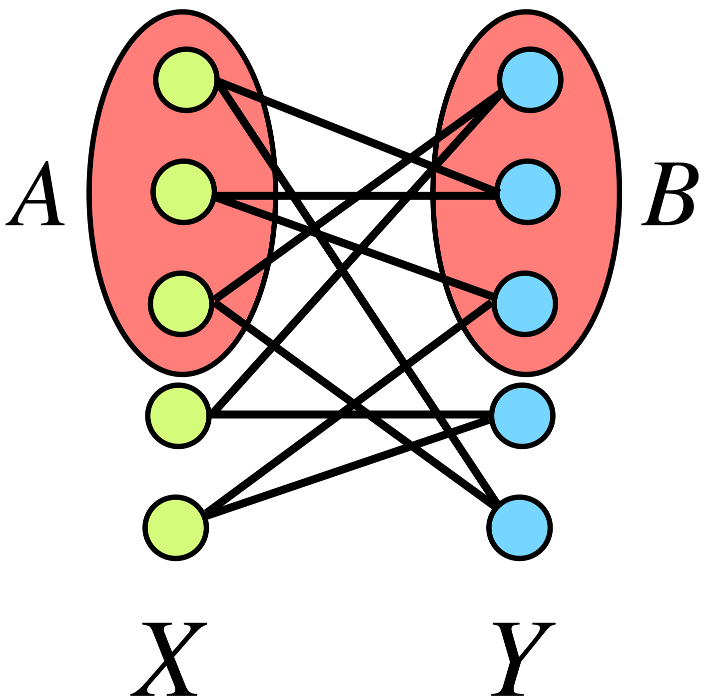
$$d(X,Y) = \frac{E(X,Y)}{|X||Y|}$$

in a random graph: $\forall A \subseteq X, B \subseteq Y$

$$\mathbf{E}[E(A,B)] = |A||B| \cdot d(X,Y)$$

$$\mathbf{E}[d(A,B)] = d(X,Y)$$

Regularity



$$E(X,Y) = \{uv \mid u \in X, v \in Y\}$$

density:

$$d(X,Y) = \frac{E(X,Y)}{|X||Y|}$$

ε-regularity: $0 < \epsilon < 1$

$$\forall A \subseteq X, B \subseteq Y \text{ with } |A| \geq \epsilon|X| \text{ and } |B| \geq \epsilon|Y|$$

$$|d(A,B) - d(X,Y)| \leq \epsilon$$

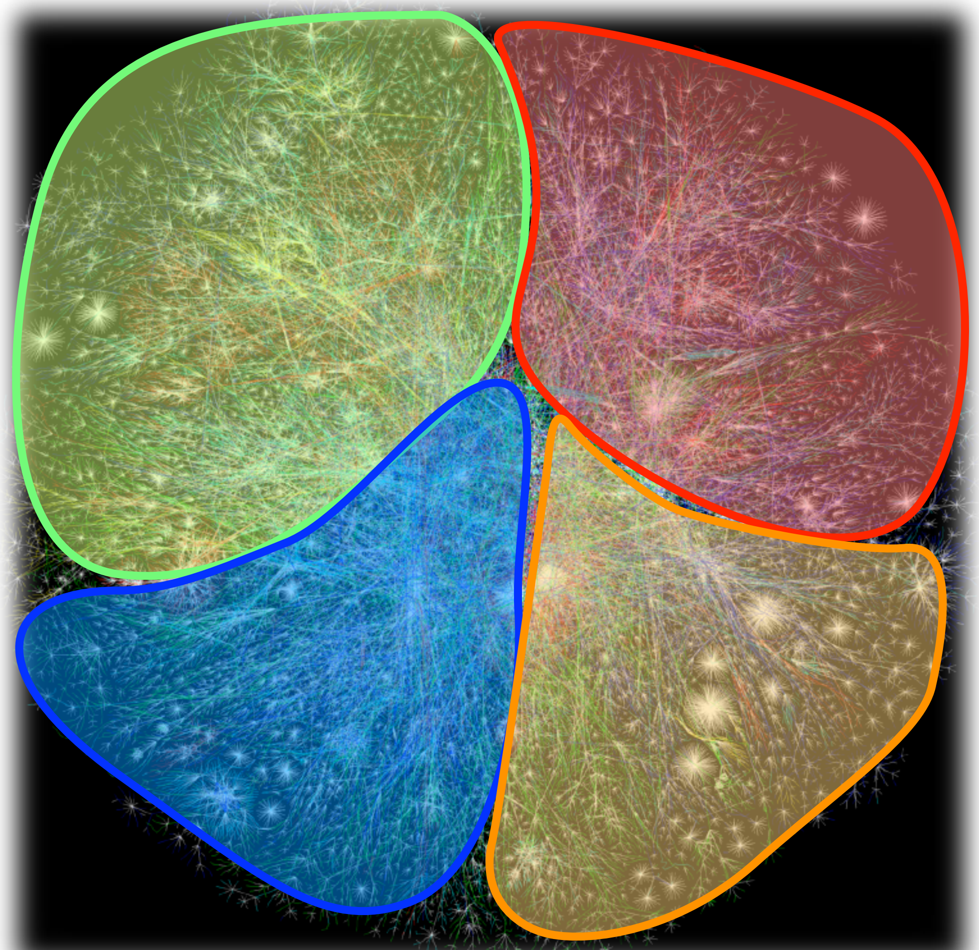
regularity = uniformity

Regular Partitions

$G(V,E)$

ε-regular partition:

$\{R, V_1, V_2, \ldots, V_k\}$ of V



1. $|R| \leq \epsilon |V|$; garbage bin
2. $|V_1| = |V_2| = \cdots = |V_k|$;
3. all but ϵk^2 pairs of (V_i, V_j) are ϵ-regular.

Theorem (Szemerédi 1978)

$\forall\, 0 < \epsilon < 1,\ m \geq 1,\ \exists\, M$ and N every $G(V, E)$ with $|V| \geq N$ has an ϵ-partition $\{R, V_1, \ldots, V_k\}$ with $m \leq k \leq M$.

every large enough graph can be partitioned into bounded # of equal-sized parts that interact mostly in a regular way

one may pick the error ε and the complexity m

Rough structure result for all graphs.

One of the most powerful tool in graph theory.

Proof of Regularity Lemma

Theorem (Szemerédi 1978)

$\forall\, 0 < \epsilon < 1,\ m \geq 1,\ \exists\, M$ and N
every $G(V, E)$ with $|V| \geq N$ has an ϵ-partition $\{R, V_1, \ldots, V_k\}$ with $m \leq k \leq M$.

classic proof: potential function + regularization

mean square density   partition $P = \{R, V_1, \ldots, V_k\}$

$$q(P) = \sum_{i,j} p_i p_j d^2(V_i, V_j) \qquad p_i = \frac{|V_i|}{|V|}$$

$q(P) \in [0, 1]$ $q(P') \geq q(P)$ if P' refines P

Proof of Regularity Lemma

Theorem (Szemerédi 1978)

$\forall\, 0 < \epsilon < 1,\ m \geq 1,\ \exists\, M$ and N
every $G(V,E)$ with $|V| \geq N$ has an ϵ-partition $\{R, V_1, \ldots, V_k\}$ with $m \leq k \leq M$.

classic proof: potential function + regularization

mean square density   partition $P = \{R, V_1, \ldots, V_k\}$

$q(P) \in [0,1]$ $q(P') \geq q(P)$ if P' refines P

if P is not an ε-regular partition

$\exists$ a refinement P' of P to $\leq k2^k$ parts, s.t. $q(P') \geq q(P) + c\epsilon^5$

Proof of Regularity Lemma

Theorem (Szemerédi 1978)

$\forall\, 0 < \epsilon < 1,\ m \geq 1,\ \exists\, M$ and N
every $G(V,E)$ with $|V| \geq N$ has an ϵ-partition $\{R, V_1, \ldots, V_k\}$ with $m \leq k \leq M$.

a new proof by Tao (2007):

structure + pseudorandomness

Theorem (Szemerédi 1978)

$\forall\, 0 < \epsilon < 1,\ m \geq 1,\ \exists\, M$ and N
every $G(V, E)$ with $|V| \geq N$ has a partition of V into $R, V_1, \ldots, V_k$ with $m \leq k \leq M$, satisfying:

1. $|R| \leq \epsilon |V|$;
2. $|V_1| = |V_2| = \cdots = |V_k|$;
3. all but ϵk^2 pairs of (V_i, V_j) are ϵ-regular.

every large enough graph can be partitioned into bounded # of equal-sized parts that interact mostly in a regular way

one may pick the error ε and the complexity m

Applying Regularity Lemma

Goal:
a theorem holding for all large enough graphs

⬆

the theorem holds for random graphs

every large enough graph looks random in some way

Extremal Graph Theory

Fix a graph H.

$$\mathrm{ex}(n, H)$$

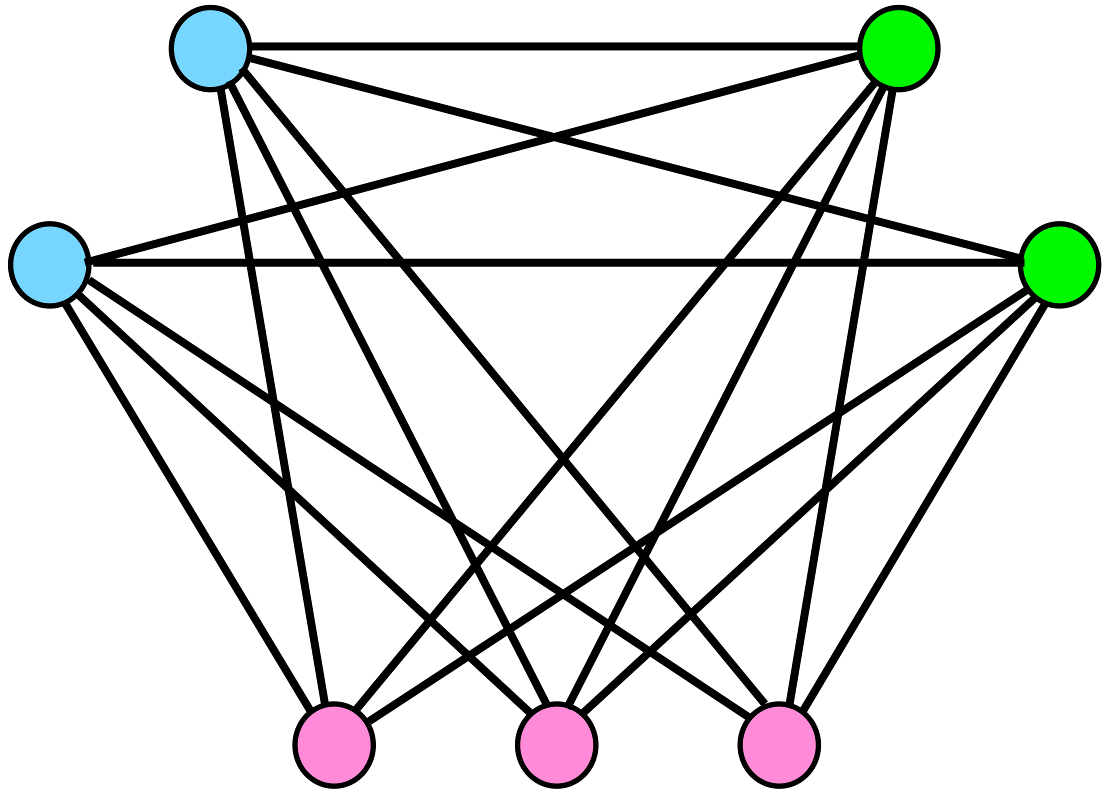
largest possible number of edges of $G \not\supseteq H$ on n vertices

$$\mathrm{ex}(n, H) = \max_{\substack{G \not\supseteq H \\ |V(G)|=n}} |E(G)|$$

$|E| > \mathrm{ex}(n, H)$ forces subgraph H in $G([n], E)$

Complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$

$K_{2,2,3}$



Turán graph $T(n,r)$

$$T(n,r) = K_{n_1,n_2,\ldots,n_r}$$

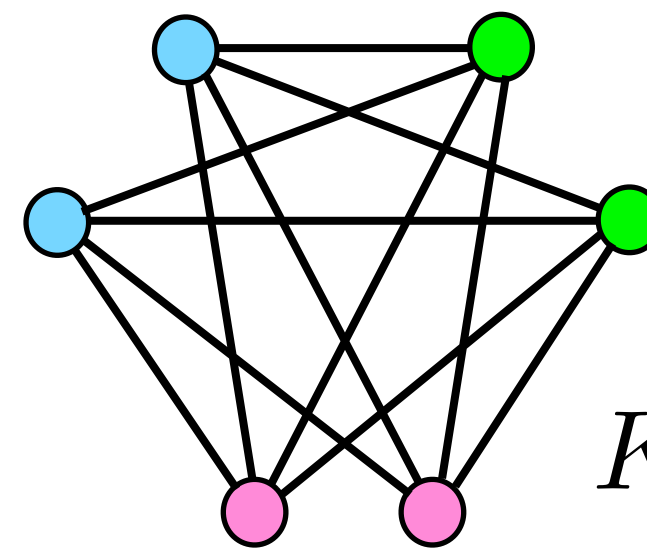
$$n_1 + n_2 + \cdots + n_r = n \qquad n_i \in \left\{ \left\lfloor \frac{n}{r} \right\rfloor, \left\lceil \frac{n}{r} \right\rceil \right\}$$

Erdős–Stone theorem

(Fundamental theorem of extremal graph theory)

$$K_s^r = K_{\underbrace{s, s, \cdots, s}_{r}} = T(rs, r)$$

complete r-partite graph
with s vertices in each part



K_2^3

Theorem (Erdős–Stone 1946)

$$\operatorname{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1) \right) n^2$$

Theorem (Erdős–Stone 1946)

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

$\mathrm{ex}(n, H)/\binom{n}{2}$ **extremal density** of subgraph H

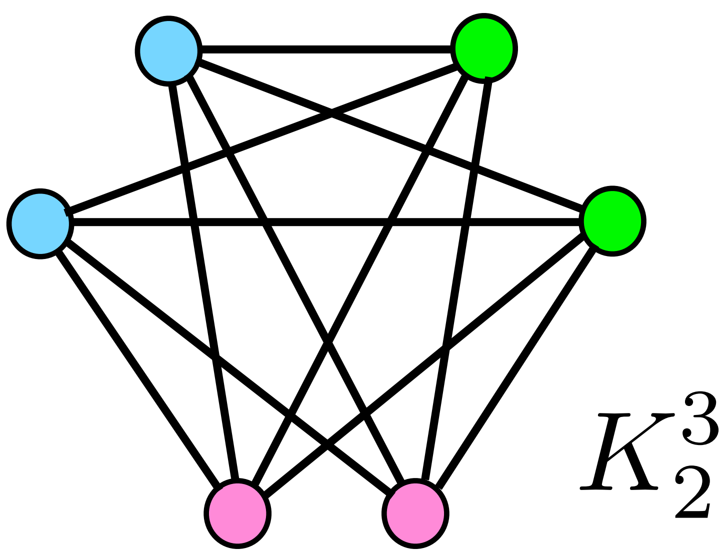
Corollary

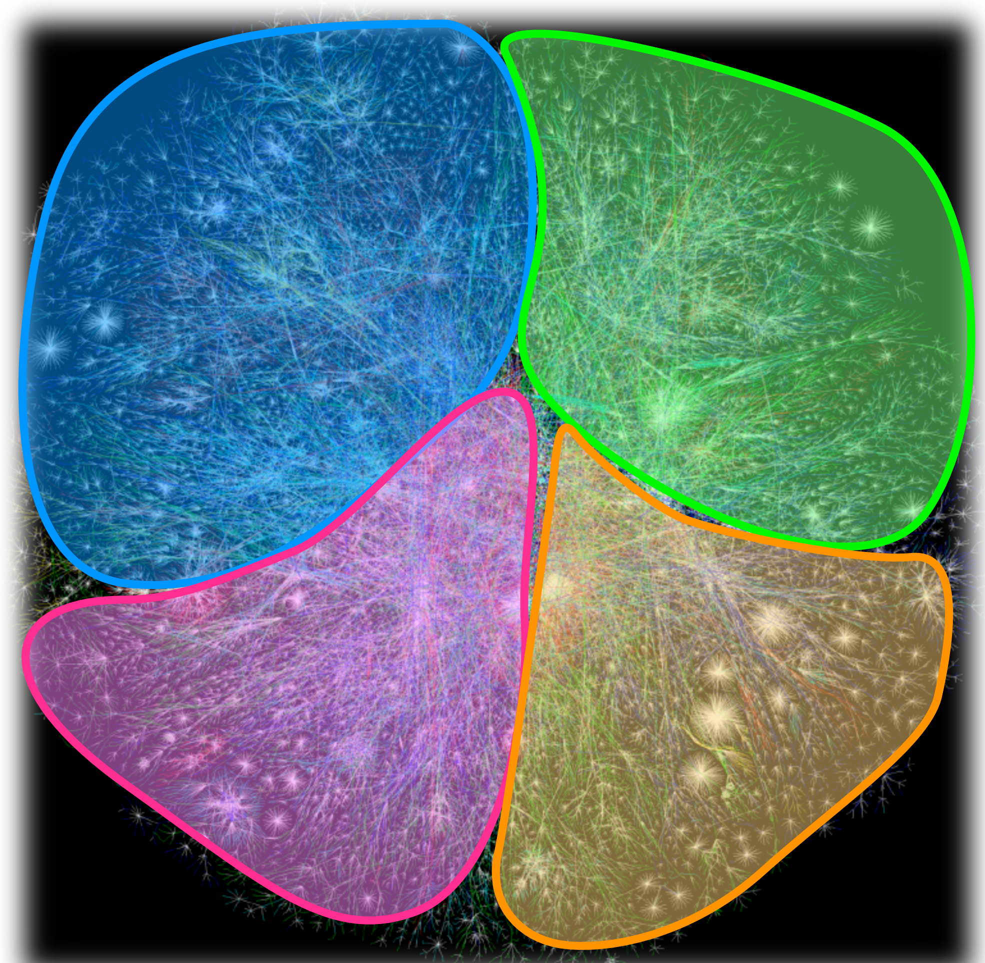
For any nonempty graph H

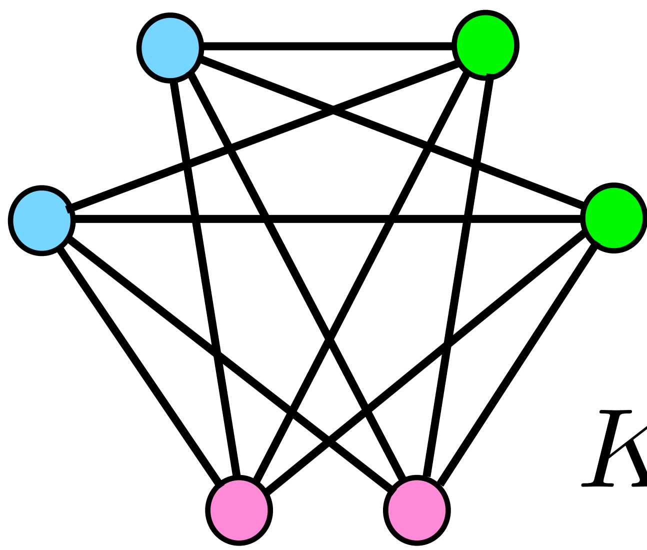
$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

Theorem (Erdős–Stone 1946)

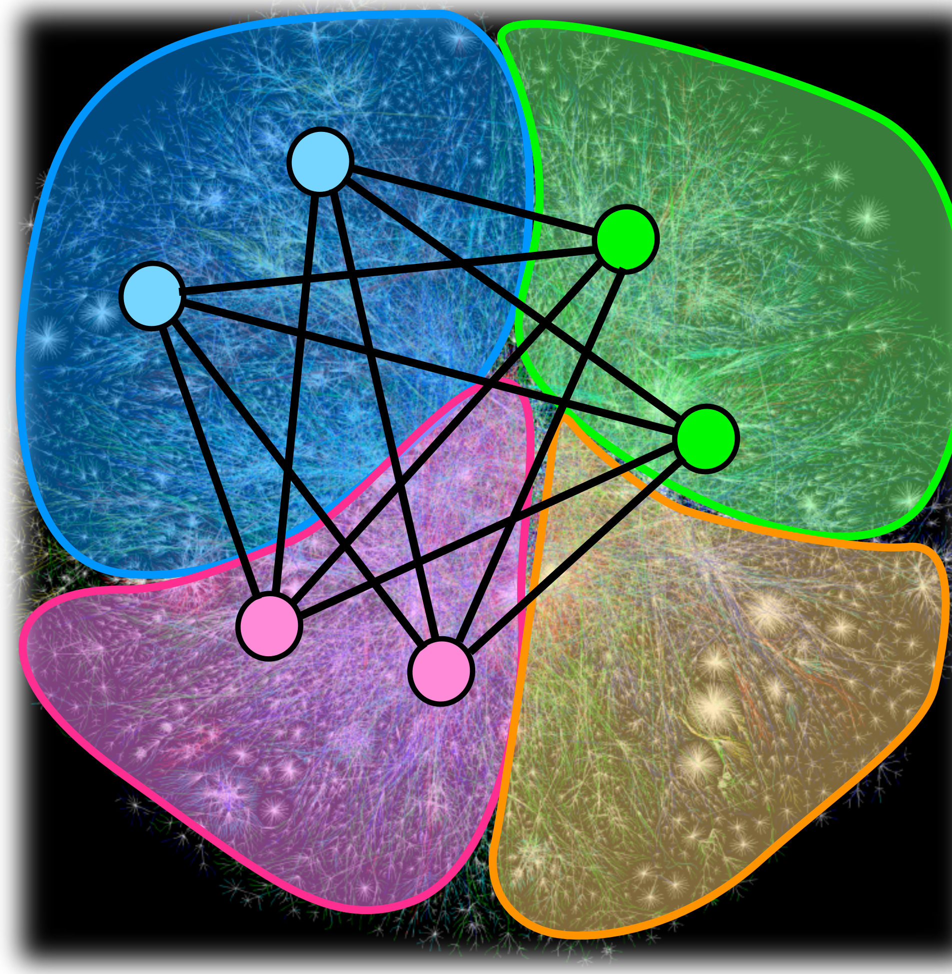
$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1) \right) n^2$$







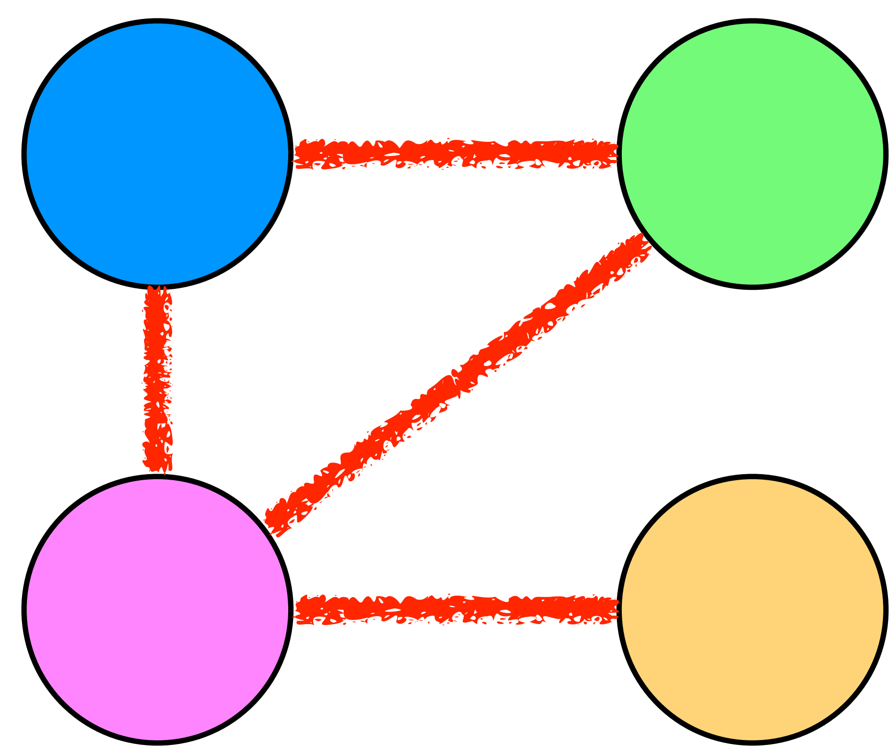
K_2^3



reduced graph
(cluster graph):

vertex ⇔ part

edge ⇔ positive density
ε-regular



with enough edge density $\exists\, K_r$ in the reduced graph

Applying Regularity Lemma

certain edge density forces subgraph H
in large enough graphs

- Edge density of G implies edge density of the reduced graph G^R.
- Apply extremal argument to the reduced graph G^R to locate a subgraph H^R.
- Embed H in H^R with guaranteed density and regularity.