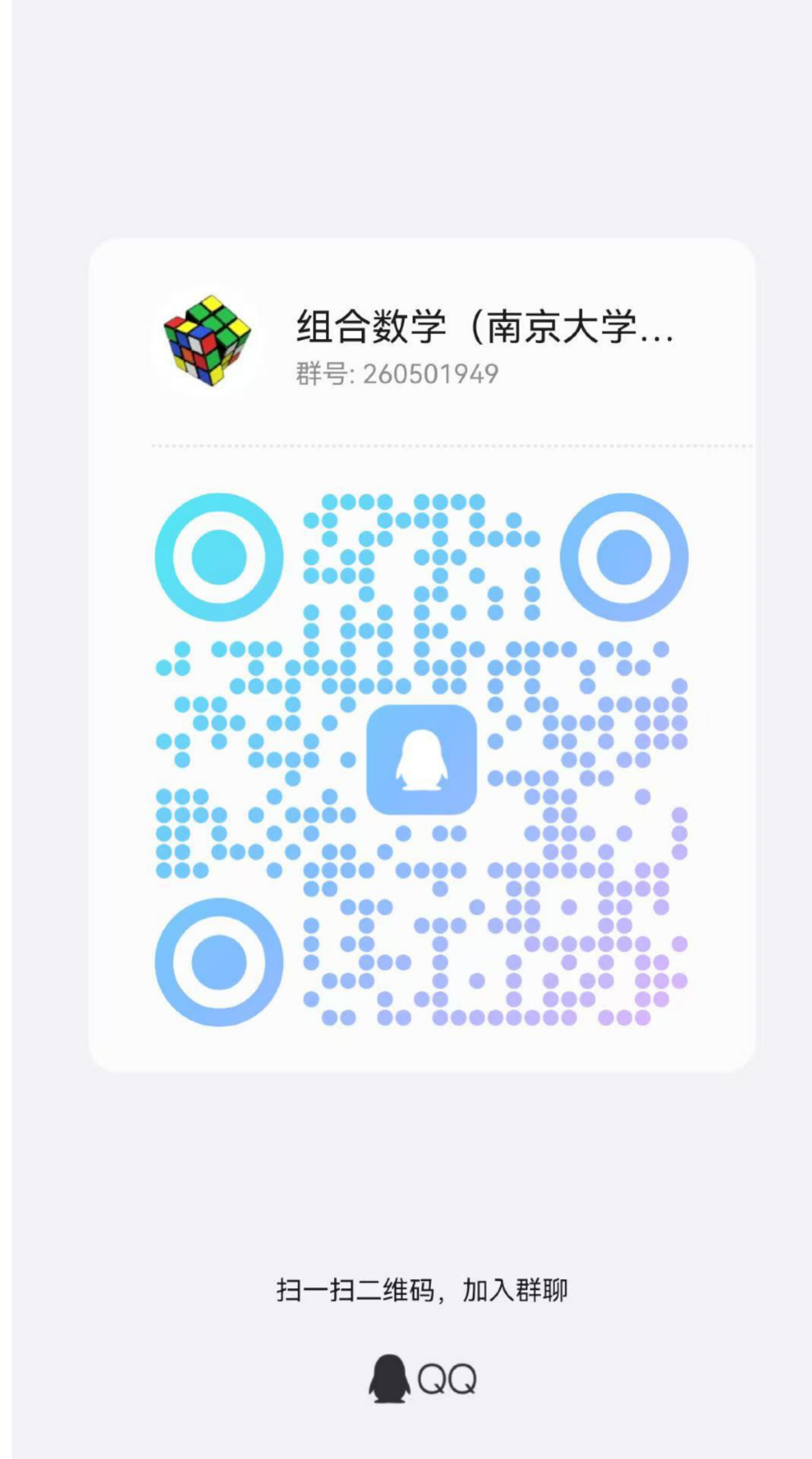
Combinatorics

Basic Enumeration

尹一通 Nanjing University, 2025 Spring

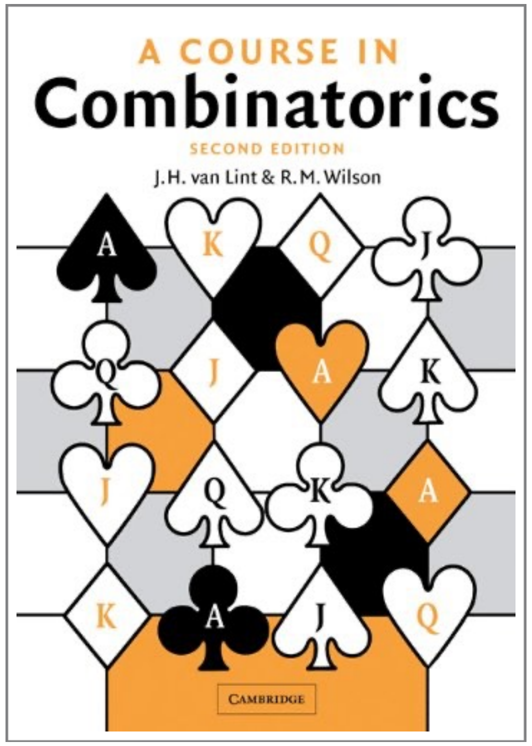
Course Info

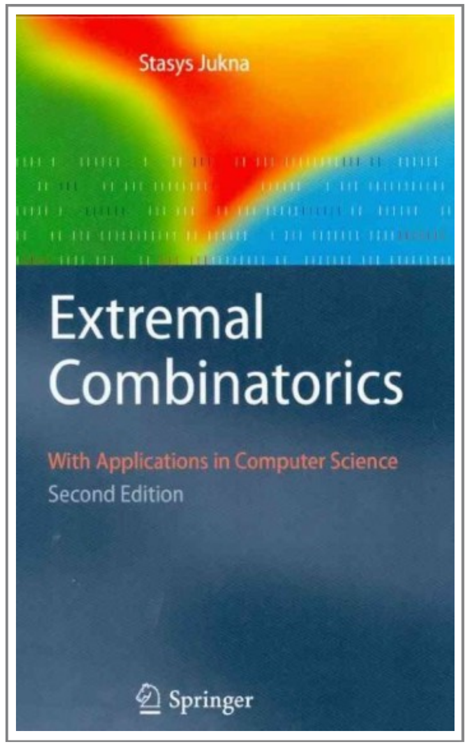
- Instructor: 尹一通
  - Email: yinyt@nju.edu.cn
- Office hour:
  - 804，Tuesday 2-4pm
  - QQ group: 260501949
- Course homepage:
  - http://tcs.nju.edu.cn/wiki/



Textbooks

- van Lint and Wilson.
  A course in Combinatorics,
  2nd Edition.

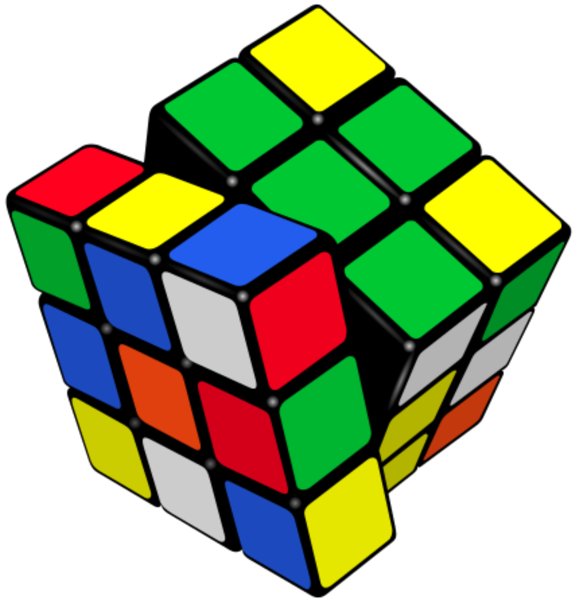




- Jukna.
  Extremal Combinatorics:
  with applications in computer science,
  2nd Edition.

Combinatorics:

A math for finite structures (*not really*)



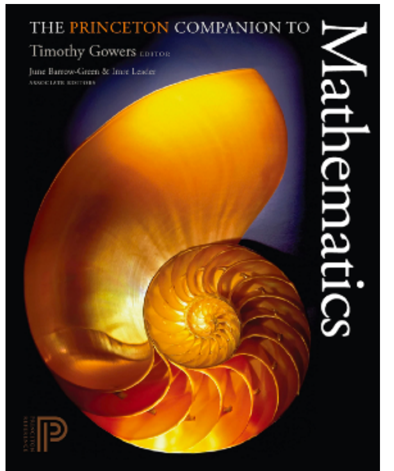
- It is difficult to give rigorous definition of ***Combinatorics***.
- According to *Wikipedia*, subfields of combinatorics include:

Problems
- *enumerative combinatorics*, *extremal combinatorics*, *combinatorial design*

Tools
- *algebraic combinatorics*, *analytic combinatorics*, *probabilistic combinatorics*, *geometric combinatorics*, *topological combinatorics*, *arithmetic combinatorics*

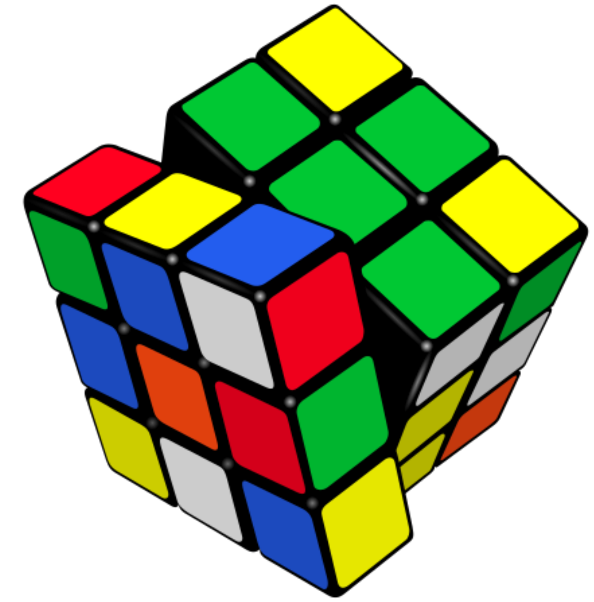
Objects
- *partition theory*, *graph theory*, *finite geometry*, *order theory*, *matroid theory*, *combinatorics on words*, *infinitary combinatorics*

- In *The Princeton Companion to Mathematics*, there are two branches of mathematics on the subject of combinatorics:
  - ***Enumerative and Algebraic Combinatorics*** (Counting)
  - ***Extremal and Probabilistic Combinatorics*** (Hungarian)



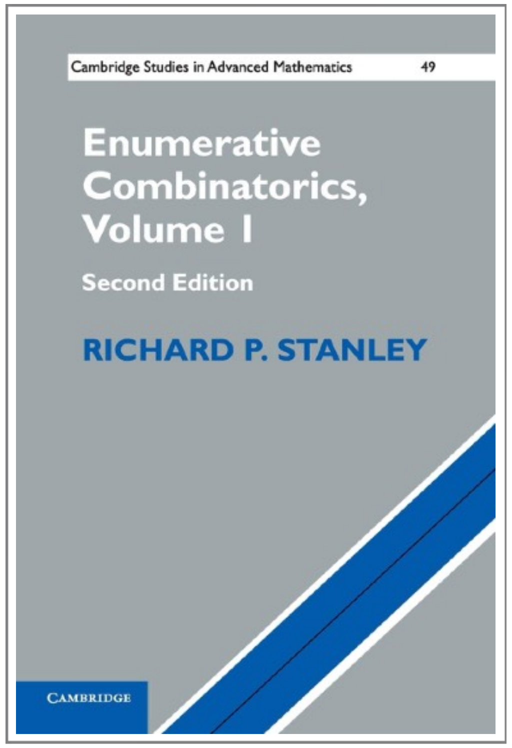
Combinatorics:

by the types of problems it studies

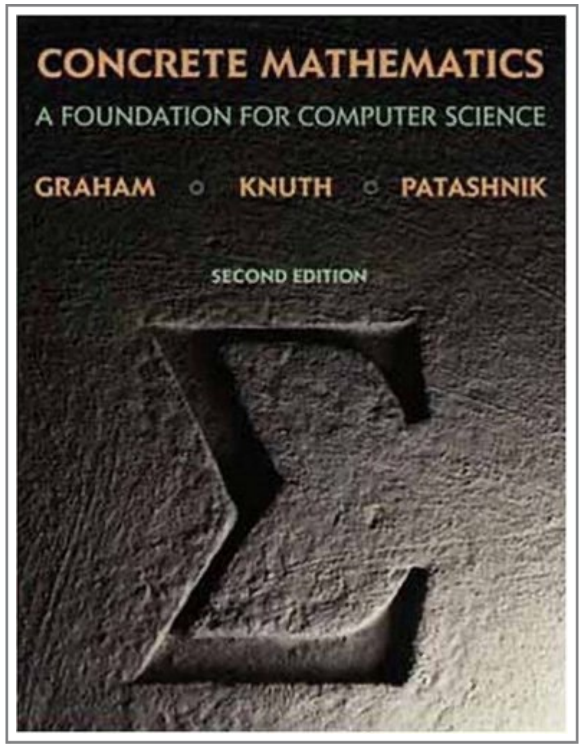


- Enumeration (counting) of finite structures (*e.g. solutions/ assignments/arrangements/configurations of finite systems*) satisfying certain given constraints.
- Existence of finite structures satisfying certain given constraints.
    - Extremal **problems**: How large/small a finite structure can be to satisfy certain given constraints?
    - Ramsey **problems**: When a finite structure becomes large enough, some regularity must show up somewhere.
- Construction (design) of such finite structures.
- Optimization: to find the best structure/solution in some sense.

References

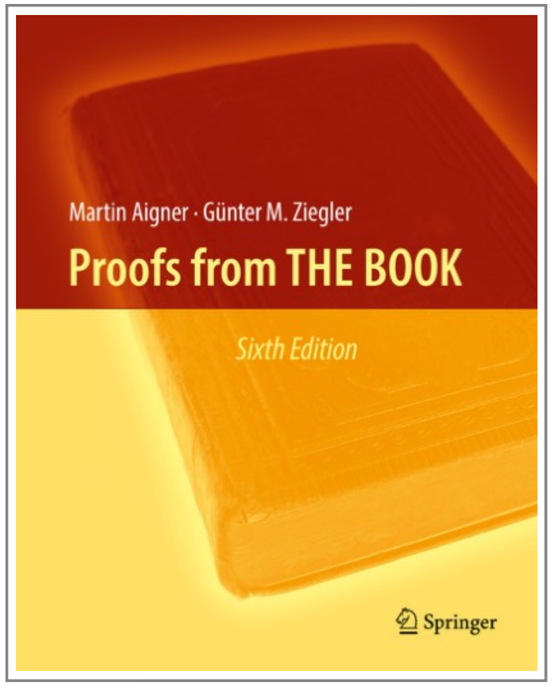


- Stanley.
  Enumerative Combinatorics,
  Volume 1, 2nd Edition.

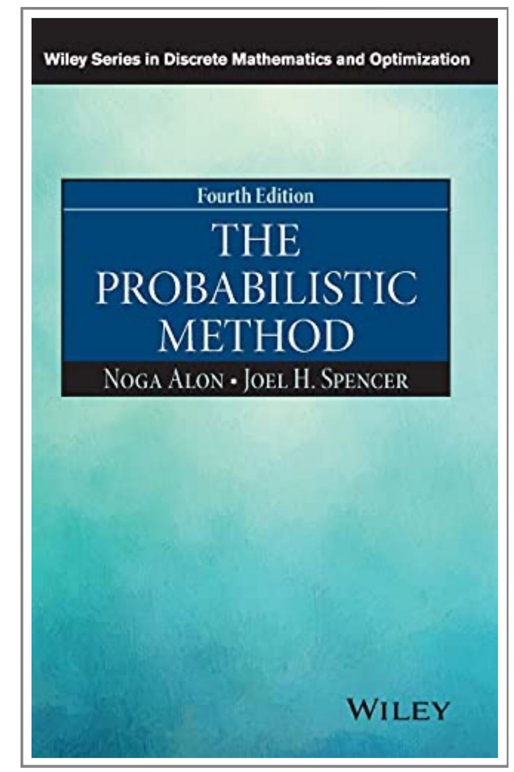


- Graham, Knuth, and Patashnik.
  Concrete Mathematics:
  A Foundation for Computer Science.

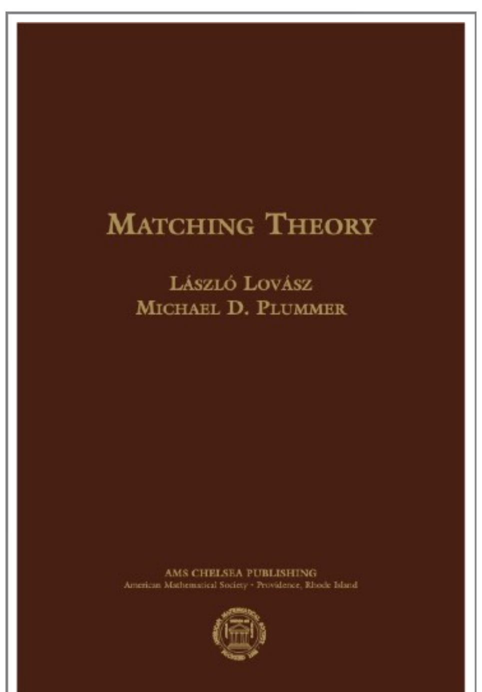
References



- Aigner and Ziegler.
  Proofs from THE BOOK,
  6th Edition.



- Alon and Spencer.
  The Probabilistic Method,
  4th Edition.



- Lovász and Plummer.
  Matching Theory.

Enumeration
(Counting)

- How many ways are there:
  - to rank n people?
  - to assign m zodiac signs to n people?
  - to choose m people out of n people?
  - to partition n people into m groups?
  - to distribute m yuan to n people?
  - to partition m yuan to n parts?
  - ... ...

Basic Enumeration:
The Twelvefold Way

Gian-Carlo Rota
(1932-1999)

The Twelvefold Way

$$f : N \to M \qquad |N| = n, \ |M| = m$$

elements of N	elements of M	any f	1-1	on-to
distinct	*distinct*			
identical	*distinct*			
distinct	*identical*			
identical	*identical*			

Knuth's version (in *TAOCP* vol.4A)

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n		
n identical balls, m distinct bins			
n distinct balls, m identical bins			
n identical balls, m identical bins			

Tuples

$$\{1, 2, \ldots, m\}$$

$[m] =$ ~~$\{0, 1, \ldots, m-1\}$~~

$$[m]^n = \underbrace{[m] \times \cdots \times [m]}_{n}$$

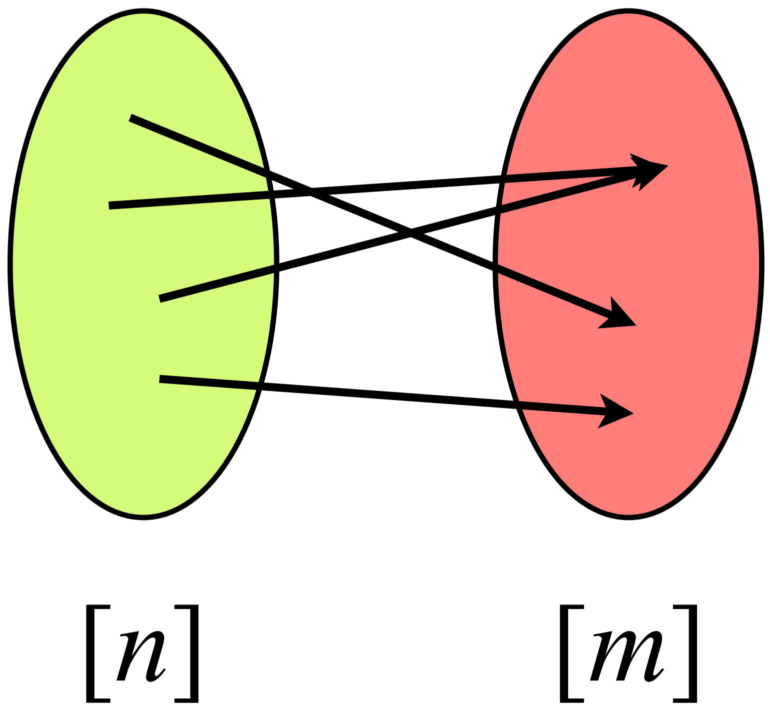
$$|[m]^n| = m^n$$

Product rule:

For finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Functions



count the # of functions

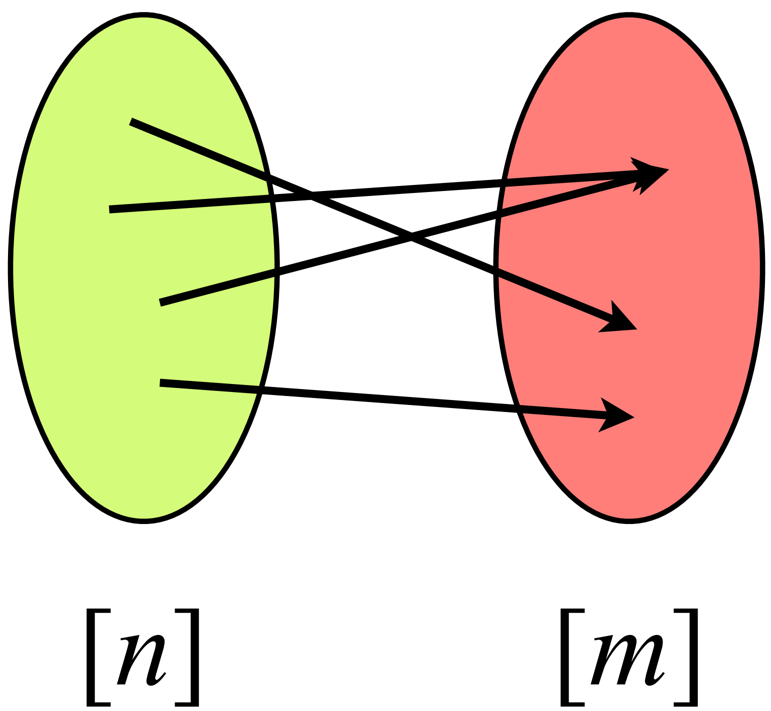
$$f : [n] \to [m]$$

$$(f(1), f(2), \ldots, f(n)) \in [m]^n$$

one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

Functions



count the # of functions

$$f : [n] \to [m]$$

one-one correspondence

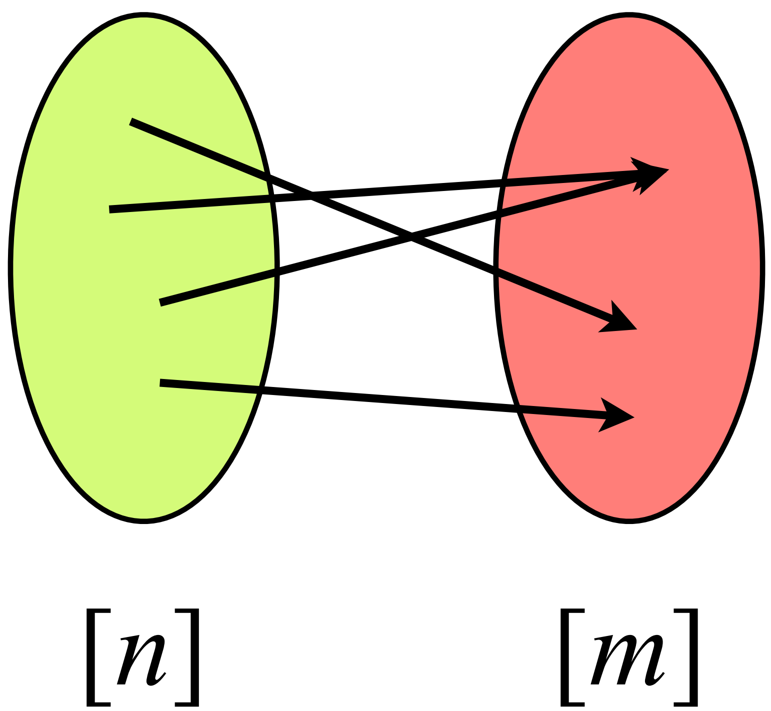
$$[n] \to [m] \Leftrightarrow [m]^n$$

Bijection rule:

For finite sets S and T

$$\exists \phi : S \xrightarrow[\text{onto}]{1-1} T \implies |S| = |T|$$

Functions



count the # of functions

$$f : [n] \to [m]$$

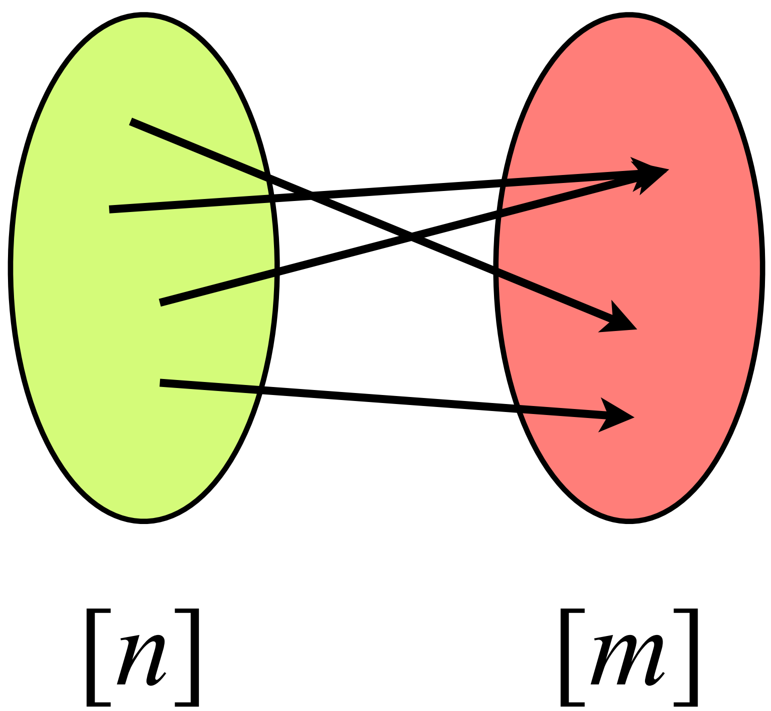
one-one correspondence

$$[n] \to [m] \Leftrightarrow [m]^n$$

$$|[n] \to [m]| = |[m]^n| = m^n$$

“*Combinatorial* proof.”

Injections



count the # of 1-1 functions

$$f : [n] \xrightarrow{1\text{-}1} [m]$$

one-to-one correspondence

$$\pi = (f(1), f(2), \ldots, f(n))$$

n-permutation: $\pi \in [m]^n$ of distinct elements

$$(m)_n = m(m-1)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

"m lower factorial n"

Subsets

subsets of { 1, 2, 3 }:

∅,

{1}, {2}, {3},

{1, 2}, {1, 3}, {2, 3},

{1, 2, 3}

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Combinatorial proof:

A subset $S \subseteq [n]$ corresponds to a string of n bits, where the i-th bit indicates whether $i \in S$.

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

Combinatorial proof:

$$S \subseteq [n] \Longleftrightarrow \chi_S \in \{0,1\}^n \quad \chi_S(i) = \begin{cases} 1 & i \in S \\ 0 & i \not\in S \end{cases}$$

one-to-one correspondence

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| =$$

A not-so-combinatorial proof:

Let $f(n) = \left|2^{[n]}\right|$

$$\boxed{f(n) = 2f(n-1)}$$

$$f(n) = \left|2^{[n]}\right|$$

$$\boxed{f(n) = 2f(n-1)}$$

$$2^{[n]} = \{S \subseteq [n] \mid n \notin S\} \cup \{S \subseteq [n] \mid n \in S\}$$

$$\left|2^{[n]}\right| = \left|2^{[n-1]}\right| + \left|2^{[n-1]}\right| = 2f(n-1)$$

Sum rule:

For **disjoint** finite sets S and T

$$|S \cup T| = |S| + |T|$$

Subsets

$$[n] = \{1, 2, \ldots, n\}$$

Power set: $2^{[n]} = \{S \mid S \subseteq [n]\}$

$$\left|2^{[n]}\right| = 2^n$$

A not-so-combinatorial proof:

Let $f(n) = \left|2^{[n]}\right|$

$$\boxed{f(n) = 2f(n-1)} \qquad f(0) = |2^{\emptyset}| = 1$$

Basic Rules for Counting

Sum rule:

For **disjoint** finite sets S and T

$$|S \cup T| = |S| + |T|$$

Product rule:

For finite sets S and T

$$|S \times T| = |S| \cdot |T|$$

Bijection rule:

For finite sets S and T

$$\exists \phi : S \xrightarrow[\text{onto}]{1-1} T \implies |S| = |T|$$

Subsets of fixed size

2-subsets of { 1, 2, 3 }: {1, 2}, {1, 3}, {2, 3}

k-uniform $$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$$

$$\binom{n}{k} = \left| \binom{[n]}{k} \right|$$

"n choose k"

Subsets of fixed size

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} = \frac{n!}{k!(n-k)!}$$

# of ordered k-subsets of $[n]$:

$$n(n-1)\cdots(n-k+1)$$

# of permutations of a k-set:

$$k(k-1)\cdots 1$$

Binomial Coefficients

Binomial coefficient: $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1. $$\binom{n}{k} = \binom{n}{n-k}$$

choose a k-subset ⇔ choose its compliment

2. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

0-subsets + 1-subsets + ... + n-subsets = all subsets

Binomial Theorem

Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Proof:

$$(1+x)^n = \underbrace{(1+x)(1+x)\cdots(1+x)}_{n}$$

$$\#\text{ of } x^k\text{:}\quad \text{choose } k \text{ factors out of } n$$

Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Let $x = 1$.

$$S = \{x_1, x_2, \ldots, x_n\}$$

$$\begin{aligned} & \text{\# of subsets of } S \text{ of odd sizes} \\ = \; & \text{\# of subsets of } S \text{ of even sizes} \end{aligned}$$

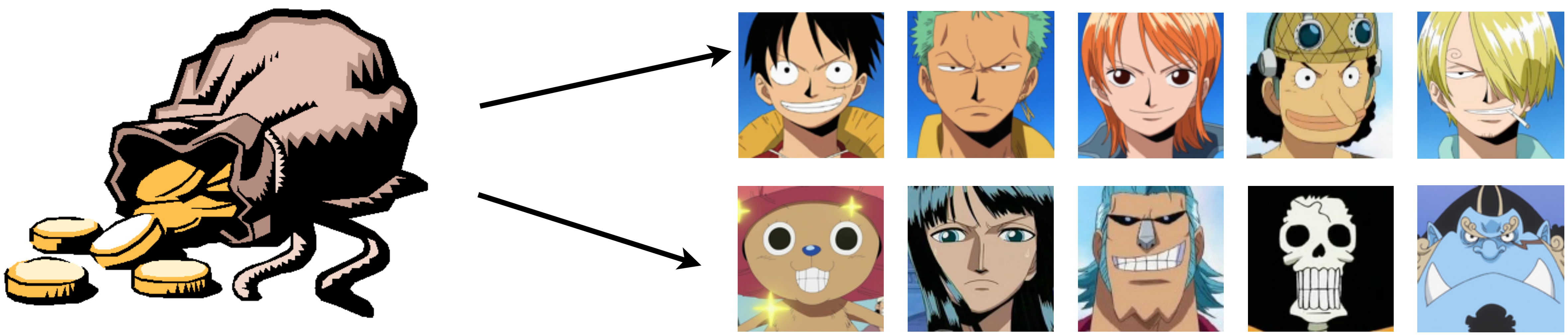
Let $x = -1$.

The Twelvefold Way

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins		$\binom{m}{n}$	
n distinct balls, m identical bins			
n identical balls, m identical bins			

Compositions of an Integer



n beli k pirates

How many ways to assign n beli to k pirates?

How many ways to assign n beli to k pirates, so that each pirate receives at least 1 beli?

Compositions of an Integer

$n \in \mathbb{Z}^+$

k-composition of n:

an ordered sum of k positive integers

a k-tuple $(x_1, x_2, \ldots, x_k)$ satisfying
$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

Compositions of an Integer

$n \in \mathbb{Z}^+$

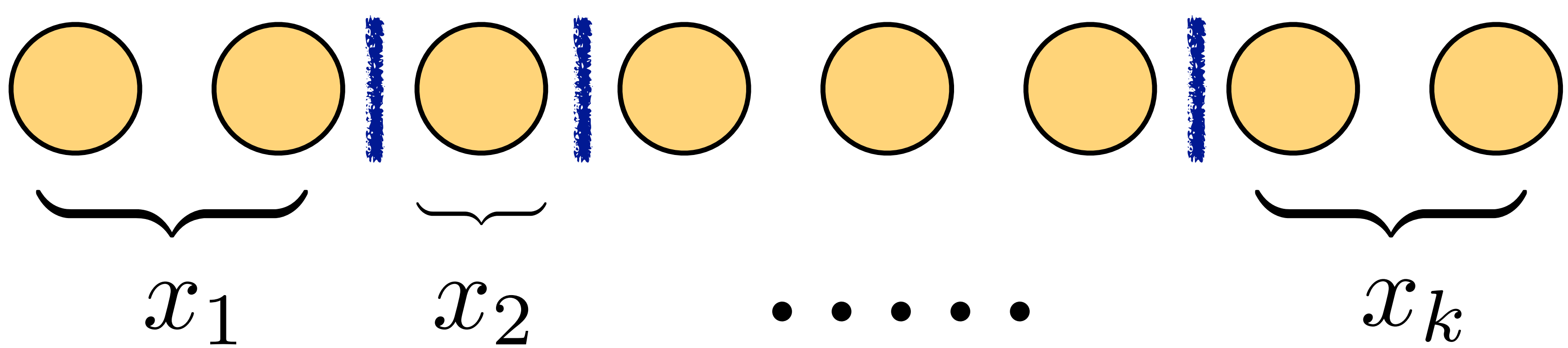
k-composition of n:

a k-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

\# of k-compositions of n? $\binom{n-1}{k-1}$

n identical balls



Compositions of an Integer

a k-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{Z}^+$

# of k-compositions of n? $\binom{n-1}{k-1}$

$$\phi((x_1, x_2, \ldots, x_k)) = \{x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3,\ \ldots,\ x_1 + x_2 + \cdots + x_{k-1}\}$$

ϕ is a 1-1 correspondence between

$\{k$-compositions of $n\}$ and $\binom{\{1,2,\ldots,n-1\}}{k-1}$

Compositions of an Integer

weak k-composition of n:

an ordered sum of k nonnegative integers

a k-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

Compositions of an Integer

weak k-composition of n:

a k-tuple $(x_1, x_2, \ldots, x_k)$ satisfying

$x_1 + x_2 + \cdots + x_k = n$ and $x_i \in \mathbb{N}$

# of weak k-compositions of n? $\binom{n+k-1}{k-1}$

$$(x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k$$

a k-composition of $n + k$

1-1 correspondence

Multisets

k-subset of S "k-combination of S *without* repetition"

3-combinations of { 1, 2, 3, 4 }

without repetition:

{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}

with repetition:

{1,1,1}, {1,1,2}, {1,1,3}, {1,1,4}, {1,2,2}, {1,3,3},
{1,4,4}, {2,2,2}, {2,2,3}, {2,2,4}, {2,3,3}, {2,4,4},
{3,3,3}, {3,3,4}, {3,4,4}, {4,4,4}

Multisets

multiset M on set S:

$$m : S \to \mathbb{N}$$

multiplicity of $x \in S$

$m(x)$: # of repetitions of x in M

cardinality $|M| = \sum_{x \in S} m(x)$

"k-combination of S with repetition" ⇔ k-multiset on S

$\left(\!\binom{n}{k}\!\right)$: # of k-multisets on an n-set

Multisets

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

$$k\text{-multiset on } S = \{x_1, x_2, \ldots, x_n\}$$

$$m(x_1) + m(x_2) + \cdots + m(x_n) = k$$

$$m(x_i) \geq 0$$

a weak n-composition of k

Multinomial Coefficients

permutations of a multiset on k elements
of size n with multiplicities $m_1, m_2, \ldots, m_k$

# of reordering of "*multinomial*"

permutations of {*a, i,i, l,l, m,m, n, o, t, u*}

assignments of n distinct balls to k distinct bins
with the i-*th* bin receiving m_i balls

multinomial coefficient $\binom{n}{m_1, \ldots, m_k}$

$$m_1 + m_2 + \cdots + m_k = n$$

Multinomial Coefficients

\# of permutations of a multiset on k elements of size n with multiplicities $m_1, m_2, \ldots, m_k$

||

\# of assignments of n distinct balls to k distinct bins with the *i-th* bin receiving m_i balls

$$\binom{n}{m_1, \ldots, m_k} = \frac{n!}{m_1! m_2! \cdots m_k!}$$

$$\binom{n}{m, n-m} = \binom{n}{m}$$

Multinomial Theorem

<u>Multinomial Theorem</u>:

$$(x_1 + x_2 + \cdots + x_k)^n$$
$$= \sum_{m_1+\cdots+m_k=n} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$$

Proof: $(x_1 + x_2 + \cdots + x_k)^n$

$$= \underbrace{(x_1 + x_2 + \cdots + x_k) \cdots\cdots (x_1 + x_2 + \cdots + x_k)}_{n}$$

$\#$ of $x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k}$:

assign n factors to k groups of sizes $m_1, m_2, \ldots, m_k$

The Twelvefold Way

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n *distinct balls,* m *distinct bins*	m^n	$(m)_n$	
n *identical balls,* m *distinct bins*	$\left(\binom{m}{n}\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n *distinct balls,* m *identical bins*			
n *identical balls,* m *identical bins*			

The Twelvefold Way

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n *distinct balls,* m *distinct bins*	m^n	$(m)_n$	
n *identical balls,* m *distinct bins*	$\binom{n+m-1}{m-1}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n *distinct balls,* m *identical bins*			
n *identical balls,* m *identical bins*			

Partitions of a Set

n pirates

k boats

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

- $A_i \neq \varnothing$;
- $A_i \cap A_j = \varnothing$;
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Partitions of a Set

$P = \{A_1, A_2, \ldots, A_k\}$ is a partition of S:

- $A_i \neq \emptyset$;
- $A_i \cap A_j = \emptyset$;
- $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

$\left\{ n \atop k \right\}$: # of k-partitions of an n-set

"Stirling number of the second kind"

$B_n = \sum_{k=1}^{n} \left\{ n \atop k \right\}$: total # of partitions of an n-set

"Bell number"

Stirling Number of the 2nd Kind

$\left\{ n \atop k \right\}$: # of k-partitions of an n-set

$$\left\{ n \atop k \right\} = k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$$

Case.1: $\{n\}$ is not a partition block

n is in one of the k blocks in a k-partition of $[n-1]$

Case.2: $\{n\}$ is a partition block

the remaining $k-1$ blocks forms a $(k-1)$-partition of $[n-1]$

The Twelvefold Way

n balls are put into m bins

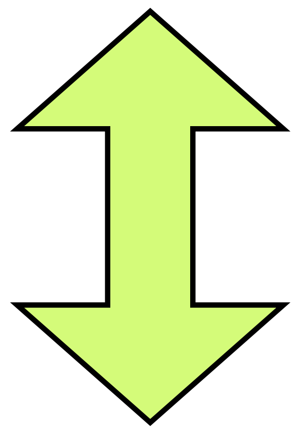
balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	
n identical balls, m distinct bins	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins			

Surjections

$$f : [n] \xrightarrow{\text{on-to}} [m]$$

$\forall i \in [m]$

$f^{-1}(i) \neq \emptyset$



$$(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(m))$$

ordered m-partition of $[n]$

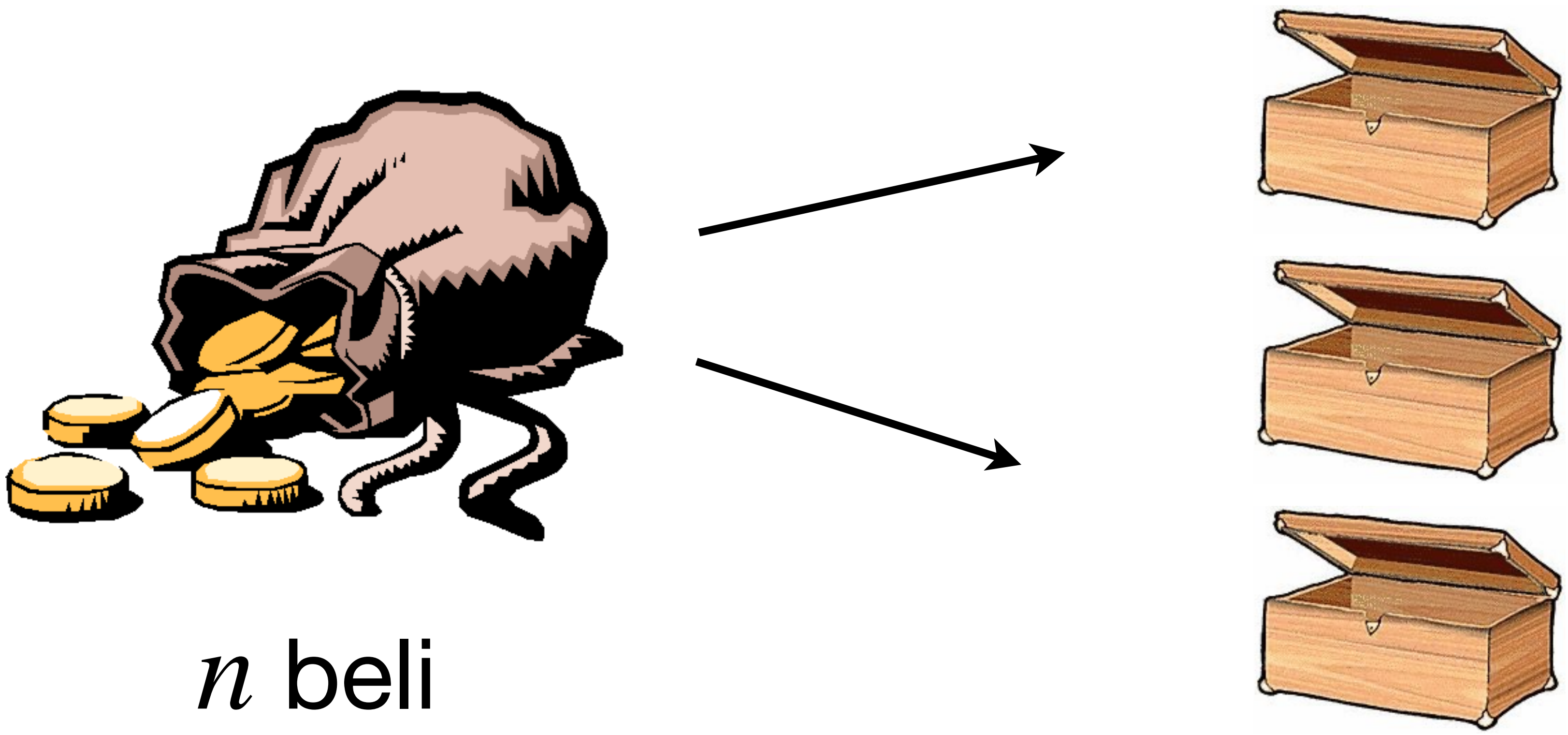
$$m! \left\{ {n \atop m} \right\}$$

The Twelvefold Way

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n *distinct balls,* m *distinct bins*	m^n	$(m)_n$	$m!\left\{ n \atop m \right\}$
n *identical balls,* m *distinct bins*	$\left(\!\binom{m}{n}\!\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n *distinct balls,* m *identical bins*	$\sum_{k=1}^{m} \left\{ n \atop k \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ n \atop m \right\}$
n *identical balls,* m *identical bins*			

Partitions of a Number



n beli

k boxes

a **partition** of number n into k parts:

an unordered sum of k positive integers

Partitions of a Number

a **partition** of n into k parts: "positive"

$n = 7$ "unordered"

{7}

{1,6}, {2,5}, {3,4}

{1,1,5}, {1,2,4}, {1,3,3}, {2,2,3}

{1,1,1,4}, {1,1,2,3}, {1,2,2,2}

{1,1,1,1,3}, {1,1,1,2,2}

{1,1,1,1,1,2}

{1,1,1,1,1,1,1}

$p_k(n)$: # of **partitions** of n into k parts

$p_k(n)$: # of **partitions** of n into k parts

integral solutions to $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$

$$p_k(n) = ?$$

$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$

$$\boxed{p_k(n) = p_{k-1}(n-1) + p_k(n-k)}$$

Case.1 $x_k = 1$

$(x_1, \ldots, x_{k-1})$ is a $(k-1)$-partition of $n-1$

Case.2 $x_k > 1$

$(x_1 - 1, \ldots, x_k - 1)$ is a k-partition of $n-k$

partition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$

composition $\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_i \geq 1 \end{cases}$

partition $\{x_1, \cdots, x_k\}$ → π (permutation) → composition $(x_1, \cdots, x_k)$

“on-to”

$$k! p_k(n) \geq \binom{n-1}{k-1}$$

partition $\{x_1, \cdots, x_k\}$ $\qquad$ $y_i = x_i + k - i$

$$x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq x_{k-1} \geq x_k \geq 1$$

$+k-1 \quad +k-2 \quad\quad +2 \quad +1$

$$y_1 > y_2 > \cdots > y_{k-2} > y_{k-1} > y_k > 1$$

π

permutation

composition of $n + \frac{k(k-1)}{2}$

$$(y_1, y_2, \ldots, y_k)$$

"1-1"

$$k!p_k(n) \leq \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$$

$$\frac{\binom{n-1}{k-1}}{k!} \le p_k(n) \le \frac{\binom{n+\frac{k(k-1)}{2}-1}{k-1}}{k!}$$

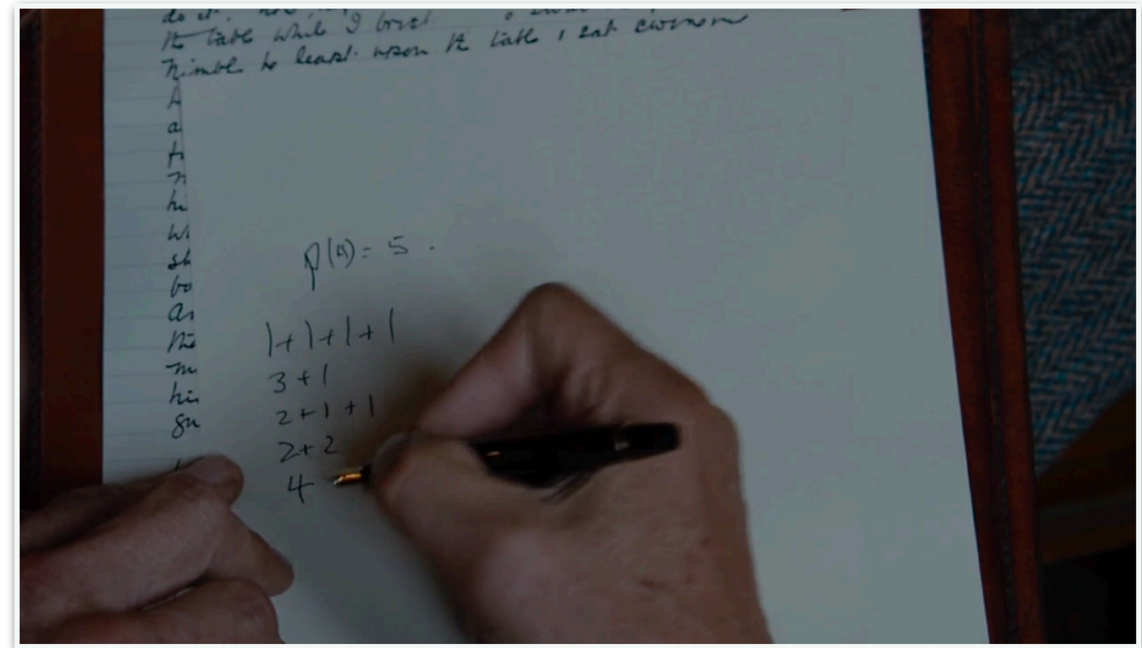
If k is fixed,

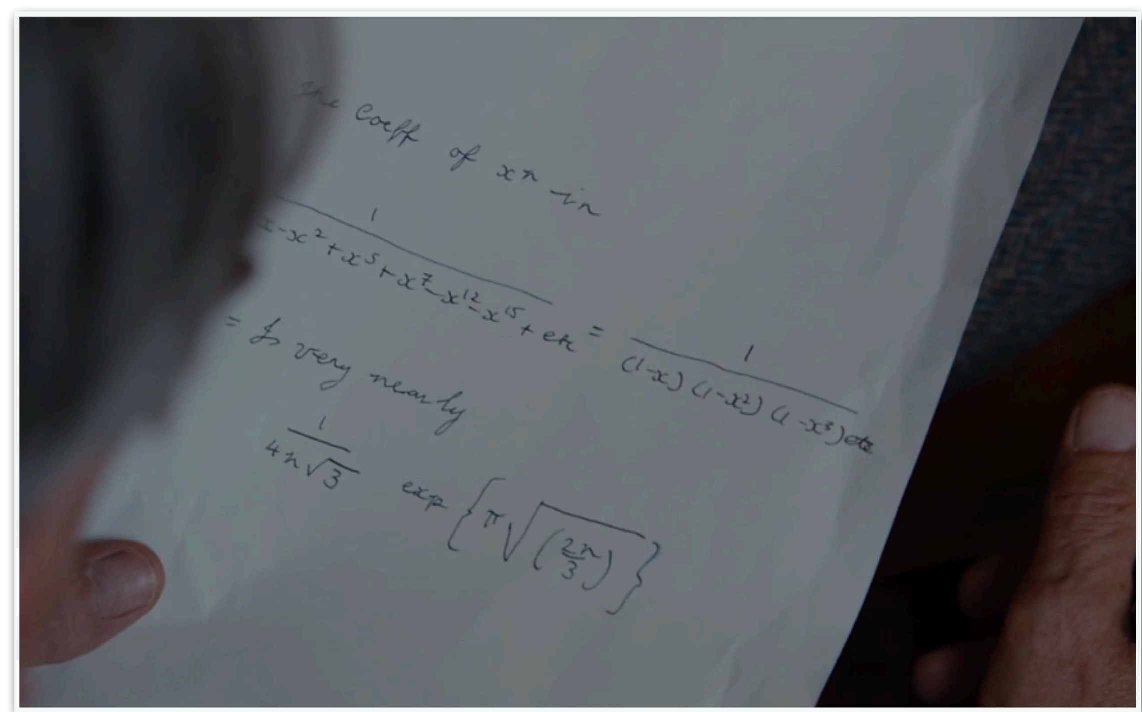
$$p_k(n) \sim \frac{n^{k-1}}{k!(k-1)!} \qquad \text{as} \quad n \to \infty$$

G. H. Hardy
(1877-1947)

Srinivasa
Ramanujan
(1887-1920)

The Man Who Knew Infinity
(2015 film)



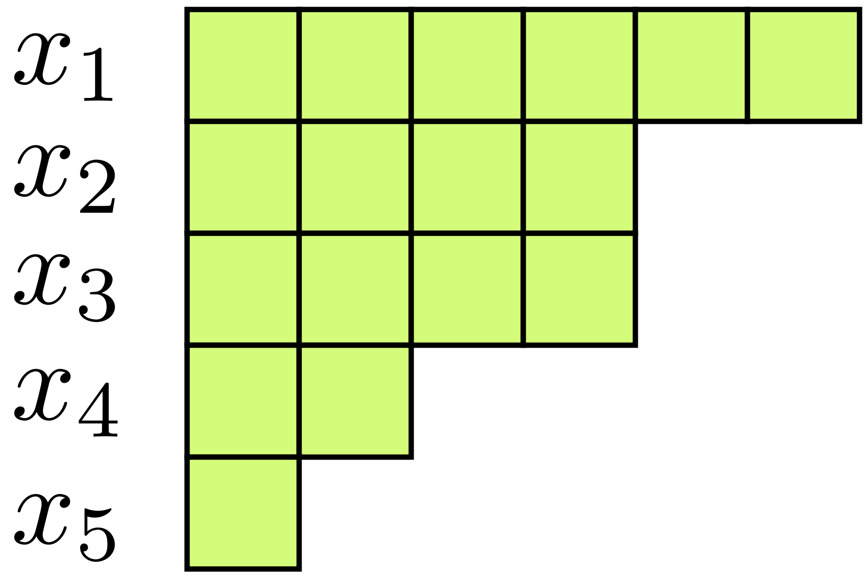


$$p(n) = \sum_{k=1}^{n} p_k(n)$$

$$\approx \frac{1}{4n\sqrt{3}} \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}$$

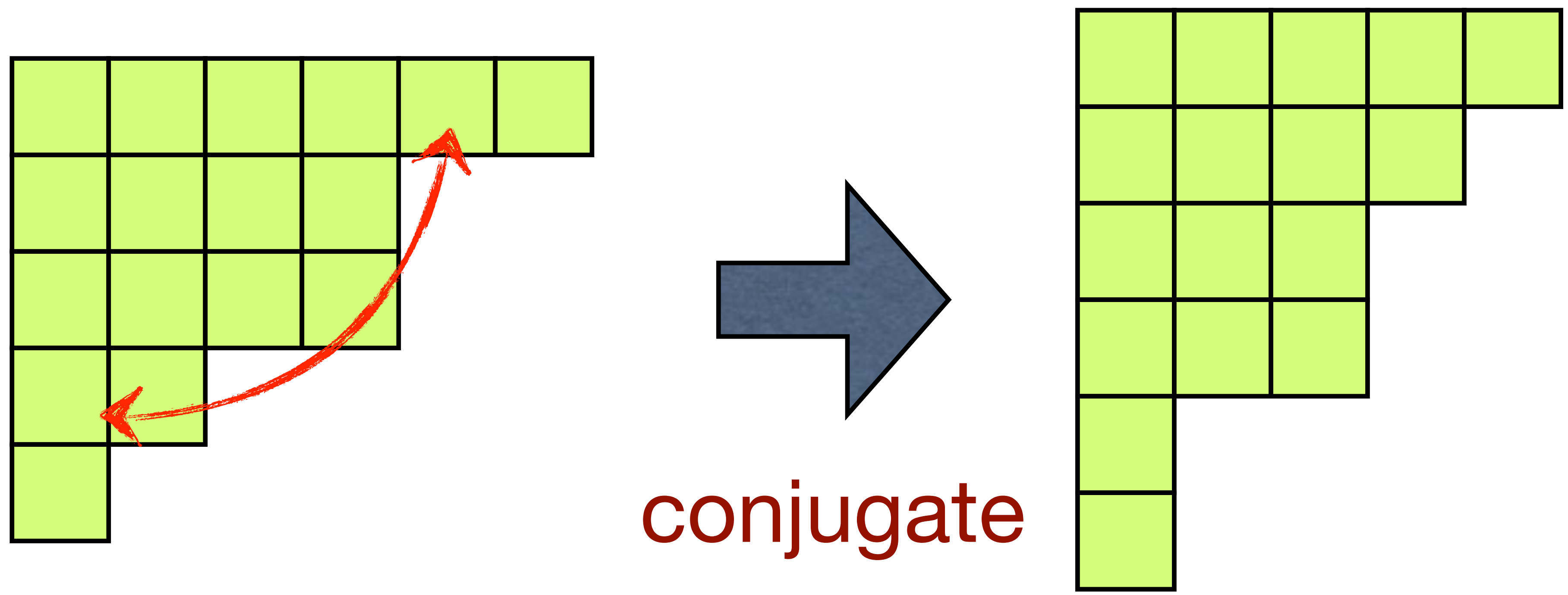
Ferrers diagram

(Young diagram)



partition

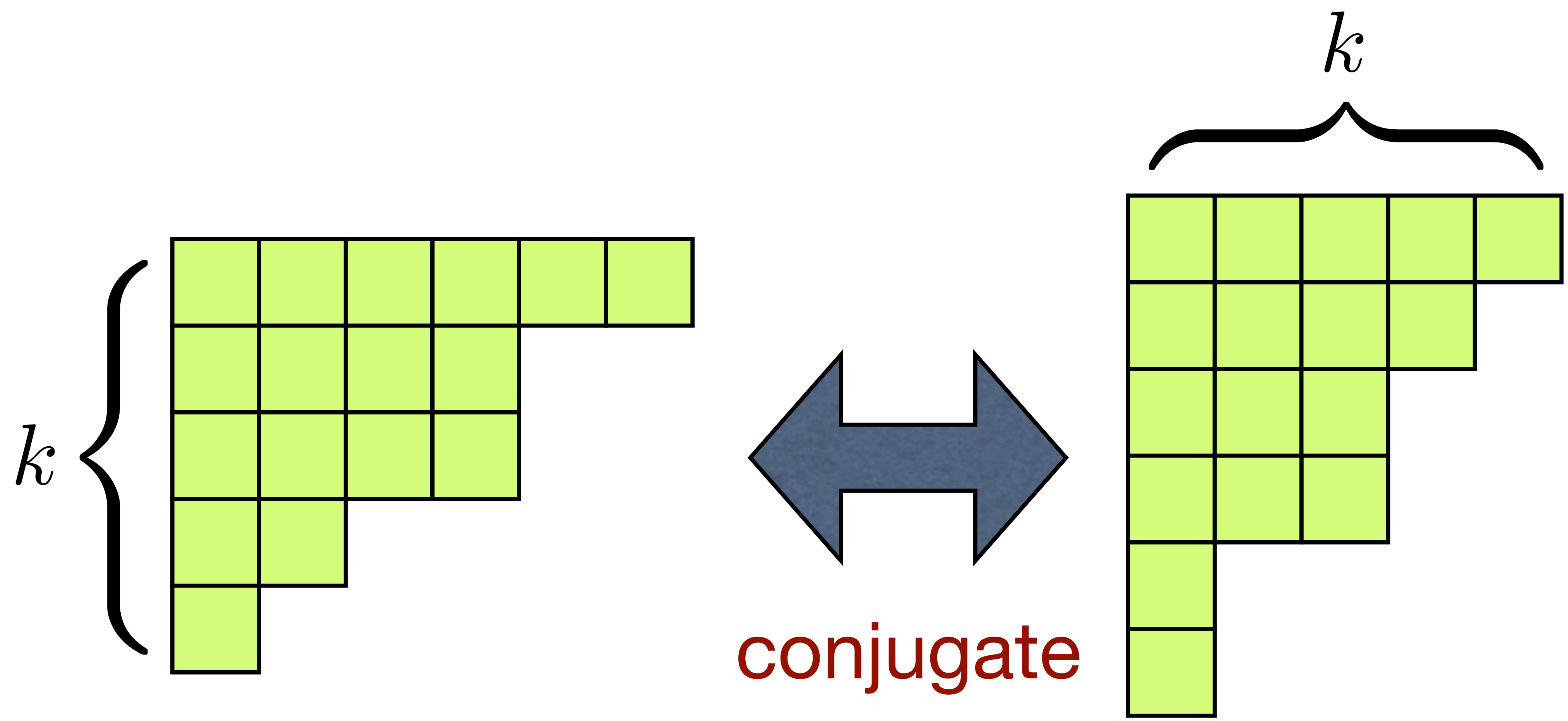
$$\begin{cases} x_1 + x_2 + \cdots + x_k = n \\ x_1 \geq x_2 \geq \cdots \geq x_k \geq 1 \end{cases}$$



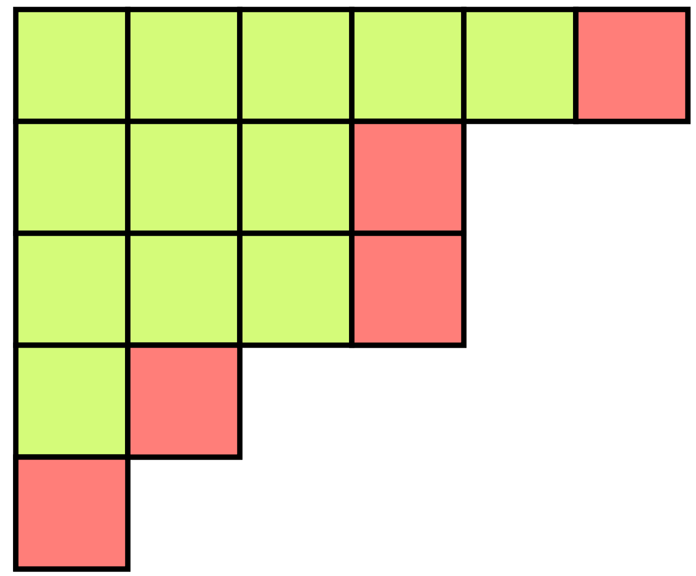
(6,4,4,2,1)

(5,4,3,3,1,1)

one-to-one correspondence



# of partitions of n into k parts = # of partitions of n with largest part k



# of partitions of n into k parts	=	# of partitions of n with largest part k

$$p_k(n) = \sum_{j=1}^{k} p_j(n-k)$$

The Twelvefold Way

n balls are put into m bins

balls per bin:	unrestricted	≤ 1	≥ 1
n distinct balls, m distinct bins	m^n	$(m)_n$	$m! \left\{ {n \atop m} \right\}$
n identical balls, m distinct bins	$\left(\kern-.3em\binom{m}{n}\kern-.3em\right)$	$\binom{m}{n}$	$\binom{n-1}{m-1}$
n distinct balls, m identical bins	$\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$\left\{ {n \atop m} \right\}$
n identical balls, m identical bins	$\sum_{k=1}^{m} p_k(n)$	$\begin{cases} 1 & \text{if } n \leq m \\ 0 & \text{if } n > m \end{cases}$	$p_m(n)$