Combinatorics

Existence Problems

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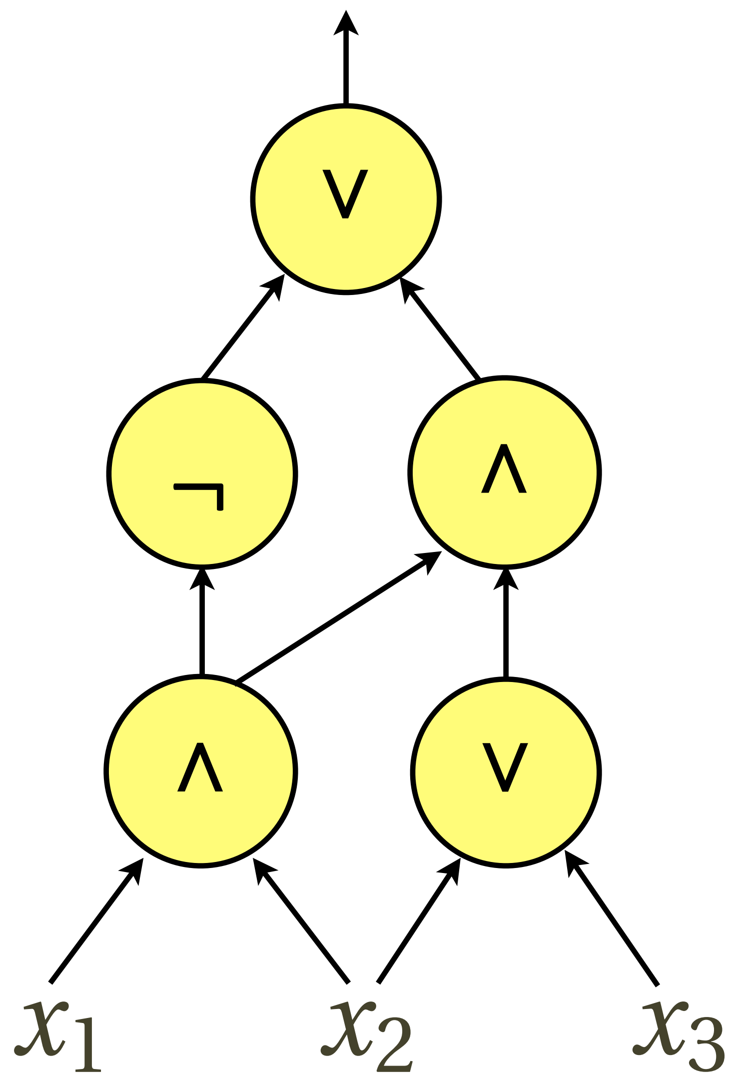
Counting Argument

Circuit Complexity

Boolean function $f: \{0,1\}^n \to \{0,1\}$

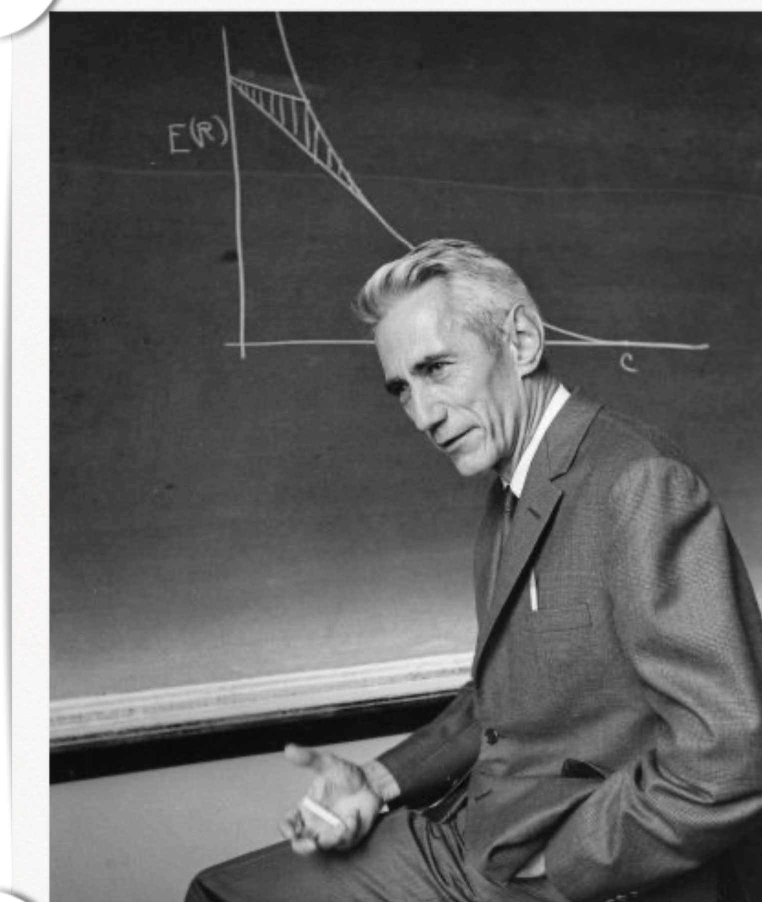
Boolean circuit



- ***DAG***
  (directed acyclic graph)
- Nodes:
  - **inputs:** $x_1, \ldots, x_n$
  - **gates:** $\wedge \ \vee \ \neg$
- **Complexity:** #gates

Theorem (Shannon 1949)

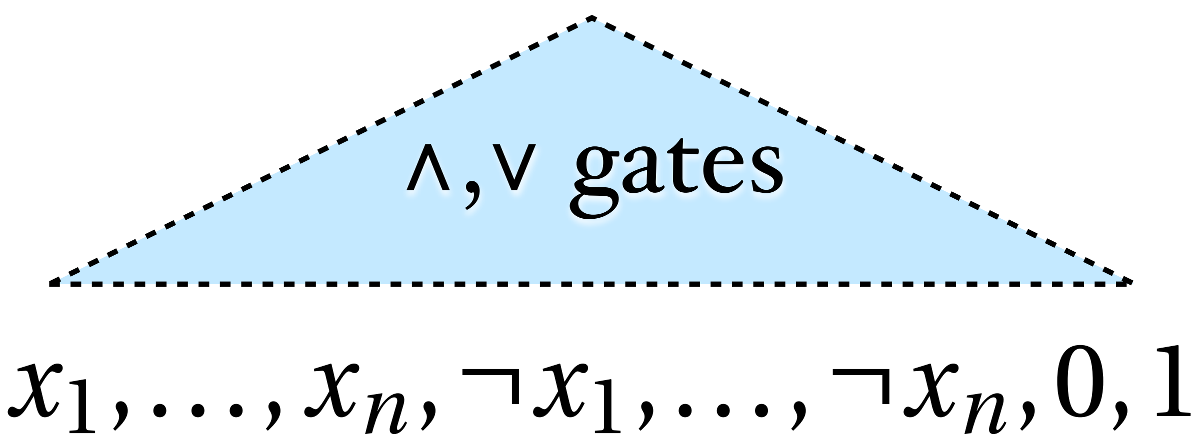
There is a boolean function $f: \{0,1\}^n \to \{0,1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.



Claude Shannon
(1916–2001)

of $f : \{0,1\}^n \to \{0,1\}$ $\left|\{0,1\}^n \to \{0,1\}\right| = 2^{2^n}$

of circuits with t gates: $< 2^t(2n + t + 1)^{2t}$



De Morgan's law:

$$\neg(A \vee B) = \neg A \wedge \neg B$$

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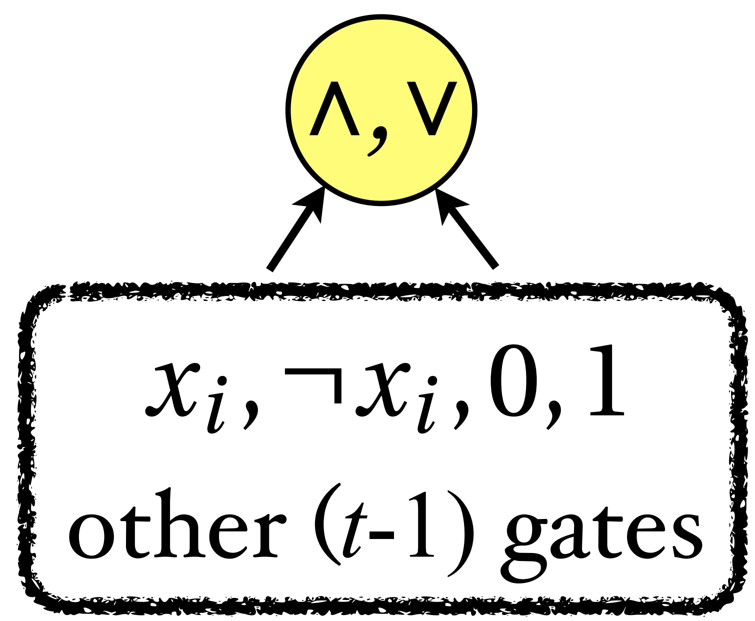


Theorem (Shannon 1949)

Almost all

~~There is a~~ boolean function $f : \{0,1\}^n \to \{0,1\}$ which cannot be computed by any circuit with $\frac{2^n}{3n}$ gates.

one circuit computes one function

$\#f$ computable by t gates ≤

#circuits with t gates ≤

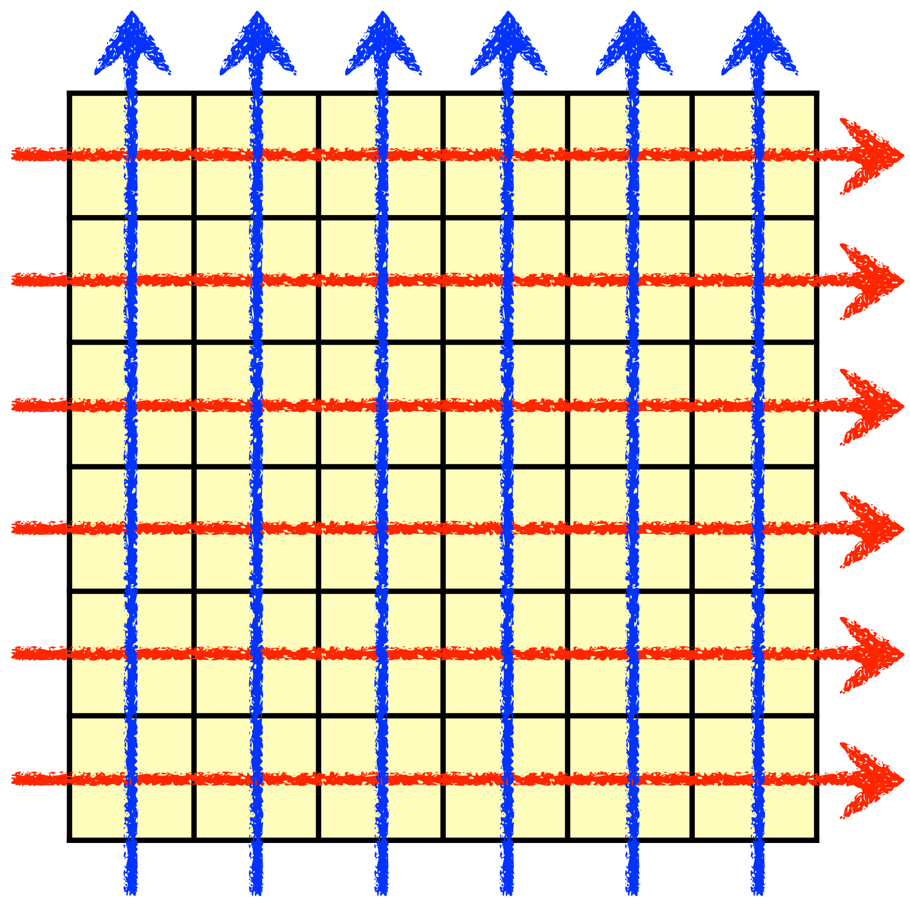
$$< 2^t(2n + t + 1)^{2t} \ll 2^{2^n} = \#f$$

for $t \leq \frac{2^n}{3n}$

Double Counting

"Count the same thing twice.
The result will be the same."

sum by row
A



sum by column
B

$A = B$

Handshaking Lemma

A party of n guests.

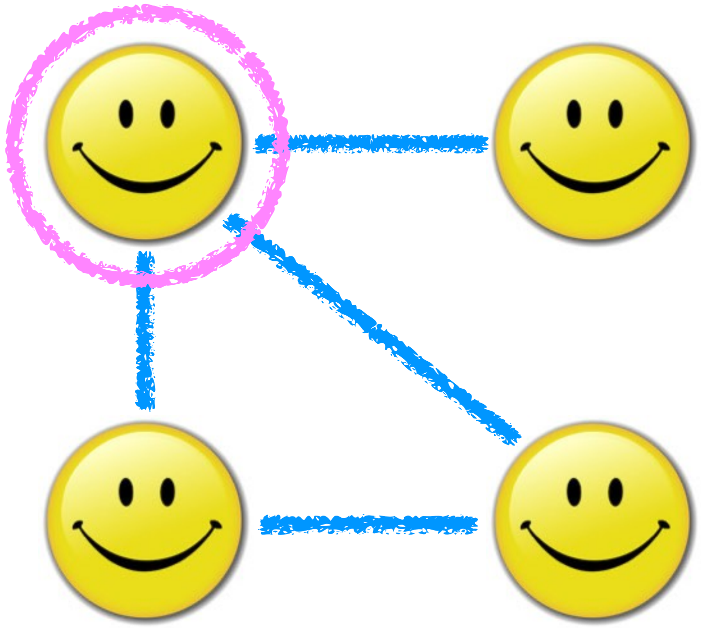
Handshaking Lemma: The number of people who shake an odd number of other people's hands is even.

Represented by graph:

n guests ⇔ n vertices

handshaking ⇔ edge

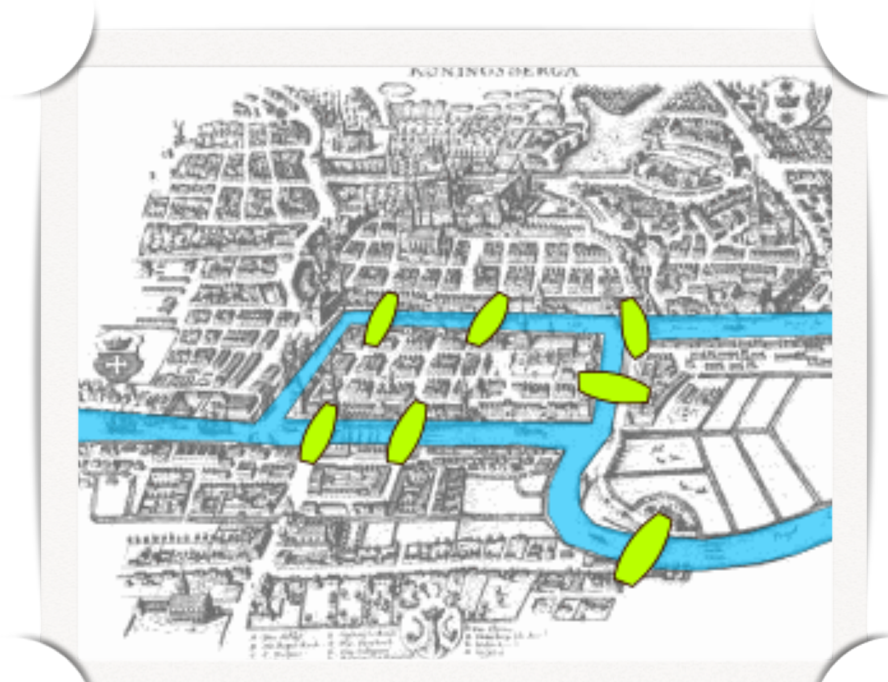
# of handshaking ⇔ degree



Handshaking Lemma (Euler 1736)

$$\sum_{v \in V} d(v) = 2|E|$$

Leonhard Euler



In the 1736 paper of *Seven Bridges of Königsberg*

Handshaking Lemma (Euler 1736)

$$\sum_{v \in V} d(v) = 2|E|$$

Count the # of edge *orientations*:

$$(u, v) : \{u, v\} \in E$$

Count by vertex:		Count by edge:
$\forall v \in V$		$\forall \{u, v\} \in E$
d directed edges	=	2 orientations
$(v, u_1) \cdots (v, u_d)$		(u, v) and (v, u)

Handshaking Lemma (Euler 1736)

$$\sum_{v \in V} d(v) = 2|E|$$

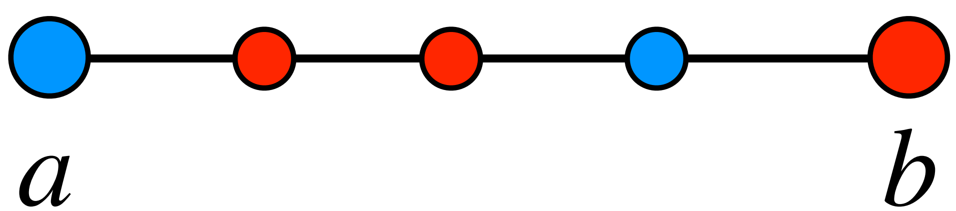
Corollary

# of odd-degree vertices is even.

Sperner's Lemma

line segment: ab divided into small segments

each endpoint: red or blue

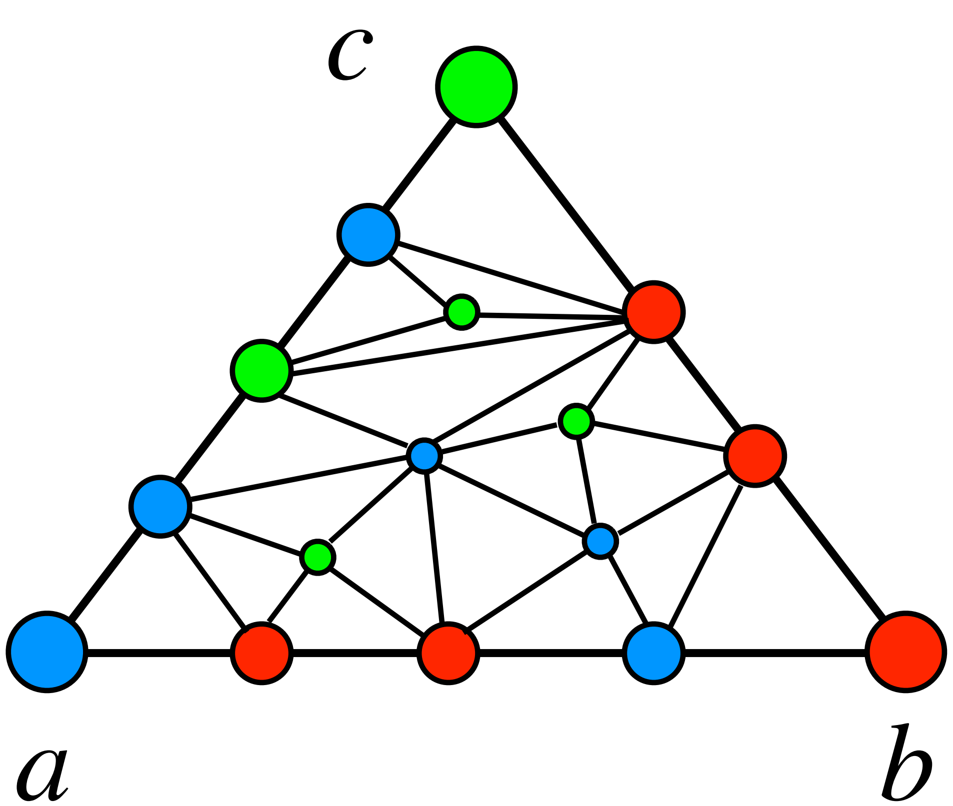


ab are colored differently

∃ small segment 

Emanuel Sperner
(1905–1980)

Sperner's Lemma



triangle: abc

triangulation

proper coloring:

3 colors red, blue, green

abc is tricolored

lines ab, bc, ac are 2-colored

Sperner's Lemma (1928)

∀ properly colored triangulation of a triangle,

∃ a properly colored small triangle.

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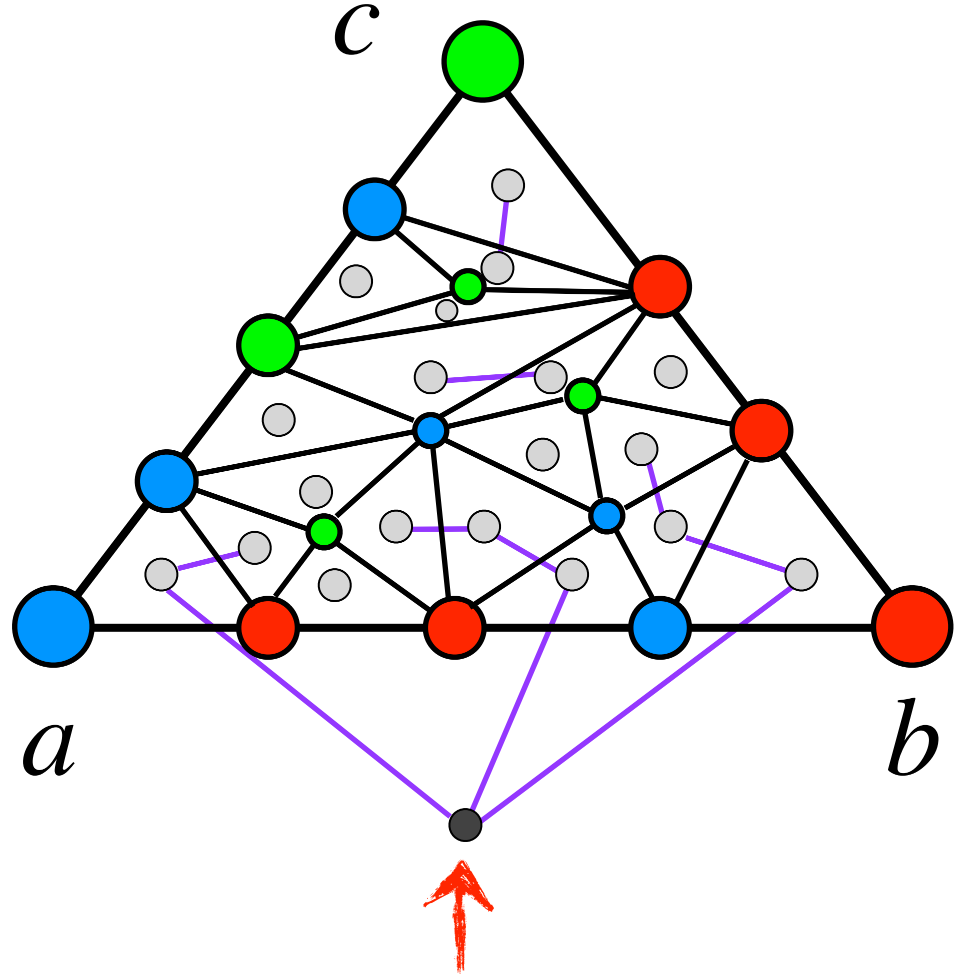
partial dual graph:

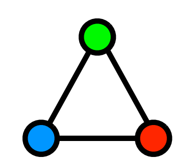


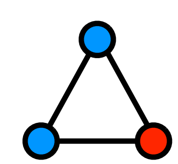
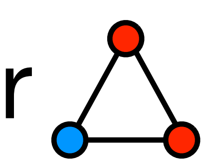
degree is odd

each △ is a vertex

the outer-space is a vertex

add an edge if 2 △ share a  edge

degree of  node: 1

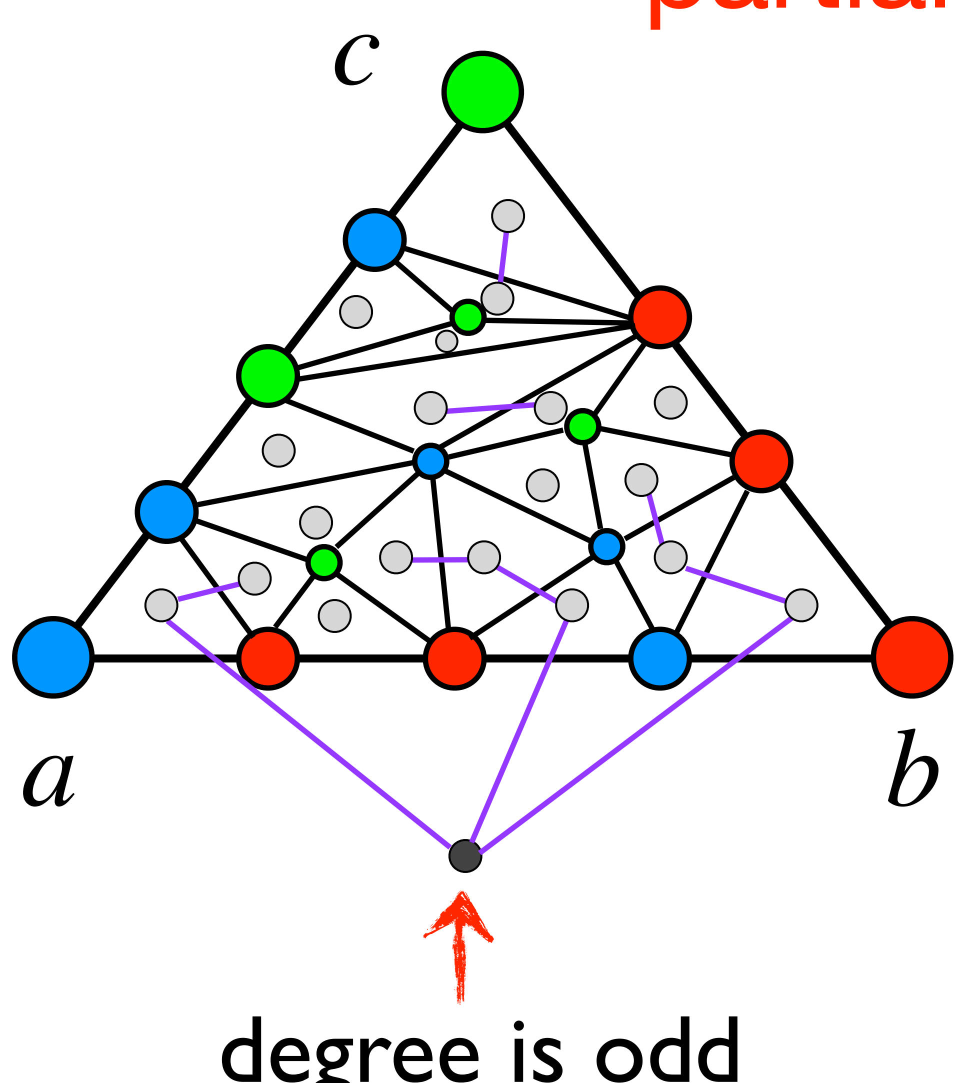
degree of  or  node: 2

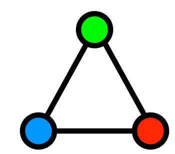
other cases: 0 degree

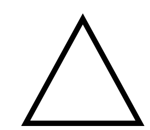
Sperner's Lemma (1928)

∀ properly colored triangulation of a triangle,
∃ a properly colored small triangle.

partial dual graph:

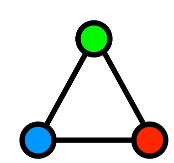


degree of  node: 1

degree of other  : even

handshaking lemma:

# of odd-degree vertices is even.

of  : odd $\neq 0$

Sperner's Lemma (1928)

∀ properly colored triangulation of a triangle,

∃ a properly colored small triangle.

high-dimension: triangle ➡ simplex

triangulation ➡ simplicial subdivision

Brouwer's fixed point theorem (1911)

∀ continuous function $f: B \to B$ of an n-dimensional ball B, ∃ a fixed point $x = f(x)$.

Averaging Principle

Pigeonhole Principle

"$n+1$ pigeons cannot sit in n holes so that every pigeon is alone in its hole."

Pigeonhole Principle

If $> mn$ objects are partitioned into n classes, then some class receives $> m$ objects.

Schubfachprinzip

"*drawer principle*"

Dirichlet Principle

Johann Peter Gustav Lejeune Dirichlet
(1805 – 1859)

Dirichlet's approximation

Approximate any irrational x
by a rational with bounded denominator.

Theorem (Dirichlet 1879)

$\forall$ irrational x and natural number n, $\exists$ a rational $\frac{p}{q}$ such that $1 \leq q \leq n$ and

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq}$$

Theorem (Dirichlet 1879)

$\forall$ irrational x and natural number n, $\exists$ a rational $\frac{p}{q}$ such that $1 \leq q \leq n$ and

$$\left| x - \frac{p}{q} \right| < \frac{1}{nq} \iff |qx - p| < \frac{1}{n}$$

fractional part: $\{x\} = x - \lfloor x \rfloor$

$(n+1)$ pigeons: $\{kx\}$ for $k = 1,2,\ldots,n+1$

n holes: $\left(0,\frac{1}{n}\right), \left(\frac{1}{n},\frac{2}{n}\right), \ldots, \left(\frac{n-1}{n},1\right)$

fractional part: $\{x\} = x - \lfloor x \rfloor$

$(n+1)$ pigeons: $\{kx\}$ for $k = 1,2,\ldots,n+1$

n holes: $\left(0,\frac{1}{n}\right),\left(\frac{1}{n},\frac{2}{n}\right),\ldots,\left(\frac{n-1}{n},1\right)$

$\exists 1 \leq b < a \leq n+1 \qquad \{ax\},\{bx\}$ in the same hole

$$|(a-b)x - (\lfloor ax \rfloor - \lfloor bx \rfloor)| = |\{ax\} - \{bx\}| < \frac{1}{n}$$

integers: $q \leq n$ $\qquad p$

$$|qx - p| < \frac{1}{n} \quad \Rightarrow \quad \left|x - \frac{p}{q}\right| < \frac{1}{nq}.$$

An *initiation* question to Mathematics

$\forall S \subseteq \{1, 2, \ldots, 2n\}$ that $|S| > n$

$\exists a, b \in S$ such that $a \mid b$

$\forall a \in \{1, 2, \ldots, 2n\}$

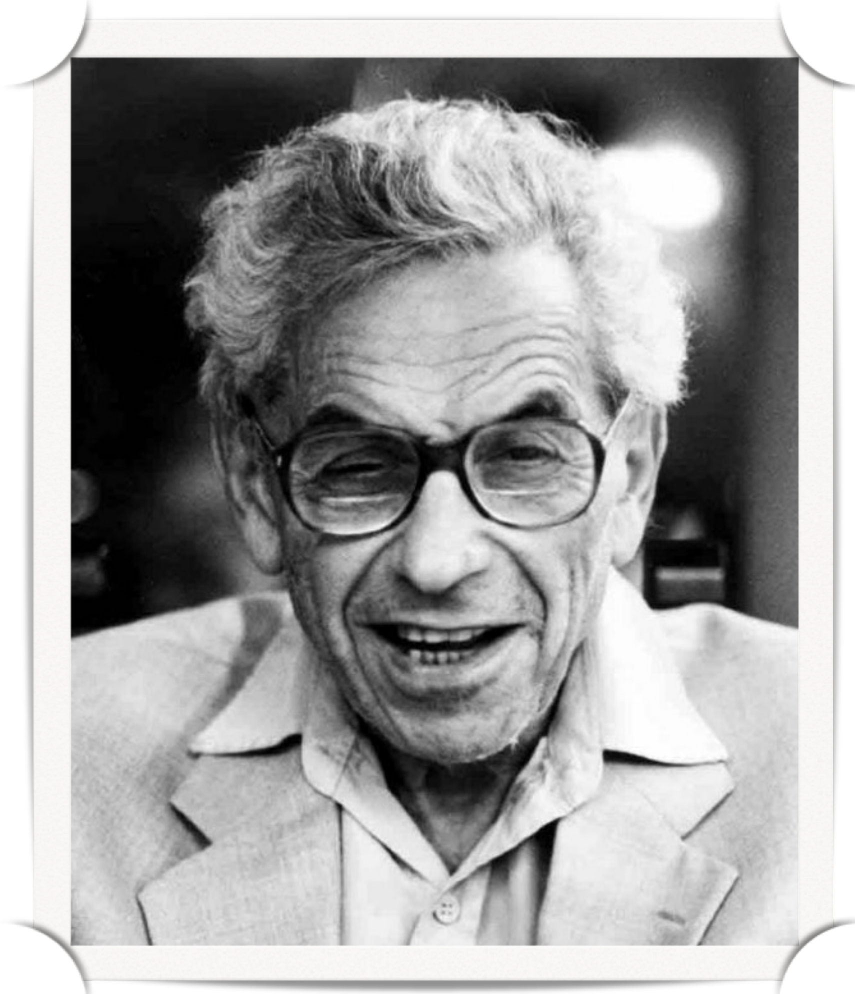
$a=2^k m$ for an odd m

$C_m = \{2^k m \mid k \geq 0, 2^k m \leq 2n\}$

$>n$ pigeons: S

n pigeonholes: $C_1, C_3, C_5, \ldots, C_{2n-1}$

$a < b \quad a, b \in C_m \Rightarrow a \mid b$



Paul Erdős
(1913—1996)

Monotonic subsequences

sequence: $(a_1, \ldots, a_n)$ of n different numbers

$1 \leq i_1 < i_2 < \cdots < i_k \leq n$

subsequence:

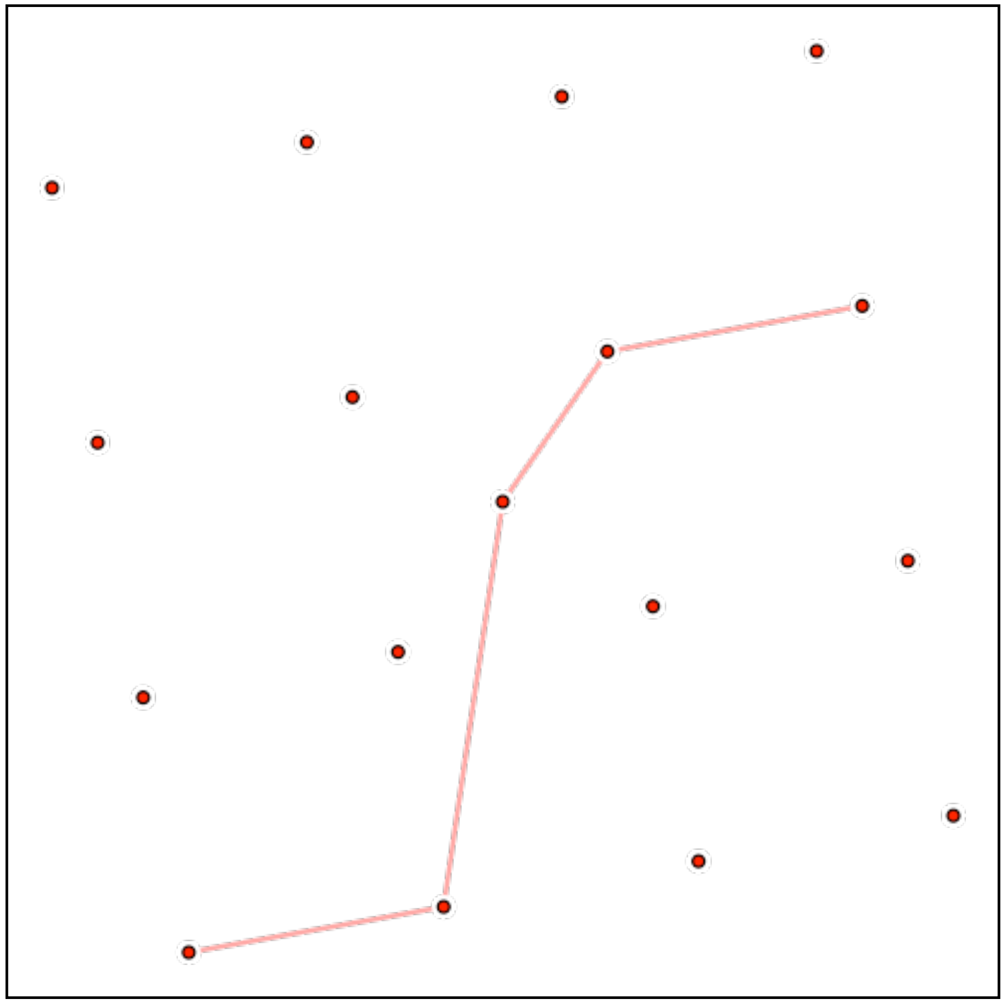
$$(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$$

increasing:

$$a_{i_1} < a_{i_2} < \ldots < a_{i_k}$$

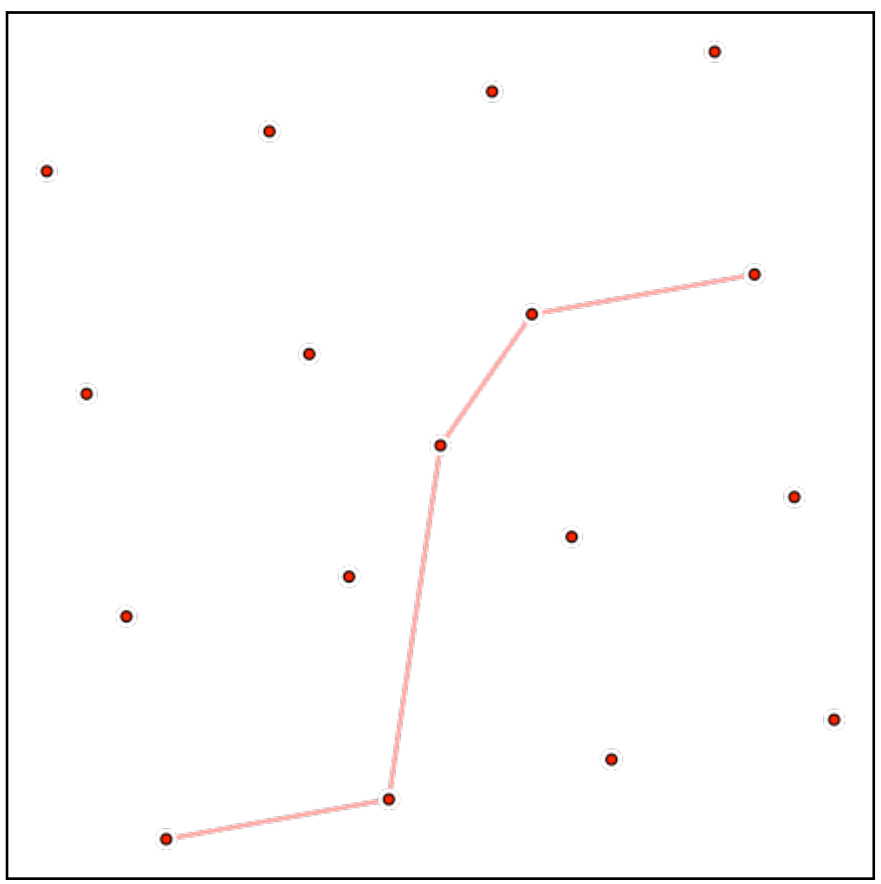
decreasing:

$$a_{i_1} > a_{i_2} > \ldots > a_{i_k}$$



Theorem (Erdős-Szekeres 1935)

A sequence of $> mn$ different numbers must contain either an increasing subsequence of length $m+1$, or a decreasing subsequence of length $n+1$.



$(a_1, \ldots, a_N)$ of N different numbers $\quad N > mn$

associate each a_i with (x_i, y_i)

x_i : length of longest *increasing* subsequence *ending* at a_i

y_i : length of longest *decreasing* subsequence *starting* at a_i

$$\forall i \neq j, \quad (x_i, y_i) \neq (x_j, y_j)$$

assume i<j

Cases.I: $a_i < a_j$ ⇒ $x_i < x_j$

Cases.2: $a_i > a_j$ ⇒ $y_i > y_j$

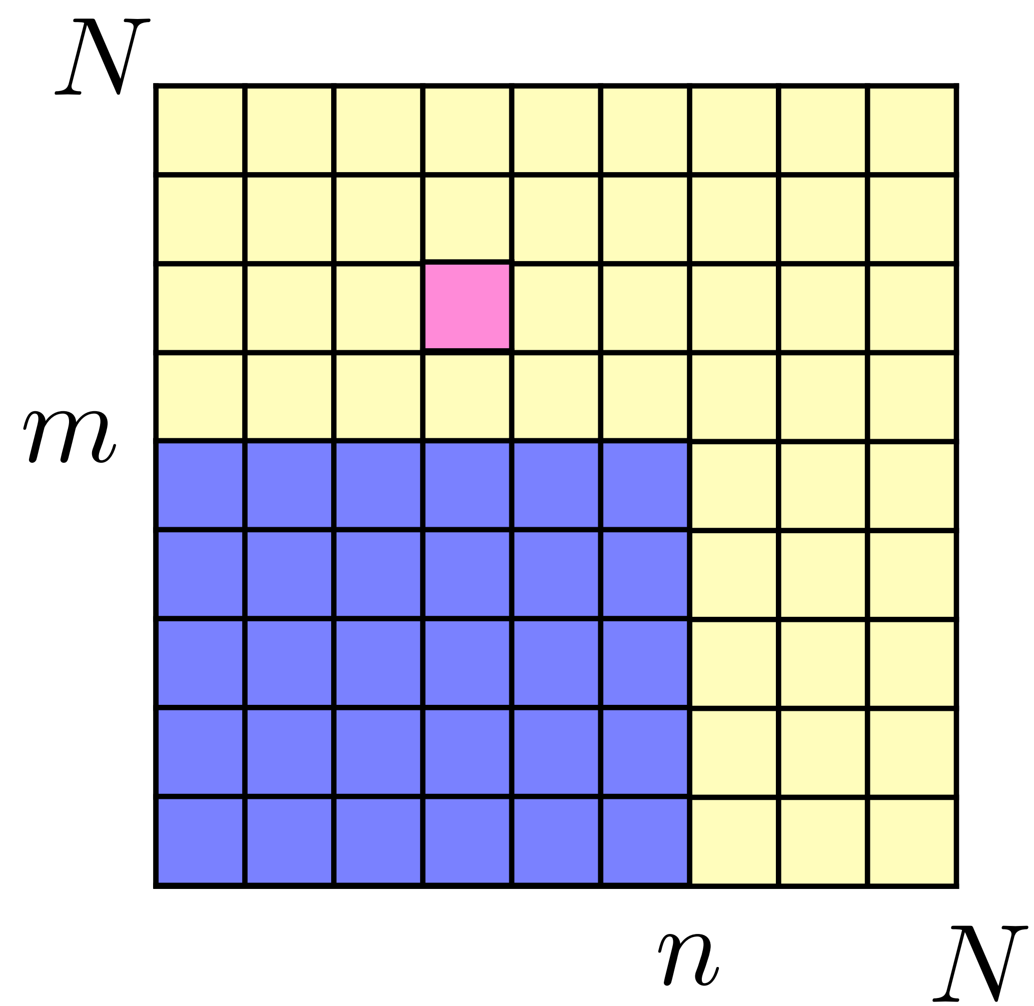
$(a_1, \ldots, a_N)$ of N different numbers $N > mn$

x_i : length of longest *increasing* subsequence *ending* at a_i

y_i : length of longest *decreasing* subsequence *starting* at a_i

"N pigeons" $(a_1, \ldots, a_N)$

a_i is in hole (x_i, y_i)



$$\forall i \neq j, \quad (x_i, y_i) \neq (x_j, y_j)$$

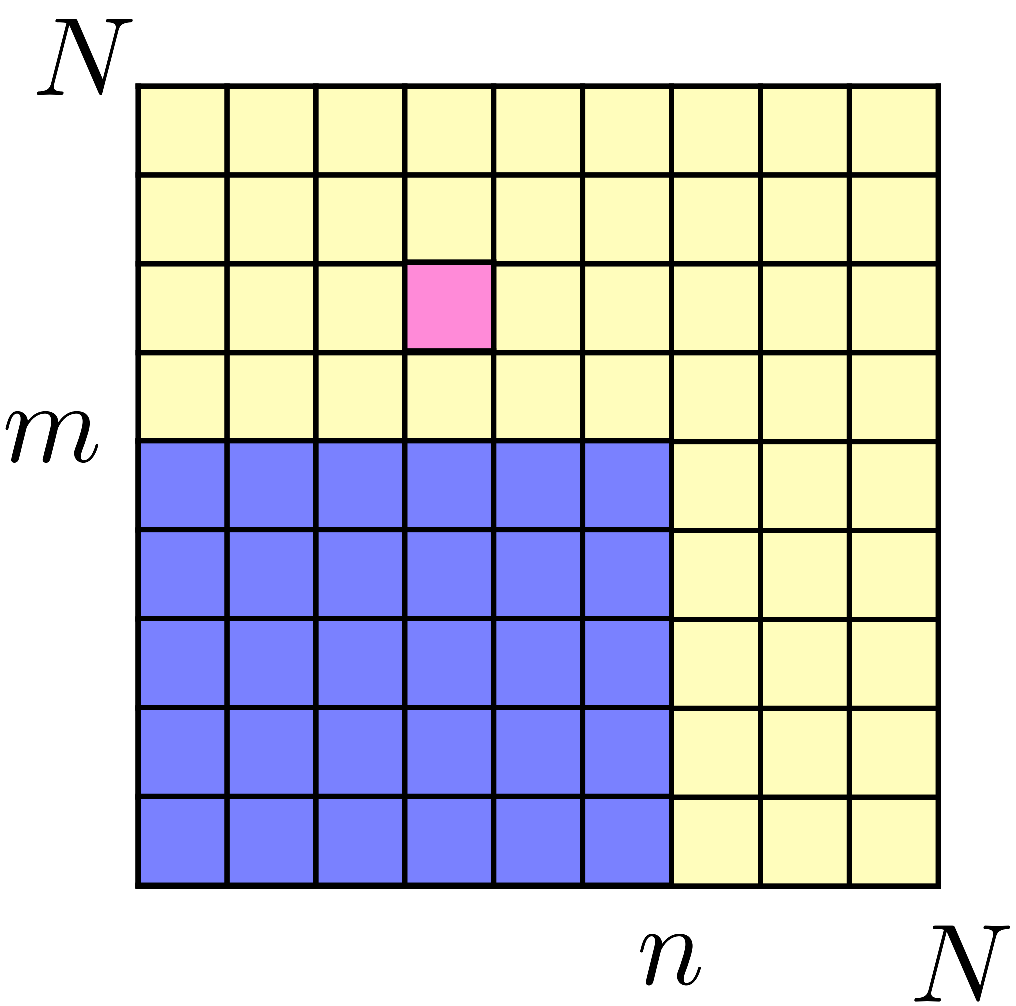
"One pigeon per each hole."

No way to put N pigeons into mn holes.

Theorem (Erdős-Szekeres 1935)

A sequence of $> mn$ different numbers must contain either an increasing subsequence of length $m+1$, or a decreasing subsequence of length $n+1$.

$(a_1, \ldots, a_N) \qquad N > mn$



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