Combinatorics

Extremal Graph Theory

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Extremal Combinatorics

"how large or how small a collection of finite objects can be, if it has to satisfy certain restrictions"

Extremal Problem:

"What is the largest number of edges that an n-vertex *cycle-free* graph can have?"

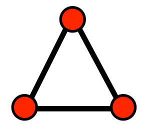
$$(n-1)$$

Extremal Graph:

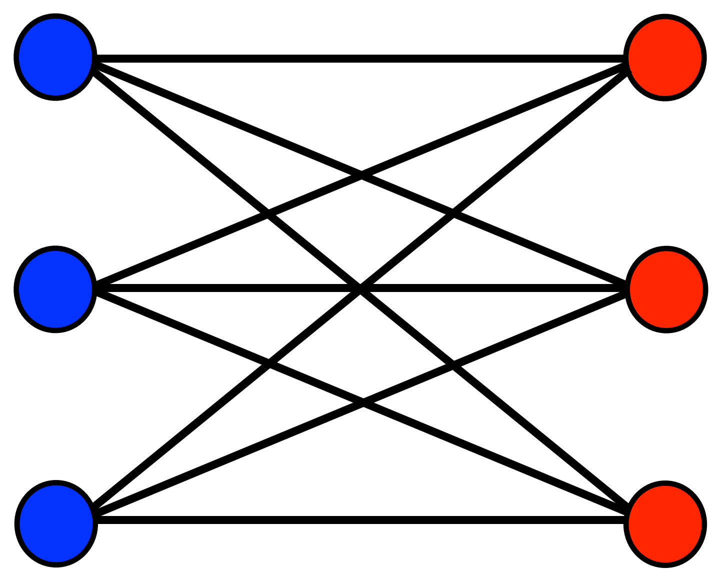
spanning tree

Triangle-Freeness

Triangle-free graph

contains no  as subgraph

Example: bipartite graph



$|E|$ is maximized for
complete balanced bipartite graph

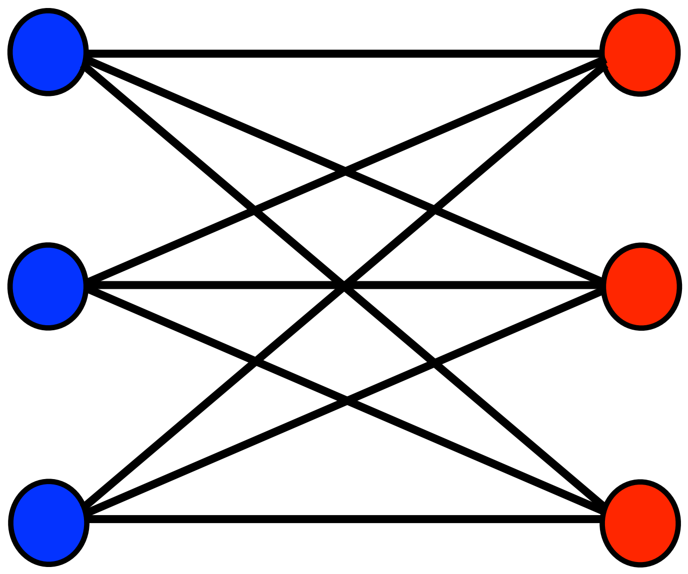
Extremal?

Mantel's Theorem

Theorem (Mantel 1907)

If $G(V,E)$ has $|V| = n$ and is triangle-free,

then $|E| \leq \dfrac{n^2}{4}$.



For n is even,
extremal graph:
$K_{\frac{n}{2},\frac{n}{2}}$

$\triangle$-free $\Longrightarrow |E| \leq n^2/4$

First Proof. Induction on n.

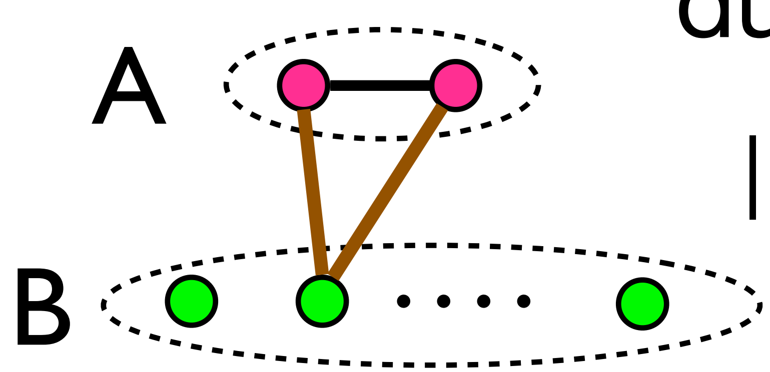
Basis: $n = 1,2.$ trivial

Induction Hypothesis: for any $n < N$

$$|E| > \frac{n^2}{4} \Longrightarrow G \supseteq \triangle$$

Induction step: for $n = N$

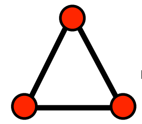
due to I.H. $|E(B)| \leq (n-2)^2/4$

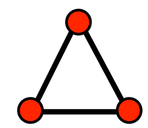


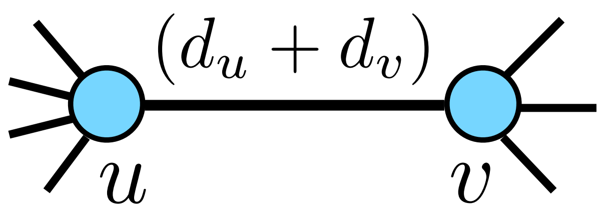
$$|E(A,B)| = |E| - |E(B)| - 1$$

$$> \frac{n^2}{4} - \frac{(n-2)^2}{4} - 1 = n - 2$$

pigeonhole!

-free $\Longrightarrow |E| \leq n^2/4$

Second Proof. -free



$$\Longrightarrow d_u + d_v \leq n, \quad \forall uv \in E$$

$$\Downarrow$$

Double counting: $\displaystyle\sum_{v \in V} d_v^2 = \sum_{uv \in E} (d_u + d_v) \leq n|E|$

Cauchy-Schwarz *(handshaking)*

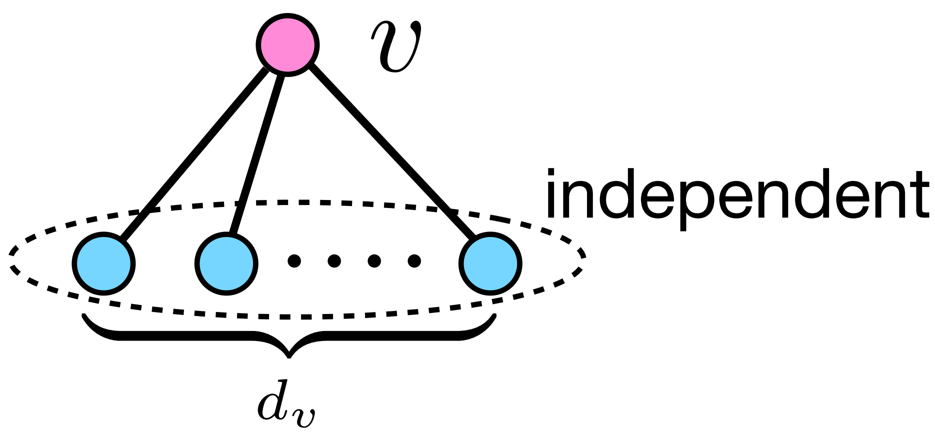
$$n^2|E| \geq n \sum_{v \in V} d_v^2 = \left(\sum_{v \in V} 1^2\right)\left(\sum_{v \in V} d_v^2\right) \geq \left(\sum_{v \in V} d_v\right)^2 = 4|E|^2$$

$$\Longrightarrow |E| \leq n^2/4$$

$\triangle$-free $\Longrightarrow |E| \leq n^2/4$

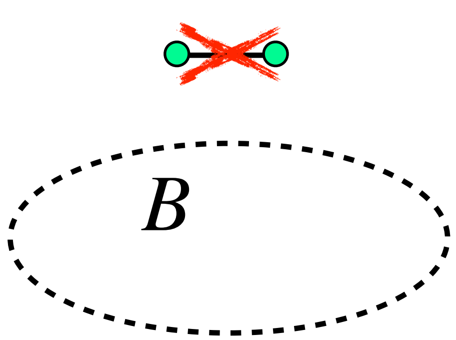
Third Proof.

A: maximum independent set $\qquad \alpha = |A|$



$$\forall v \in V, \quad d_v \leq \alpha$$

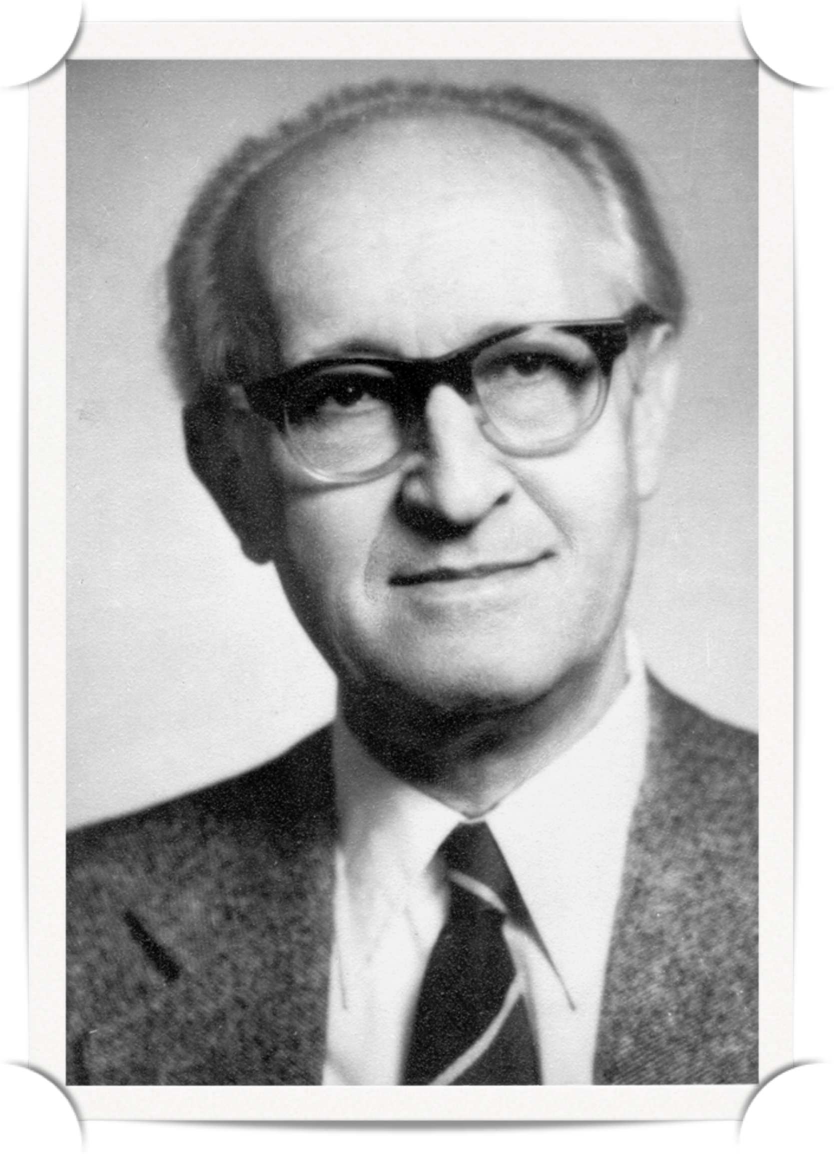
$B = V \setminus A$ $\quad$ B incident to all edges $\quad$ $\beta = |B|$



Inequality of the arithmetic and geometric mean

$$|E| \leq \sum_{v \in B} d_v \leq \alpha\beta \leq \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{n^2}{4}$$

Turán's Theorem



Paul Turán
(1910-1976)

Turán's Theorem

"Suppose G is a K_r*-free graph.*

What is the largest number of edges that G *can have?*"

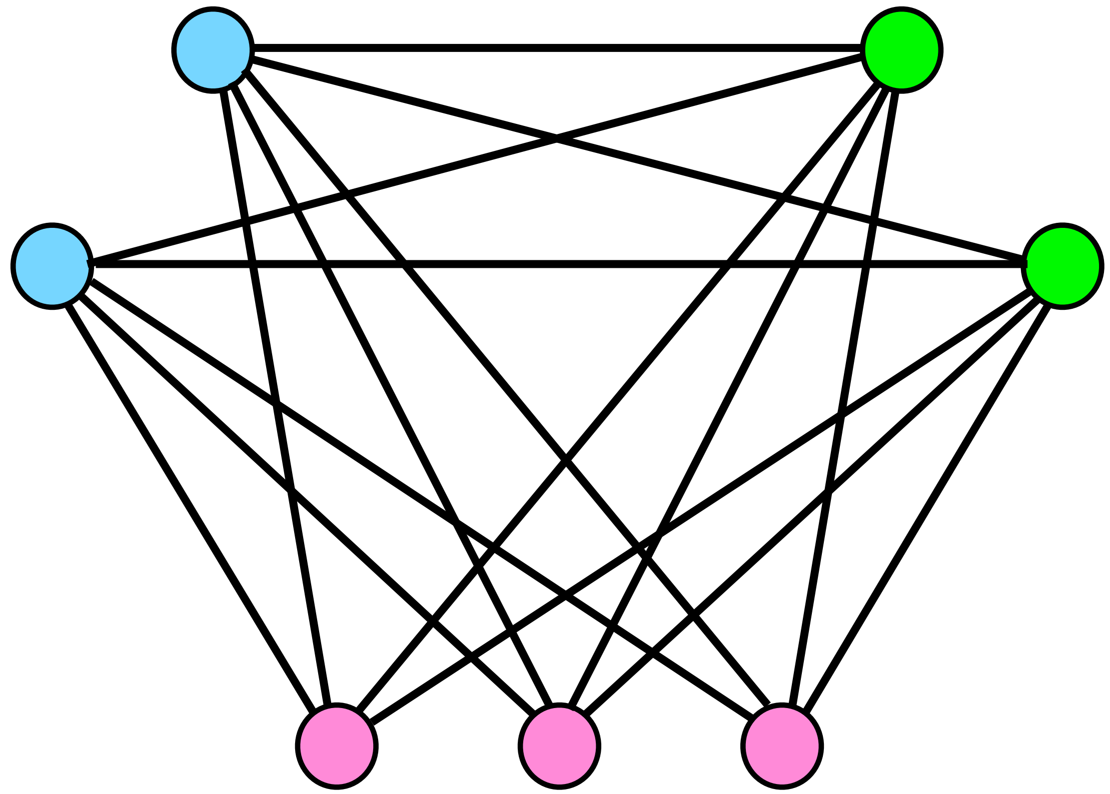
Theorem (Turán 1941)

If $G(V, E)$ has $|V| = n$ and is K_r-free, then

$$|E| \leq \frac{r-2}{2(r-1)}n^2$$

Complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$

$K_{2,2,3}$



Turán graph $T(n,r)$:

$$T(n,r) = K_{n_1,n_2,\ldots,n_r}$$

where $n_1 + \cdots + n_r = n$ and $n_i \in \left\{ \left\lfloor \frac{n}{r} \right\rfloor, \left\lceil \frac{n}{r} \right\rceil \right\}$

Turán graph $T(n, r)$:

$$T(n, r) = K_{n_1, n_2, \ldots, n_r}$$

where $n_1 + \cdots + n_r = n$ and $n_i \in \left\{ \left\lfloor \frac{n}{r} \right\rfloor, \left\lceil \frac{n}{r} \right\rceil \right\}$

$T(n, r-1)$ has no K_r

$$|T(n, r-1)| \leq \binom{r-1}{2} \left(\frac{n}{r-1} \right)^2$$

$$= \frac{r-2}{2(r-1)} n^2$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)} n^2$$

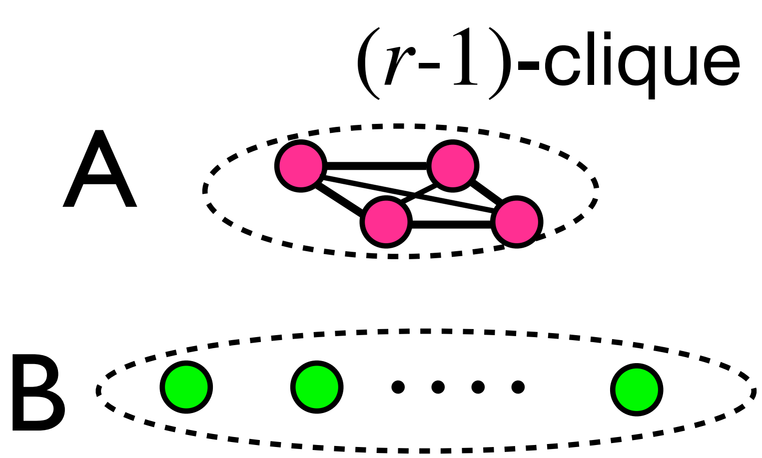
First Proof. (Induction)

Basis: $n = 1, 2, \ldots, r-1$.

Induction Hypothesis: true for any $n < N$

Induction step: for $n = N$,

suppose G is maximum K_r-free

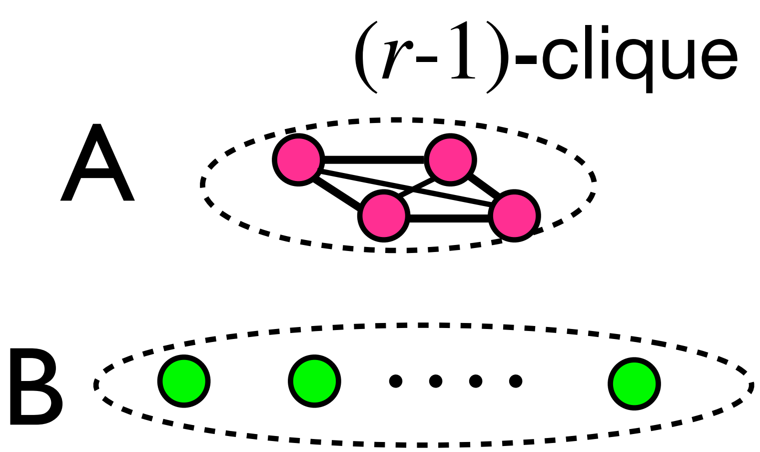


$\exists (r-1)$-clique

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)} n^2$$

First Proof. (Induction)

suppose G is maximum K_r-free



I.H.: $|E(B)| \le \frac{r-2}{2(r-1)}(n-r+1)^2$

K_r-free $\implies$ no $u \in B \sim$ all $v \in A$

$\implies |E(A,B)| \le (r-2)(n-r+1)$

$$
\begin{aligned}
|E| &= |E(A)| + |E(B)| + |E(A,B)| \\
&\le \binom{r-1}{2} + \frac{r-2}{2(r-1)}(n-r+1)^2 + (r-2)(n-r+1) \\
&= \frac{r-2}{2(r-1)} n^2
\end{aligned}
$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)}n^2$$

Second Proof. (weight shifting)

Assign each vertex v a weight $w_v > 0$ s.t. $\sum_{v \in V} w_v = 1$

Evaluate $S(\overrightarrow{w}) = \sum_{uv \in E} w_u w_v$

Let $W_u = \sum_{v \sim u} w_v$ For $u \nsim v$ that $W_u \ge W_v$

$$(w_u + \epsilon)W_u + (w_v - \epsilon) \ge w_u W_u + w_v W_v$$

shifting all weight of v to $u \implies S(\overrightarrow{w})$ non-decreasing

$S(\overrightarrow{w})$ is maximized $\implies$ all weights on a **clique**

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)} n^2$$

Second Proof. (weight shifting)

Assign each vertex v a weight $w_v > 0$ s.t. $\sum_{v \in V} w_v = 1$

Evaluate $S(\vec{w}) = \sum_{uv \in E} w_u w_v \le \binom{r-1}{2} \frac{1}{(r-1)^2}$

$S(\vec{w})$ is maximized $\implies$ all weights on a **clique**

when all $w_v = \dfrac{1}{n}$

$$S(\vec{w}) = \sum_{uv \in E} w_u w_v = \frac{|E|}{n^2}$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)} n^2$$

Third Proof. (The probabilistic method)

clique number $\omega(G)$: size of the largest clique

$$\omega(G) \ge \sum_{v \in V} \frac{1}{n - d_v}$$

random permutation π of V

$S = \{v \mid \pi_u < \pi_v \implies u \sim v\}$

is a **clique**

Linearity of expectation:

$$\mathbb{E}[|S|] = \sum_{v \in V} \Pr[v \in S] \ge \sum_{v \in V} \Pr[\forall u \nsim v : \pi_u \ge \pi_v]$$

$$= \sum_{v \in V} \frac{1}{n - d_v}$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)}n^2$$

Third Proof. (The probabilistic method)

$$\omega(G) \ge \sum_{v\in V} \frac{1}{n-d_v}$$

Cauchy-Schwarz

$$n = \sum_{v\in V} 1 \le \left(\sum_{v\in V} \frac{1}{n-d_v}\right)\left(\sum_{v\in V} (n-d_v)\right)$$

$$\le \omega(G) \sum_{v\in V} (n-d_v) = (r-1)(n^2 - 2|E|)$$

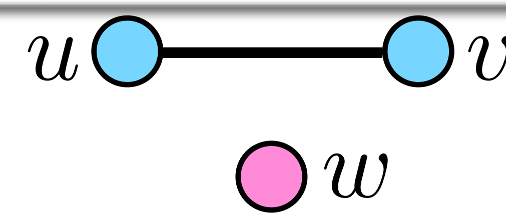
(handshaking)

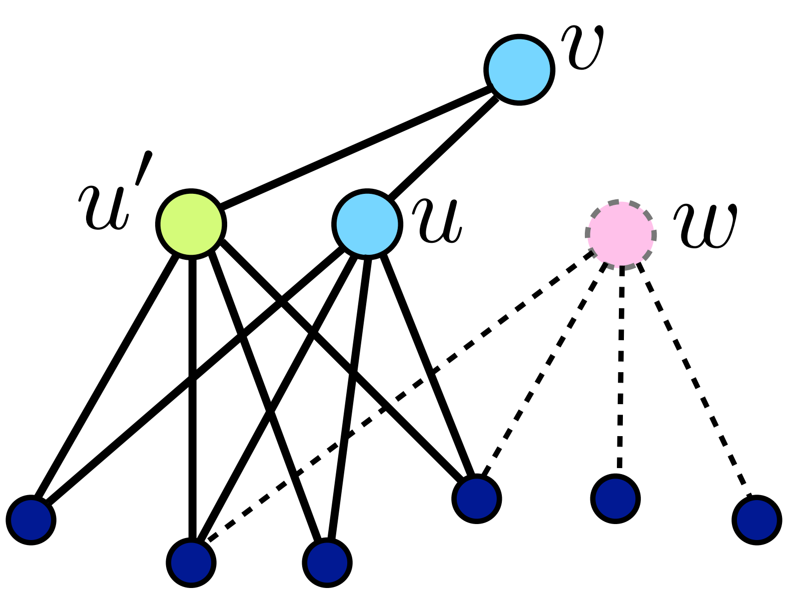
$$\implies |E| \le \frac{r-2}{2(r-1)}n^2$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)}n^2$$

Fourth Proof.

Suppose G is K_r-free with **maximum** edges.

G does not have 



By contradiction.

Case.1 $d_w < d_u$ or $d_w < d_v$

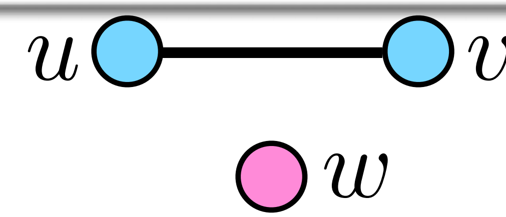
duplicate u, delete w, still K_r-free

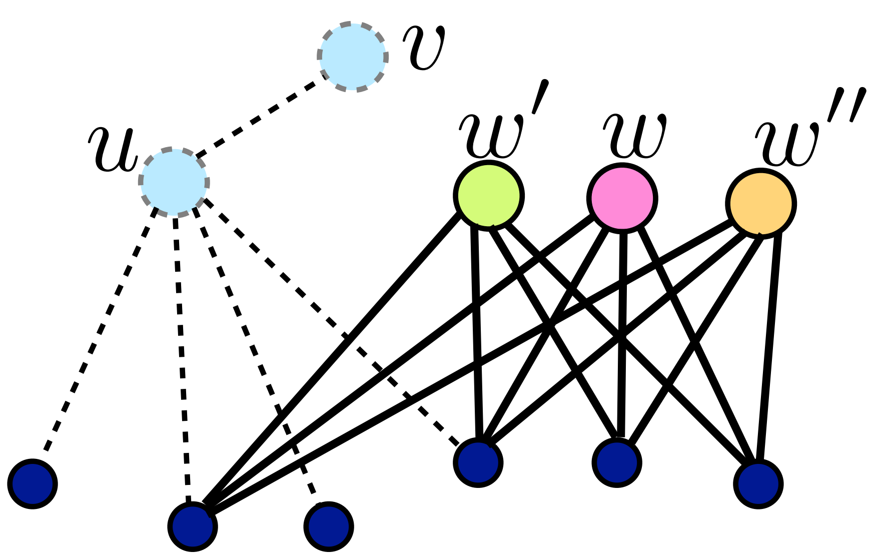
$$|E'| = |E| + d_u - d_w > |E|$$

$$K_r\text{-free} \implies |E| \leq \frac{r-2}{2(r-1)}n^2$$

Fourth Proof.

Suppose G is K_r-free with maximum edges.

G does not have 



Case.2 $d_w \geq d_u \wedge d_w \geq d_v$

delete u, v, duplicate w, twice

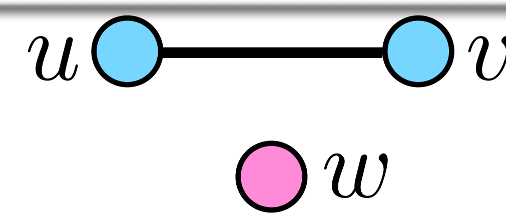
still K_r-free

$$|E'| = |E| + 2d_w - (d_u + d_v - 1) > |E|$$

$$K_r\text{-free} \implies |E| \le \frac{r-2}{2(r-1)}n^2$$

Fourth Proof.

Suppose G is K_r-free with **maximum** edges.

G does not have 

$u \not\sim v$ is an **equivalence relation**

G is a complete multipartite graph

optimize $K_{n_1, n_2, \ldots, n_{r-1}}$

subject to $n_1 + n_2 + \cdots + n_r = n$

Turán's Theorem (clique)

If $G(V, E)$ has $|V| = n$ and is K_r-free, then

$$|E| \leq \frac{r-2}{2(r-1)} n^2$$

Turán's Theorem (independent set)

If $G(V, E)$ has $|V| = n$ and $|E| = m$, then G has an independent set of size

$$\geq \frac{n^2}{2m + n}$$

Parallel Max

- compute max of n distinct numbers
  - computation model: parallel, comparison-based
- **1-round algorithm:** $\binom{n}{2}$ comparisons of all pairs
- **lower bound for one-round:**
  - $\binom{n}{2}$ comparisons are required in the worst case

 **adversary argument**

Parallel Max

- 2-round algorithm:
  - divide n numbers into k groups of n/k each
  - *1st round*: find max of each group;
    $k\binom{n/k}{2}$ comparisons
  - *2nd round*: find the max of the *k* maxes
    $\binom{k}{2}$ comparisons
- total comparisons: $k\binom{n/k}{2} + \binom{k}{2} = O\left(n^{4/3}\right)$
  for $k = n^{2/3}$

3-round?

optimal?

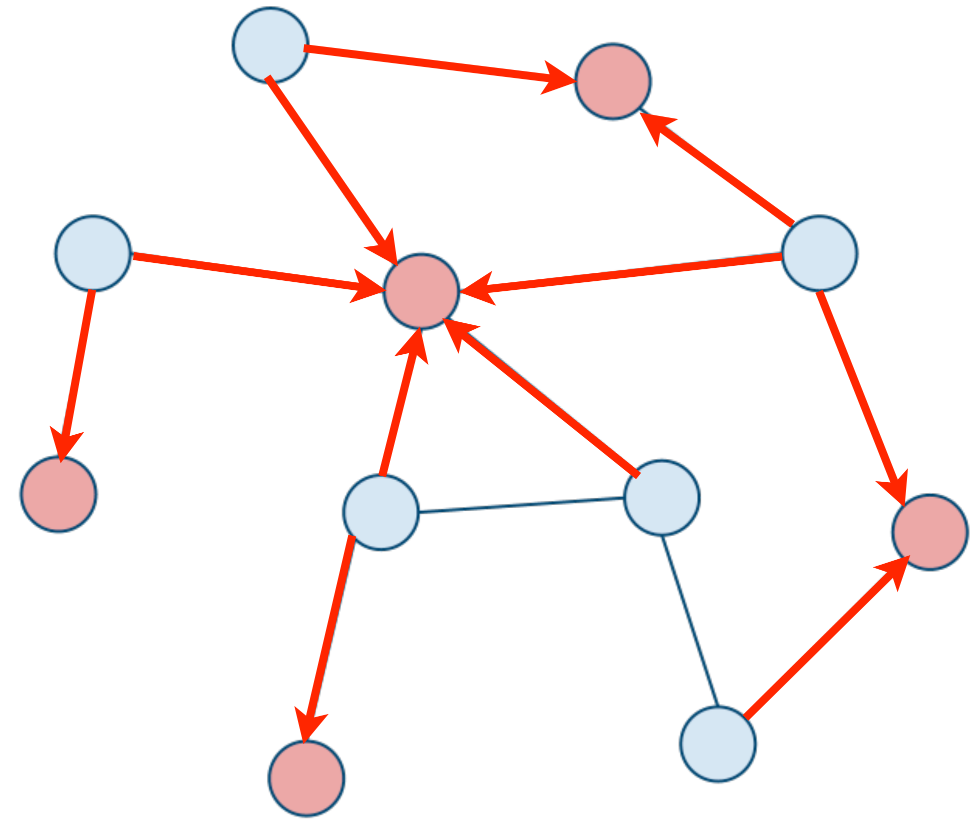
1st round:

Alg: m comparisons

choose an independent set

of size $\geq \dfrac{n^2}{2m+n}$ **(Turán)**

make them local maximal



2nd round:

a parallel max problem of size $\geq \dfrac{n^2}{2m+n}$

requires $\geq \binom{\frac{n^2}{2m+n}}{2}$ comparisons

total comparisons $\geq m + \binom{\frac{n^2}{2m+n}}{2} = \Omega(n^{4/3})$

Fundamental Theorem
of Extremal Graph Theory

Extremal Graph Theory

Fix a graph H.

$$\mathrm{ex}(n, H)$$

largest possible number of edges of $G \not\supseteq H$ on n vertices

$$\mathrm{ex}(n, H) = \max_{\substack{G \not\supseteq H \\ |V(G)|=n}} |E(G)|$$

Turán's Theorem

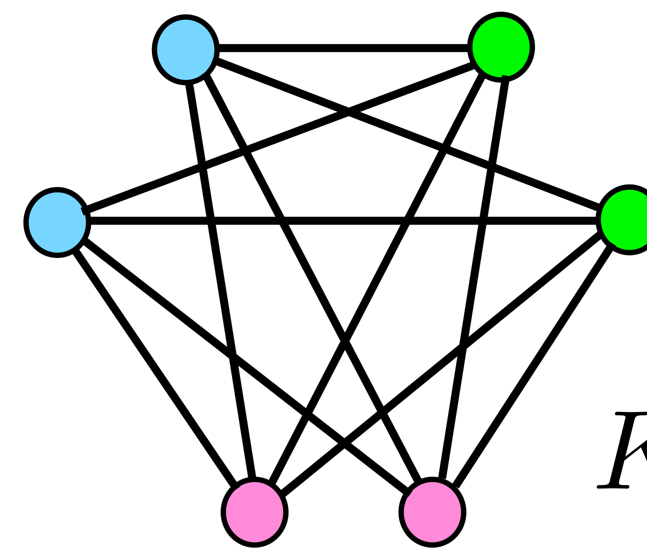
$$\mathrm{ex}(n, K_r) = |T(n, r-1)| \leq \frac{r-2}{2(r-1)} n^2$$

Erdős–Stone theorem

(Fundamental theorem of extremal graph theory)

$$K_s^r = K_{\underbrace{s, s, \cdots, s}_{r}} = T(rs, r)$$

complete r-partite graph
with s vertices in each part



K_2^3

Theorem (Erdős–Stone 1946)

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

Theorem (Erdős–Stone 1946)

$$\mathrm{ex}(n, K_s^r) = \left(\frac{r-2}{2(r-1)} + o(1) \right) n^2$$

$\mathrm{ex}(n, H) / \binom{n}{2}$ extremal density of subgraph H

Corollary

For any nonempty graph H

$$\lim_{n \to \infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

$$\chi(H) = r$$

$$H \not\subseteq T(n, r-1) \text{ for any } n$$

$$\mathrm{ex}(n,H) \geq |T(n,r-1)|$$

$$H \subseteq K_s^r \text{ for sufficiently large } s$$

$$\mathrm{ex}(n,H) \leq \mathrm{ex}(n, K_s^r)$$

$$= \left(\tfrac{r-2}{2(r-1)} + o(1)\right) n^2$$

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$$

$$\chi(H) = r$$

$$|T(n,r-1)| \leq \mathrm{ex}(n,H) \leq \left(\frac{r-2}{2(r-1)} + o(1)\right) n^2$$

$$\frac{r-2}{r-1} - o(1) \leq \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \leq \frac{r-2}{r-1} + o(1)$$

Cycles

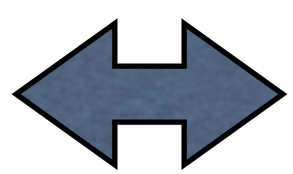
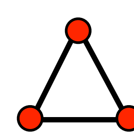
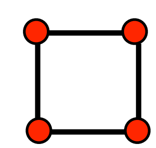
Girth

girth $g(G)$: length of the shortest cycle in G

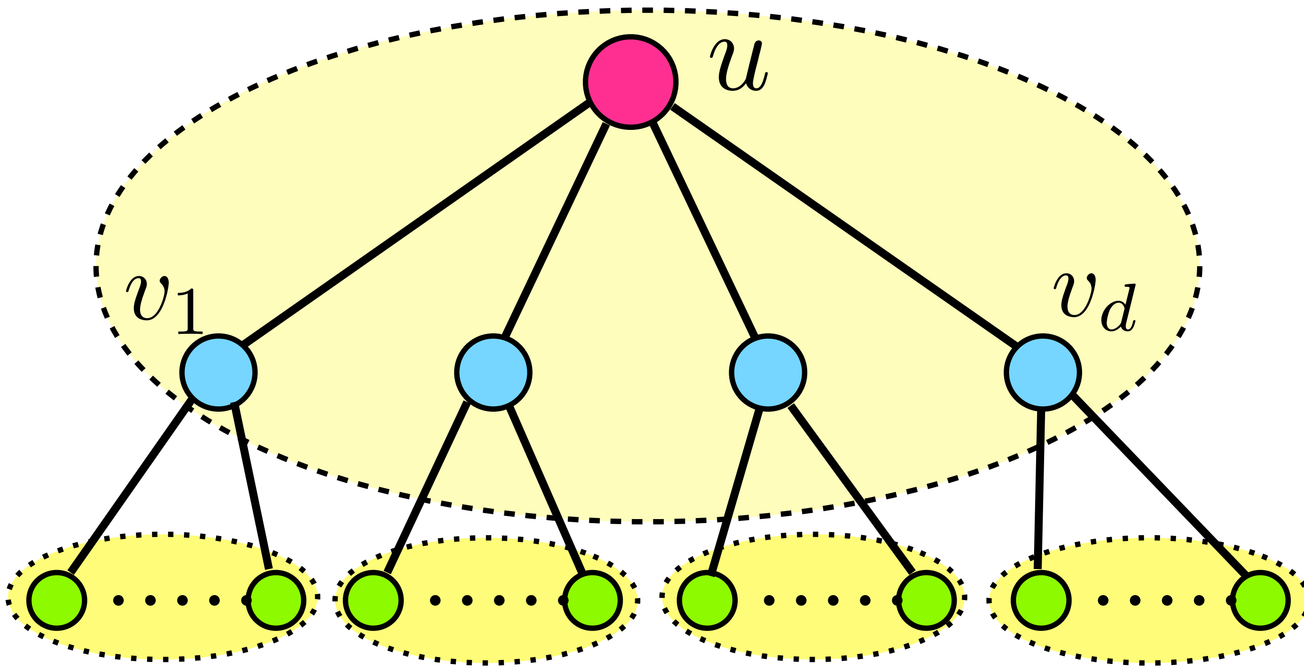
Theorem

If $G(V,E)$ has $|V| = n$ and girth $g(G) \geq 5$,

then $$|E| \leq \frac{1}{2} n\sqrt{n-1}$$

$g(G) \geq 5$  - and -free

$$g(G) \geq 5 \;\Rightarrow\; |E| \leq \frac{1}{2} n \sqrt{n-1}$$



$$d = d(u)$$

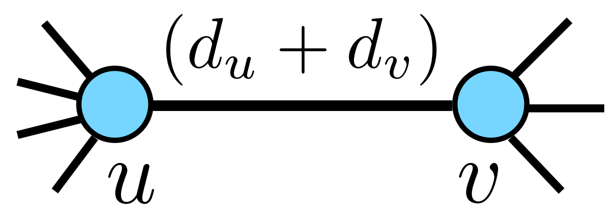
disjoint sets

$$(d+1) + (d(v_1) - 1) + \cdots + (d(v_d) - 1) \leq n$$

$$\sum_{v : v \sim u} d(v) \leq n - 1$$

$$g(G) \geq 5 \;\Rightarrow\; |E| \leq \tfrac{1}{2} n\sqrt{n-1}$$

$$\forall u \in V, \quad \sum_{v : v \sim u} d(v) \leq n - 1$$



$$n(n-1) \geq \sum_{u \in V} \sum_{v : v \sim u} d(v) \quad = \sum_{v \in V} d(v)^2$$

$$\geq \frac{\left(\sum_{v \in V} d(v)\right)^2}{n} = \frac{4|E|^2}{n}$$

Cauchy-Schwarz

Hamiltonian Cycle

Dirac's Theorem

$$\forall v \in V,\ d_v \geq \frac{n}{2} \quad \Rightarrow \quad G(V,E) \text{ is Hamiltonian.}$$

By contradiction, suppose G is the maximum non-Hamiltonian graph with $\forall v \in V,\ d_v \geq \frac{n}{2}$

adding 1 edge $\Longrightarrow$ Hamiltonian

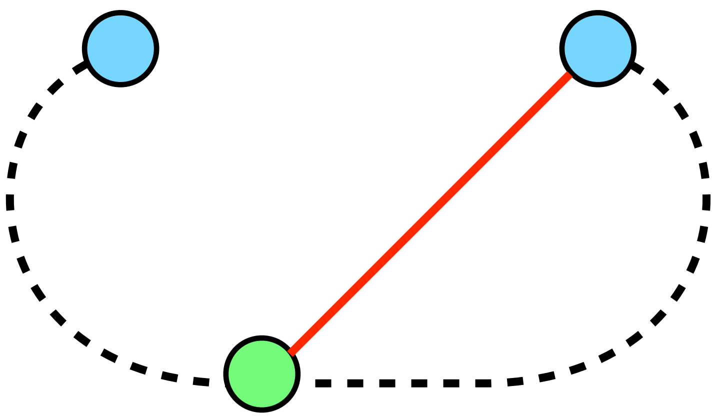
$\exists$ a Hamiltonian path

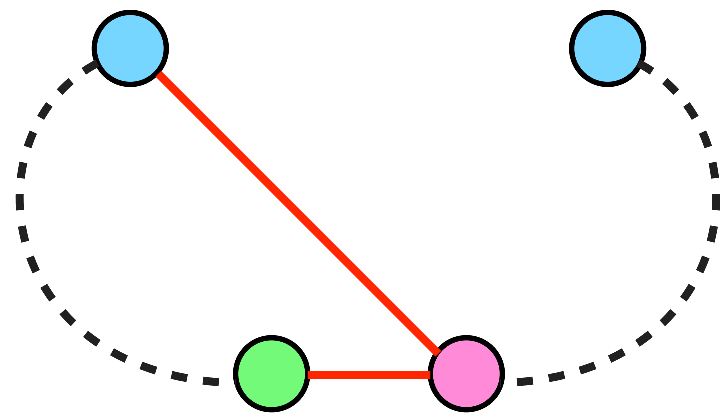
say $v_1 v_2 \cdots v_n$

G is non-Hamiltonian $\quad\quad \forall v \in V,\ d_v \geq \frac{n}{2}$

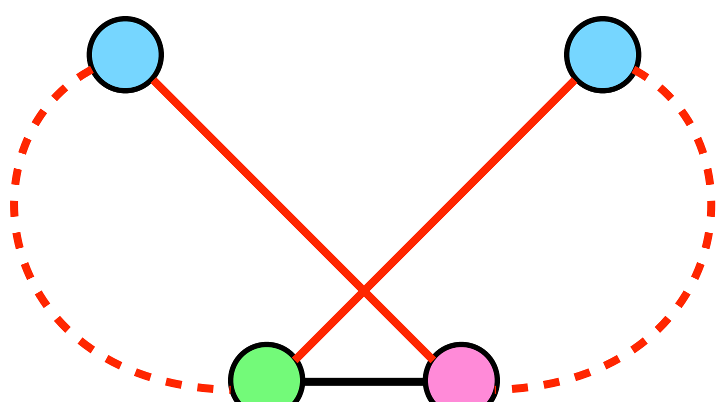
$\exists$ a Hamiltonian path $\quad\quad v_1 v_2 \cdots v_n$

$\{i \mid v_i \sim v_n\}$ $\quad\quad \{i \mid v_{i+1} \sim v_1\}$





$\geq \frac{n}{2} + \frac{n}{2}$ pigeons in $\{1, 2, \ldots, n-1\}$



Contradiction!