

Combinatorics

Generating Function

尹一通 Nanjing University, 2025 Spring

Compositions by 1 and 2

of compositions of n
with summands from
 $\{1, 2\}$

of $(x_1, x_2, \dots, x_k)$
for some $k \leq n$
 $x_1 + \dots + x_k = n$
 $x_i \in \{1, 2\}$

$$a_n = a_{n-1} + a_{n-2} \quad a_0 = 1 \quad a_1 = 1$$

Case.1 $x_k = 1$ $x_1 + \dots + x_{k-1} = n - 1$

Case.2 $x_k = 2$ $x_1 + \dots + x_{k-1} = n - 2$

Dominoes

domino: $\overset{2}{\underset{1}{\text{■}}}$

of tilings $\overset{2}{\text{■}} \underbrace{\begin{array}{|c|c|c|c|} \hline \text{■} & \text{■} & \text{■} & \text{■} \\ \hline \text{■} & \text{■} & \text{■} & \text{■} \end{array}}_n \dots \begin{array}{|c|} \hline \text{■} \\ \hline \text{■} \end{array}$

$$a_n = a_{n-1} + a_{n-2} \quad a_0 = 1 \quad a_1 = 1$$

Case.1 $\text{■} \dots \overset{2 \times (n-1)}{\text{■}}$

Case.2 $\begin{array}{|c|} \hline \text{■} \\ \hline \text{■} \end{array} \dots \overset{2 \times (n-2)}{\text{■}}$

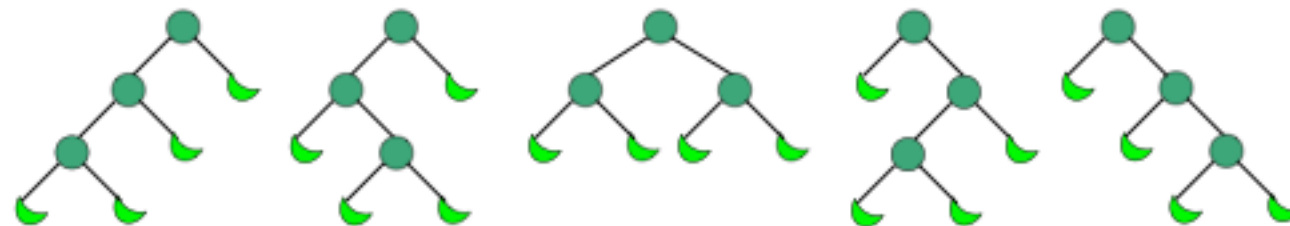
Fibonacci number

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

full **parenthesization** of $n + 1$ factors

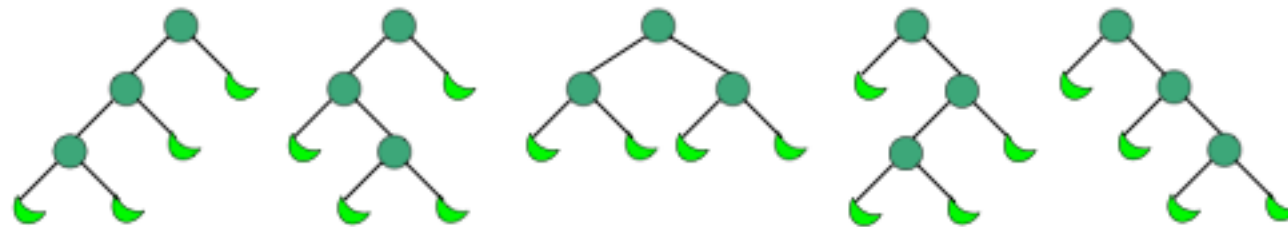
$((ab)c)d$ $(a(bc))d$ $(ab)(cd)$ $a((bc)d)$ $a(b(cd))$

full binary trees with $n + 1$ leaves

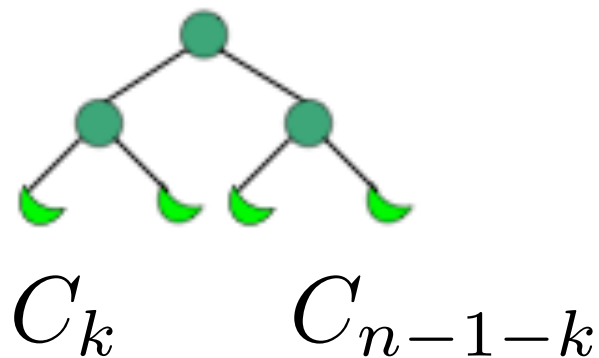


Catalan Number

C_n : # of full binary trees with $n + 1$ leaves



Recursion:



$$C_0 = 1 \quad \text{for } n \geq 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

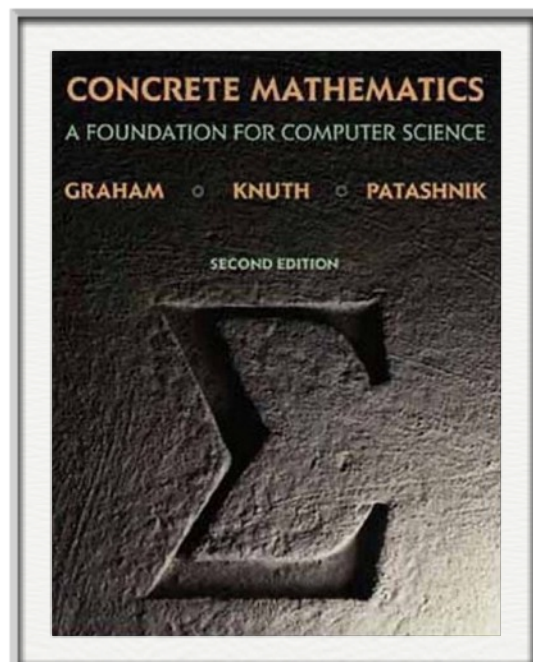
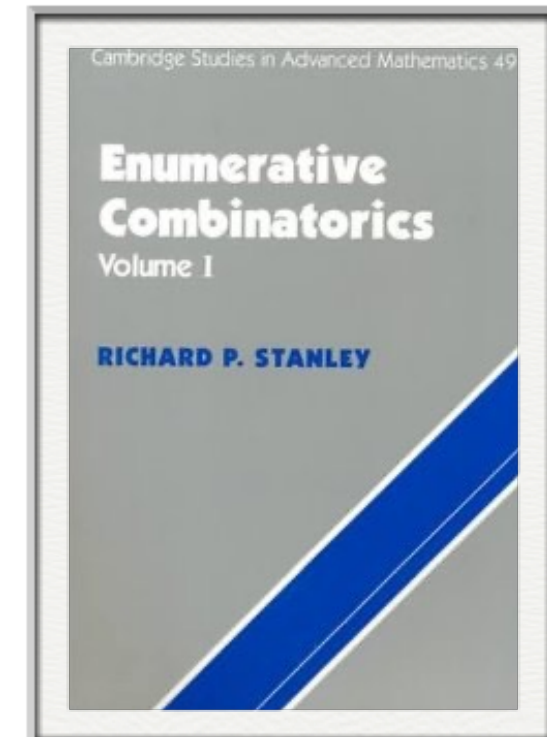
Generating Function

Generating Function

1.1 How to Count

3

4. The most useful but most difficult to understand method for evaluating $f(i)$ is to give its *generating function*. We will not develop in this chapter a rigorous abstract theory of generating functions, but will instead content ourselves with an informal discussion and some examples. Informally, a generating function is



Generating Functions

THE MOST POWERFUL WAY to deal with sequences of numbers, as far as anybody knows, is to manipulate infinite series that “generate” those sequences. We’ve learned a lot of sequences and we’ve seen a few generating functions; now we’re ready to explore generating functions in depth, and to see how remarkably useful they are.

generate

~~enumerate~~ all subsets of

$$\{ \text{red circle}, \text{blue circle}, \text{green circle} \}$$

$$(x^0 + x^1)(x^0 + x^1)(x^0 + x^1)$$

$$= x^0 x^0 x^0 + x^0 x^0 x^1 + x^0 x^1 x^0 + x^0 x^1 x^1 \\ + x^1 x^0 x^0 + x^1 x^0 x^1 + x^1 x^1 x^0 + x^1 x^1 x^1$$

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

coefficient of x^k : # of k -subsets

$$\{ \text{red circle}, \text{red circle}, \text{red circle}, \\ \text{blue circle}, \text{blue circle}, \text{blue circle}, \text{blue circle}, \\ \text{green circle}, \text{green circle}, \text{green circle}, \text{green circle}, \text{green circle} \}$$

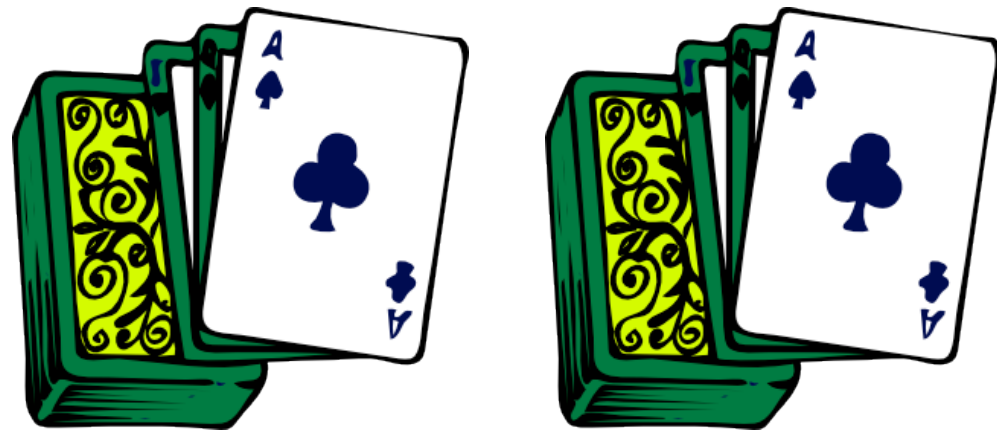
$$(1 + x + x^2 + x^3)$$

$$(1 + x + x^2 + x^3 + x^4)$$

$$(1 + x + x^2 + x^3 + x^4 + x^5)$$

$$= 1 + 3x + 6x^2 + 10x^3 + 14x^4 + 17x^5 + 18x^6 \\ + 17x^7 + 14x^8 + 10x^9 + 6x^{10} + 3x^{11} + x^{12}$$

Double Decks



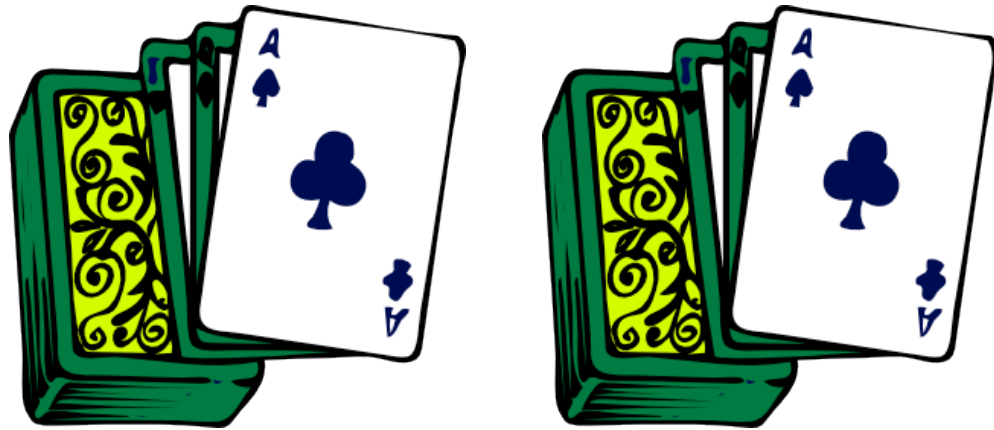
choose m cards from
2 decks of n cards

$$(x_1^0 + x_1^1 + x_1^2) (x_2^0 + x_2^1 + x_2^2) \cdots (x_n^0 + x_n^1 + x_n^2)$$

of m -order terms

coefficient of x^m in $(1 + x + x^2)^n$

Double Decks



choose m cards from
2 decks of n cards

$$\begin{aligned} (1 + (x + x^2))^n &= \sum_k \binom{n}{k} (x + x^2)^k \\ &= \sum_k \binom{n}{k} x^k \sum_{\ell \leq k} \binom{k}{\ell} x^\ell = \sum_k \sum_{\ell \leq k} \binom{n}{k} \binom{k}{\ell} x^{k+\ell} \\ &= \sum_m \left(\sum_{\ell} \binom{n}{m-\ell} \binom{m-\ell}{\ell} \right) x^m \end{aligned}$$

Multisets

multisets on $S = \{x_1, x_2, \dots, x_n\}$

$$(1 + x_1 + x_1^2 + \dots)(1 + x_2 + x_2^2 + \dots) \cdots (1 + x_n + x_n^2 + \dots)$$

$$= \sum_{m: S \rightarrow \mathbb{N}} \prod_{x_i \in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \dots)^n = \sum_{m \in \mathbb{N}^n} x^{m_1 + \dots + m_n} = \sum_{k \geq 0} \binom{n}{k} x^k$$

|| **geometric**

$$(1 - x)^{-n} = \sum_{k \geq 0} \frac{(-n)(-n-1) \cdots (-n-k+1)(-1)^k}{k!} x^k$$

Taylor

Multisets

multisets on $S = \{x_1, x_2, \dots, x_n\}$

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$$= \sum_{m: S \rightarrow \mathbb{N}} \prod_{x_i \in S} x_i^{m(x_i)}$$

$$(1 + x + x^2 + \dots)^n = \sum_{m \in \mathbb{N}^n} x^{m_1 + \dots + m_n} = \sum_{k \geq 0} \left(\binom{n}{k} \right) x^k$$

||

$$(1 - x)^{-n} = \sum_{k \geq 0} \binom{n + k - 1}{k} x^k$$

$$\left(\binom{n}{k} \right) = \binom{n + k - 1}{k}$$

Ordinary Generating Function (OGF)

$$\{a_n\} \qquad a_0, a_1, a_2, \dots$$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$[x^n]G(x) = a_n$$

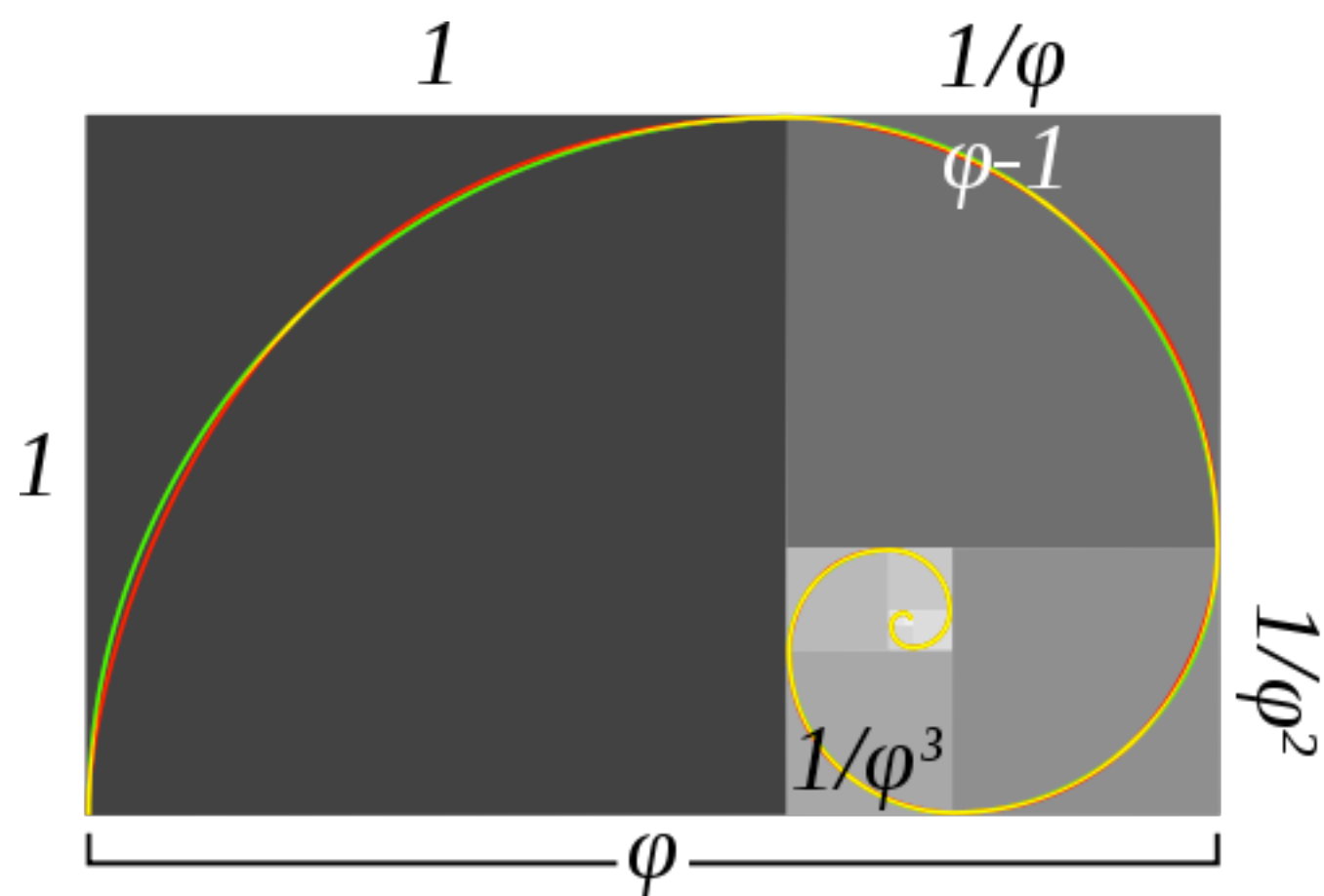
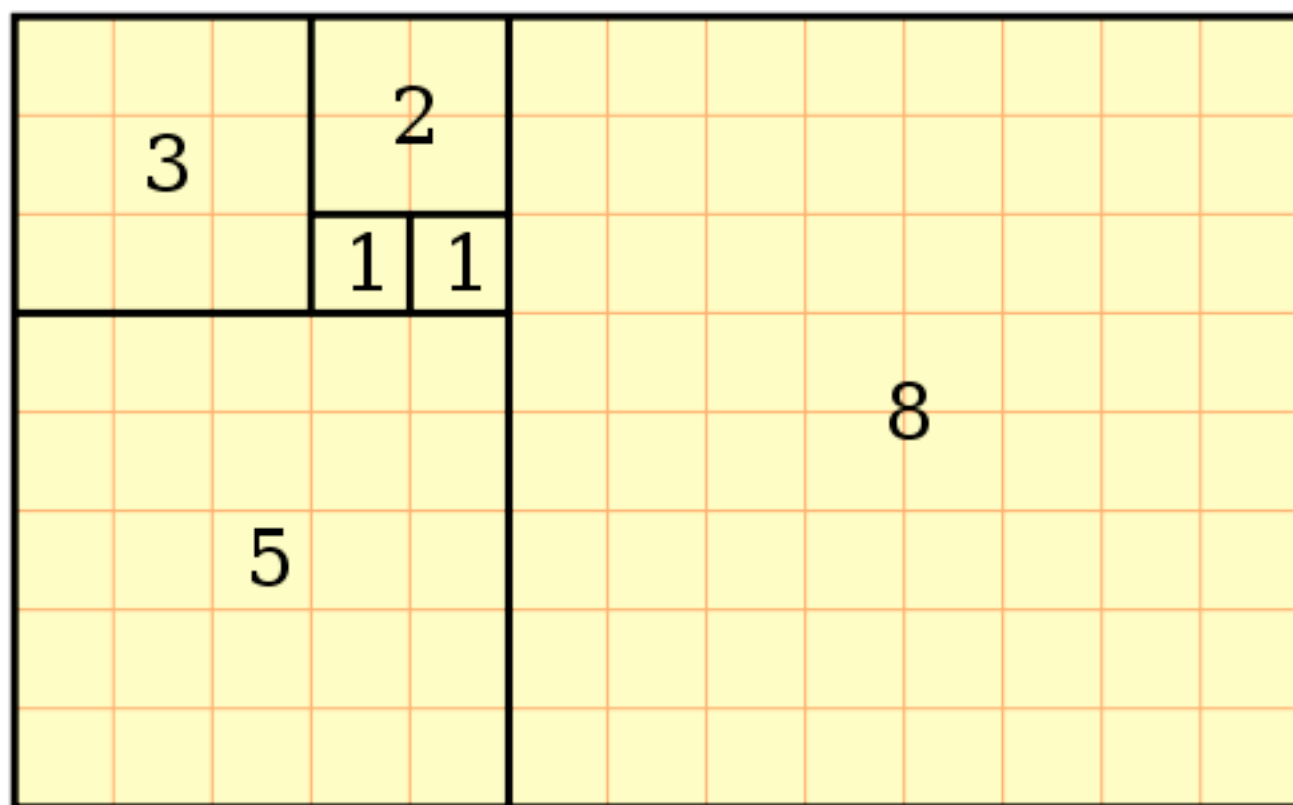
Fibonacci number

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Fibonacci number

$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right)$$



$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

generating function:

$$G(x) = \sum_{n \geq 0} F_n x^n$$

recursion:

$$G(x) = F_0 + F_1 x + \sum_{n \geq 2} F_n x^n = x + \sum_{n \geq 2} F_{n-1} x^n + \sum_{n \geq 2} F_{n-2} x^n$$

$$\sum_{n \geq 2} F_{n-1} x^n = \sum_{n \geq 1} F_{n-1} x^n = \sum_{n \geq 0} F_n x^{n+1} = xG(x)$$

$$\sum_{n \geq 2} F_{n-2} x^n = \sum_{n \geq 0} F_n x^{n+2} = x^2 G(x)$$

identity: $G(x) = x + (x + x^2)G(x)$

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generating function:

$$G(x) = \sum_{n \geq 0} F_n x^n$$

identity: $G(x) = x + (x + x^2)G(x)$

solution: $G(x) = \frac{x}{1 - x - x^2} \quad =? \quad \text{Taylor ?}$

denote $\phi = \frac{1 + \sqrt{5}}{2} \quad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x}$$

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2, \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0. \end{cases}$$

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solution:

$$\begin{aligned} G(x) &= \frac{x}{1 - x - x^2} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x} \\ &= \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\phi x)^n - \frac{1}{\sqrt{5}} \sum_{n \geq 0} (\hat{\phi} x)^n \\ &= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n \end{aligned}$$

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$$\phi = \frac{1 + \sqrt{5}}{2} \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2}$$

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Ordinary Generating Function (OGF)

$$\{a_n\} \qquad a_0, a_1, a_2, \dots$$

generating function:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$\frac{1}{1 - \alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

Formal Power Series

formal power series: $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$\mathbb{C}[[x]]$: ring of formal power series
with complex coefficient

inverse

$$F(x)G(x) = 1 \quad \Rightarrow \quad F(x) = G(x)^{-1} = \frac{1}{G(x)}$$

$$(1 - \alpha x) \left(\sum_{n=0}^{\infty} \alpha^n x^n \right) = 1 \quad \Rightarrow \quad \frac{1}{1 - \alpha x} = \sum_{n=0}^{\infty} \alpha^n x^n$$

Generatingfunctionology

“Generatingfunctionology”

1. Recurrence:

$$a_0 = 0 \quad a_1 = 1 \quad a_n = a_{n-1} + a_{n-2}$$

2. Manipulation:

$$\begin{aligned} G(x) &= \sum_{n \geq 0} a_n x^n = x + \sum_{n \geq 1} a_{n-1} x^n + \sum_{n \geq 2} a_{n-2} x^n \\ &= x + (x + x^2)G(x) \end{aligned}$$

3. Solving:

$$\begin{aligned} G(x) &= \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x} \\ &= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n \end{aligned}$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \quad F(x) = \sum_{n \geq 0} f_n x^n$$

shift:

$$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition:

$$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution:

$$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^n f_k g_{n-k} x^n$$

differentiation:

$$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

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3. Solving:

expending!

$$\begin{aligned} G(x) &= \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \phi x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{1 - \hat{\phi} x} \\ &= \sum_{n \geq 0} \frac{1}{\sqrt{5}} \left(\phi^n - \hat{\phi}^n \right) x^n \end{aligned}$$

Generating Function Expansion

Taylor's expansion:

$$G(x) = \sum_{n \geq 0} \frac{G^{(n)}(0)}{n!} x^n$$

Geometric sequence:

$$\frac{a}{1 - bx} = a \sum_{n \geq 0} (bx)^n$$

$$G(x) = \frac{a_1}{1 - b_1 x} + \frac{a_2}{1 - b_2 x} + \cdots + \frac{a_k}{1 - b_k x}$$

$$[x^n]G(x) = a_1 b_1^n + a_2 b_2^n + \cdots + a_k b_k^n$$

Generating Function Expansion

Binomial theorem: Newton's formula

$$(1+x)^{\alpha} = \sum_{n \geq 0} \binom{\alpha}{n} x^n$$

$$((1+x)^{\alpha})^{(n)} = \alpha(\alpha-1)(\alpha-2) \cdots (\alpha-n+1)(1+x)^{\alpha-n}$$

generalized binomial coefficient:

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2) \cdots (\alpha-n+1)}{n!}$$

Changing Money

$$(\text{壹}, \text{伍})_n = \sum_{k=0}^n \text{壹}_k \text{伍}_{n-k}$$



壹_n : # of ways to change n yuan using 壹圆

$$\sum_{n \geq 0} \text{壹}_n x^n = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

伍_n : # of ways to change n yuan using 伍圆

$$\sum_{n \geq 0} \text{伍}_n x^n = 1 + x^5 + x^{10} + \dots = \frac{1}{1-x^5}$$

Changing Money

$$(\text{壹}, \text{伍})_n = \sum_{k=0}^n \text{壹}_k \text{伍}_{n-k}$$



$$\sum_{n \geq 0} \text{壹}_n x^n = \frac{1}{1-x} \quad \sum_{n \geq 0} \text{伍}_n x^n = \frac{1}{1-x^5}$$

$$\begin{aligned} \sum_{n \geq 0} (\text{壹}, \text{伍})_n x^n &= \sum_{n \geq 0} \sum_{k=0}^n \text{壹}_k \text{伍}_{n-k} x^n \\ &= \frac{1}{(1-x)(1-x^5)} \end{aligned}$$

convolution!

Changing Money



$$\sum_{n \geq 0} (\text{壹}, \text{伍}, \text{拾}, \text{贰拾}, \text{伍拾}, \text{壹佰})_n x^n$$

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})}$$

$$\begin{aligned}
& \sum_{n \geq 0} (\text{壹}, \text{伍}, \text{拾}, \text{贰拾}, \text{伍拾}, \text{壹佰})_n x^n \\
&= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{20})(1-x^{50})(1-x^{100})} \\
&= (1+x+x^2+x^3+\dots+x^{99}) \cdot (1+x^5+x^{10}+x^{15}+\dots+x^{95}) \\
&\quad \cdot (1+x^{10}+x^{20}+x^{30}+\dots+x^{90}) \cdot (1+x^{20}+x^{40}+x^{60}+x^{80}) \\
&\quad \cdot (1+x^{50}) \cdot \boxed{\frac{1}{(1-x^{100})^6}}
\end{aligned}$$

Newton's formula

||

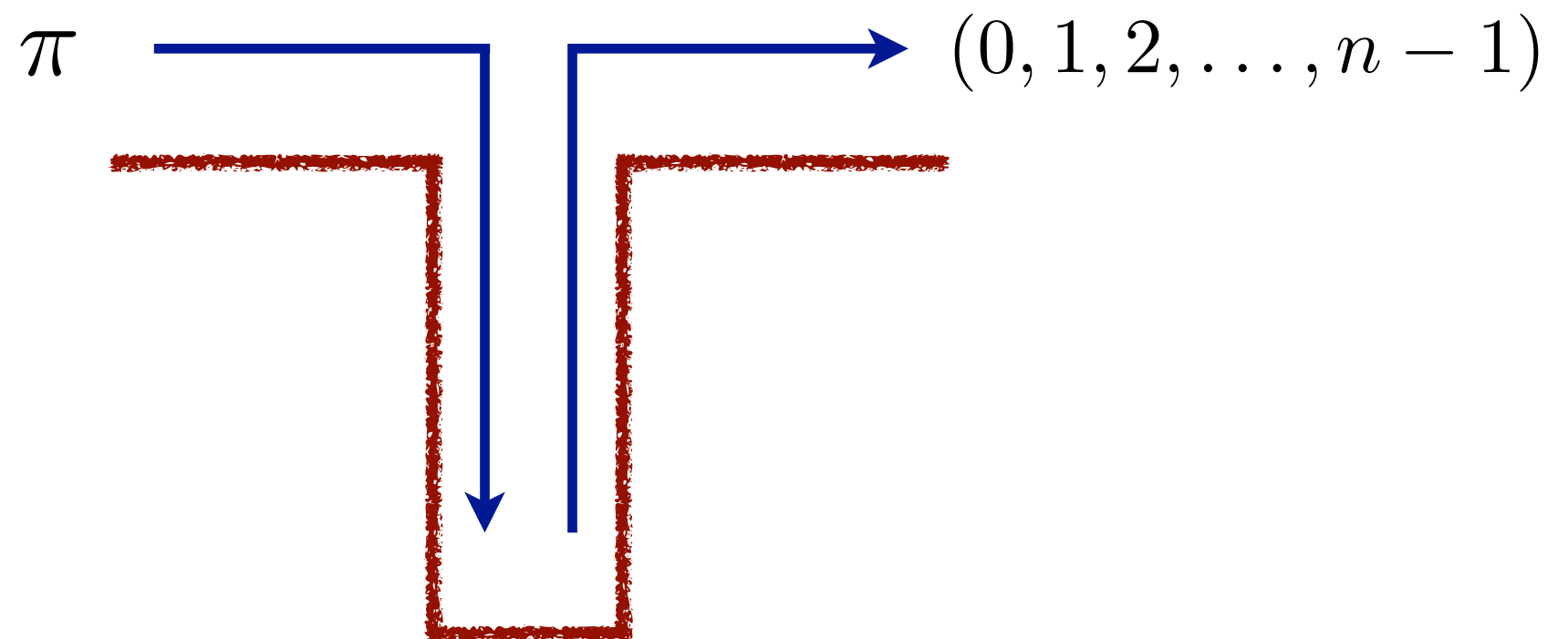
$$\sum_{n \geq 0} \binom{-6}{n} (-x^{100})^n$$

Catalan Number

n pairs of matching parenthesis

$((()))$ $()(())$ $()()()$ $(())()$ $((()))$

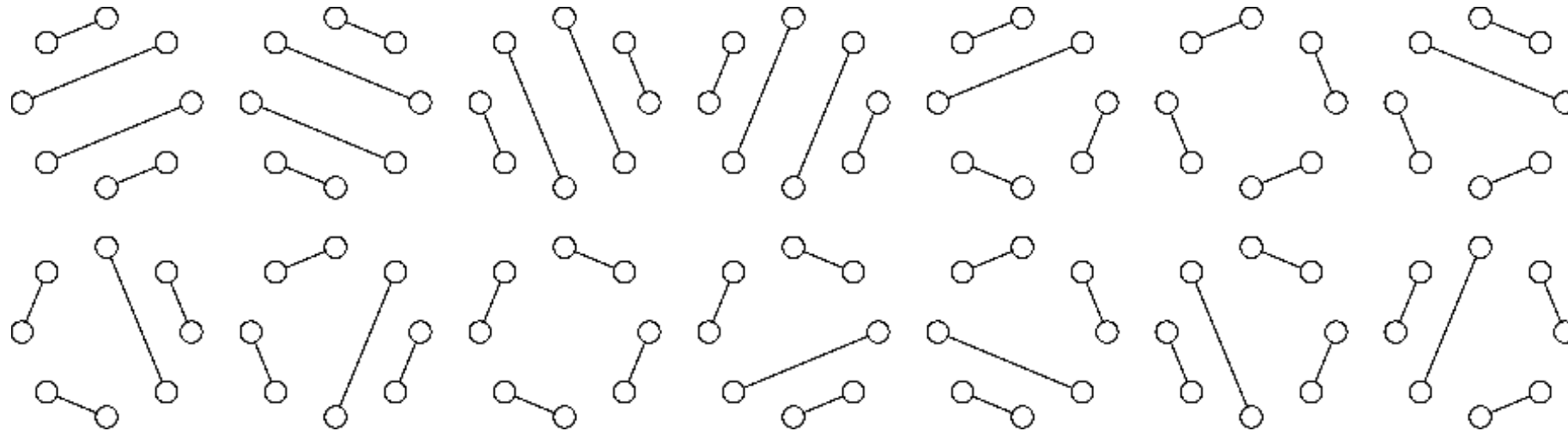
stack-sortable permutations of $[n]$



n pairs of matching parenthesis

((())) ()() ()() (()) (())

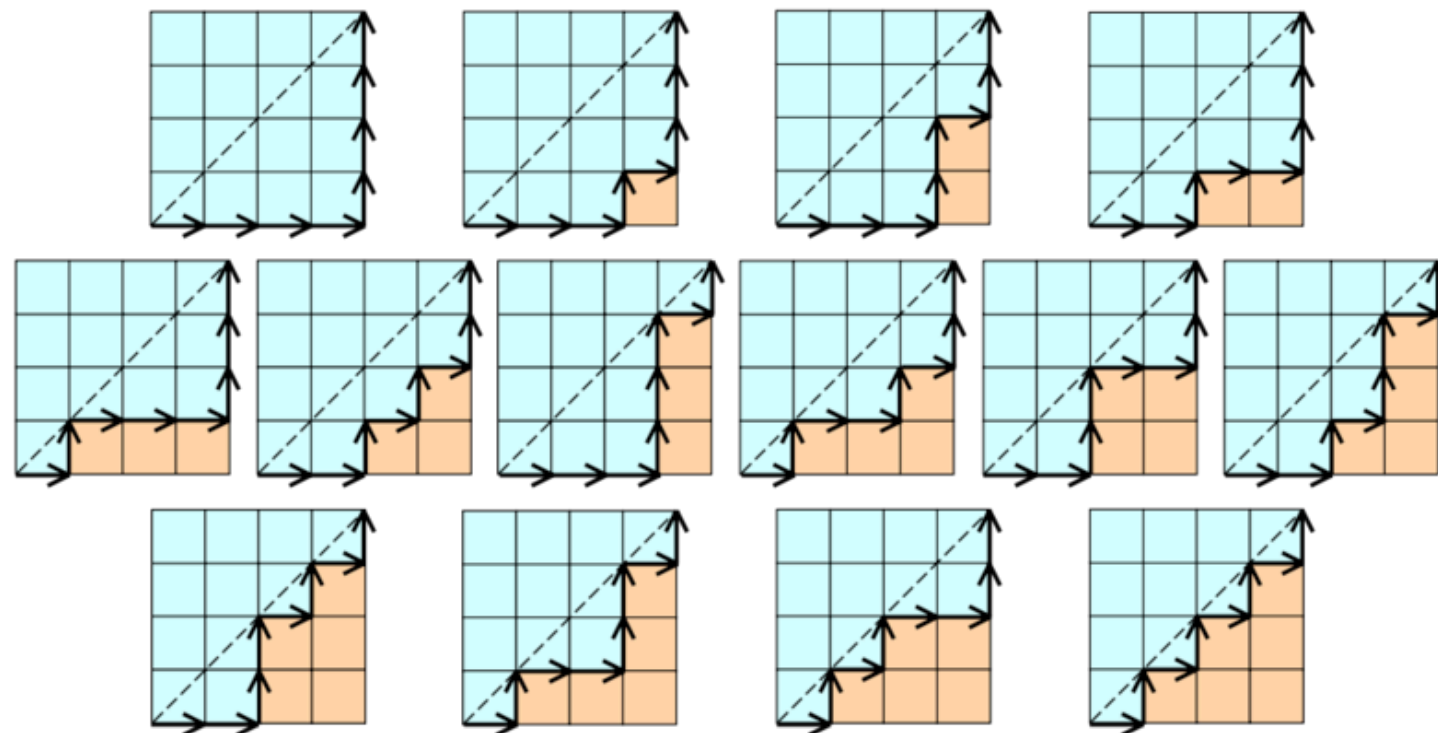
$2n$ people around a circular table shake hands,
no arms cross each other



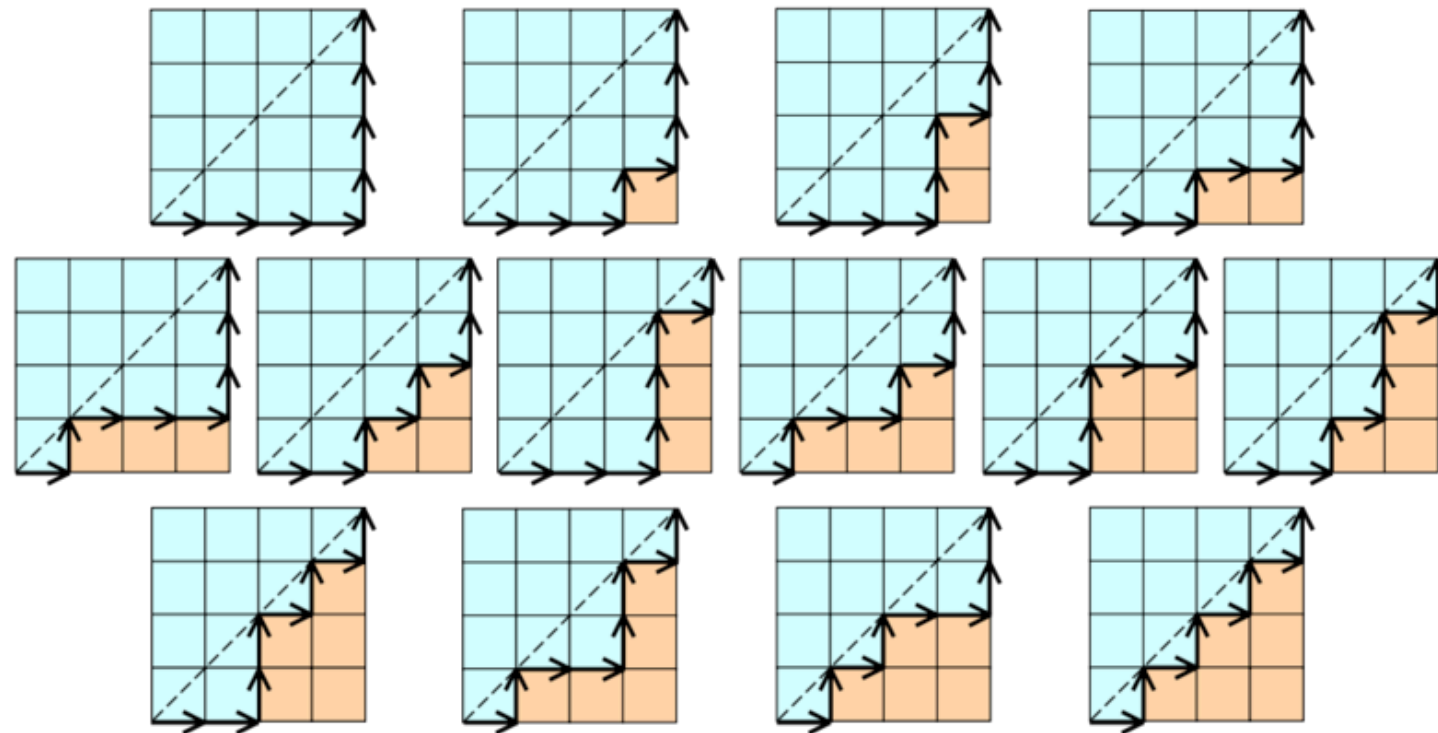
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monotone paths along $n \times n$ grids
below diagonal



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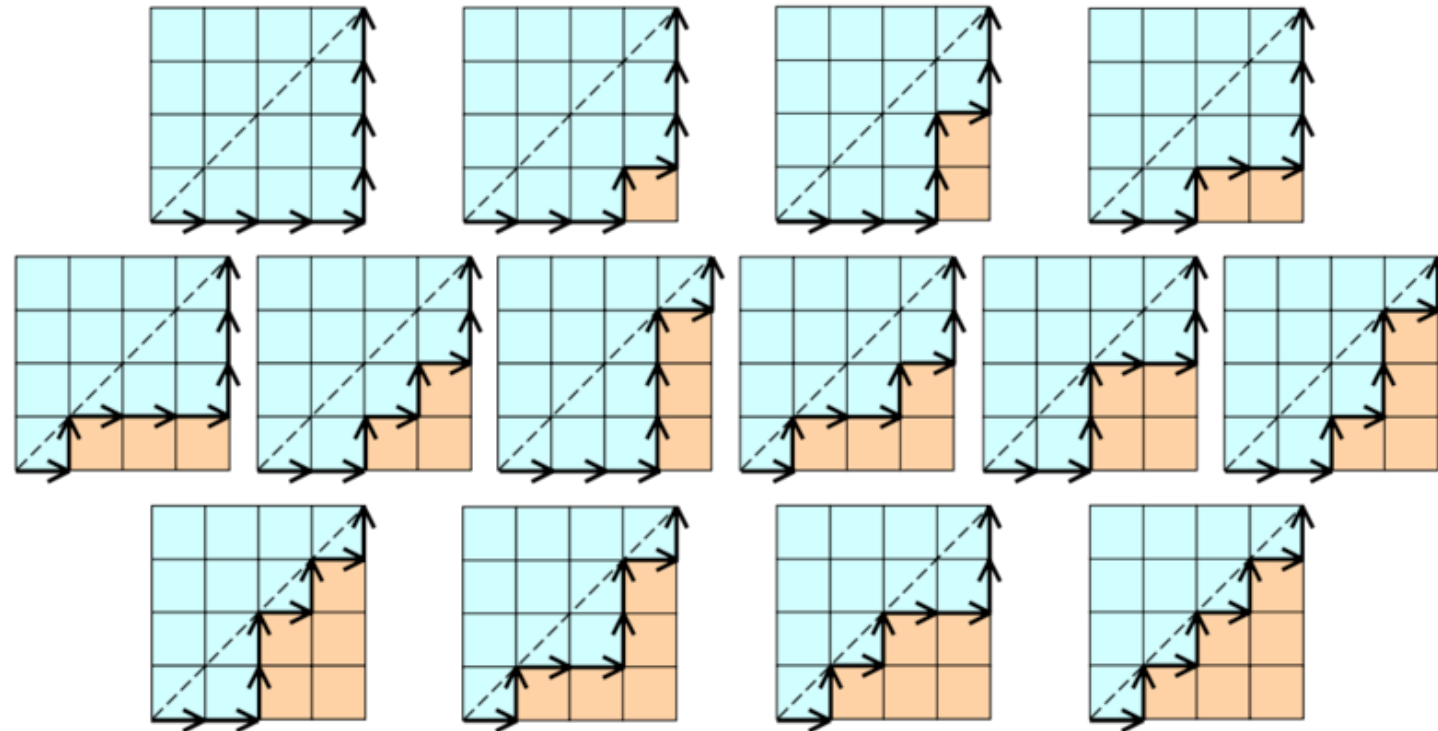


monotonically non-decreasing function:

$$f : [n] \rightarrow [n]$$

satisfying $f(i) \leq i$ for all $i \in [n]$

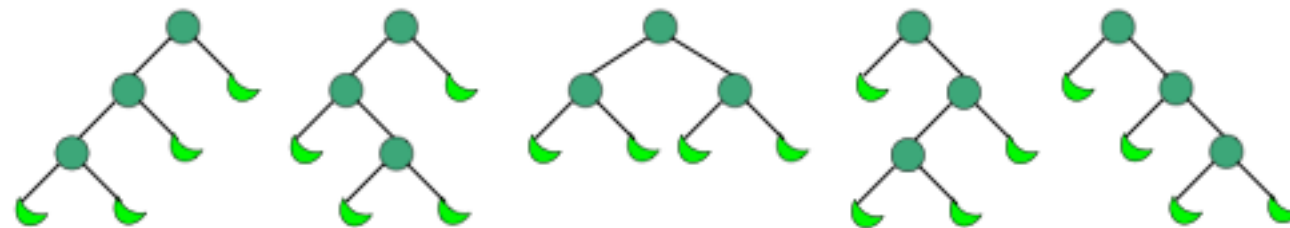
monotone paths along $n \times n$ grids
below diagonal



sequence of n $(+1)$ s and n (-1) s,
every prefix-sum is nonnegative

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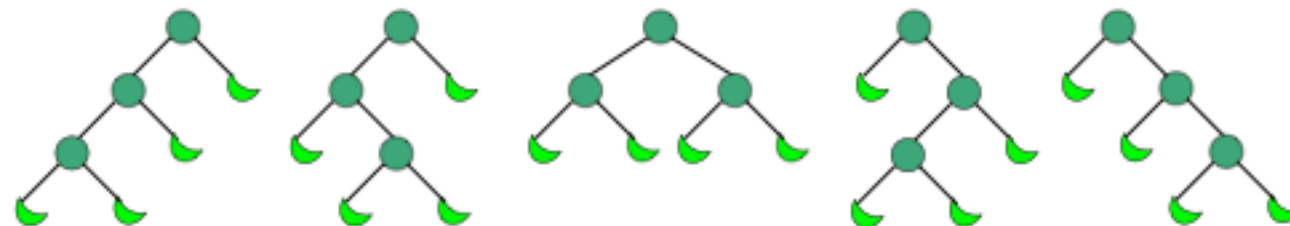
full binary trees with $n + 1$ leaves



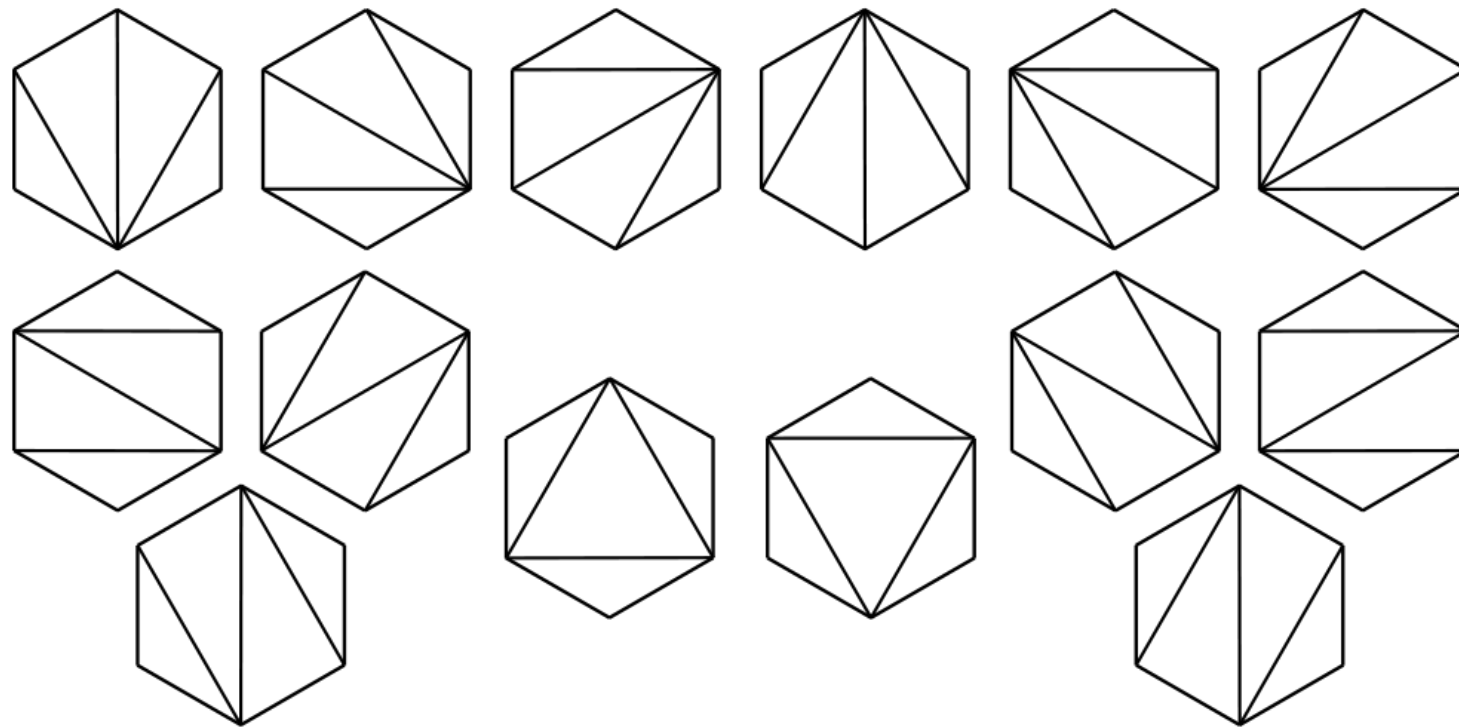
full **parenthesization** of $n + 1$ factors

$((ab)c)d$ $(a(bc))d$ $(ab)(cd)$ $a((bc)d)$ $a(b(cd))$

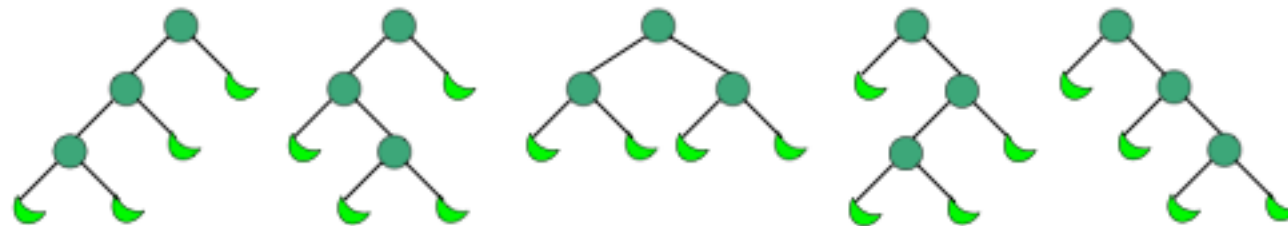
full binary trees with $n + 1$ leaves



triangulations of a convex $(n + 2)$ -gon:

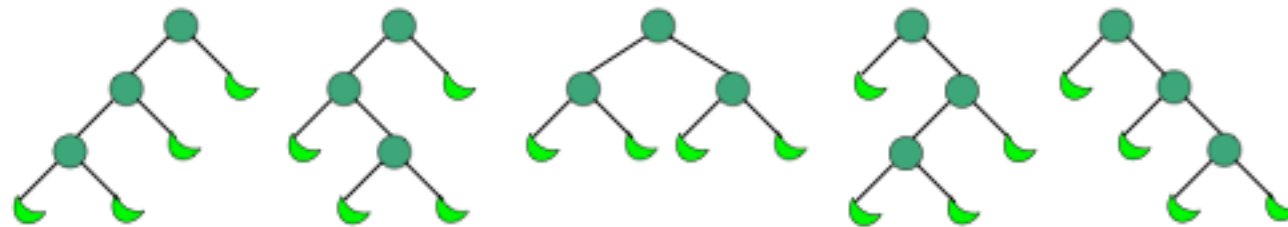


full binary trees with $n + 1$ leaves

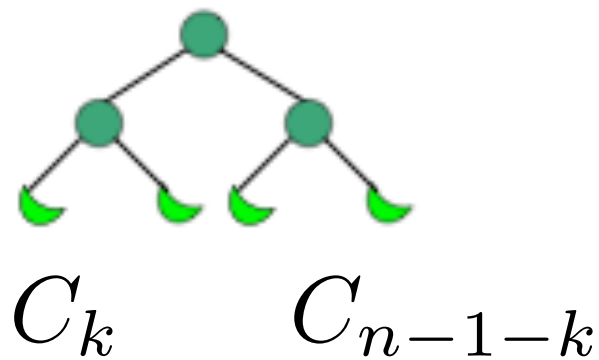


Catalan Number

C_n : # of full binary trees with $n + 1$ leaves



Recursion:



$$C_0 = 1 \quad \text{for } n \geq 1,$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

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Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \quad F(x) = \sum_{n \geq 0} f_n x^n$$

shift:

$$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

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Recursion:

$$C_0 = 1$$

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$$G(x)^2 = \sum_{n \geq 0} \sum_{k=0}^n C_k C_{n-k} x^n$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

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$$G(x)^2 = \sum_{n \geq 0} \sum_{k=0}^n C_k C_{n-k} x^n$$

$$xG(x)^2 = \sum_{n \geq 0} \sum_{k=0}^n C_k C_{n-k} x^{n+1}$$

$$= \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n$$

Manipulation:

$$G(x) = \sum_{n \geq 0} C_n x^n = C_0 + \sum_{n \geq 1} \sum_{k=0}^{n-1} C_k C_{n-1-k} x^n = 1 + xG(x)^2$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

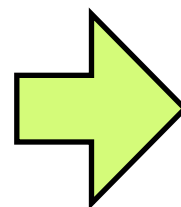
Manipulation:

$$G(x) = 1 + xG(x)^2$$

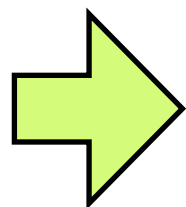
Solving:

$$G(x) = \frac{1 \pm (1 - 4x)^{1/2}}{2x}$$

$$G(x) = \sum_{n \geq 0} C_n x^n$$



$$\lim_{x \rightarrow 0} G(x) = C_0 = 1$$



$$G(x) = \frac{1 - (1 - 4x)^{1/2}}{2x}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1 - (1 - 4x)^{1/2}}{2x}$$

Expanding:

Newton's formula

$$\begin{aligned} (1 - 4x)^{1/2} &= \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n \geq 1} \binom{1/2}{n} (-4x)^n \\ &= 1 + \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^{n+1} = 1 - 4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n \end{aligned}$$

Recursion:

$$C_0 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}$$

Manipulation:

$$G(x) = 1 + xG(x)^2$$

Solving:

$$G(x) = \frac{1 - (1 - 4x)^{1/2}}{2x}$$

Expanding:

$$(1 - 4x)^{1/2} = 1 - 4x \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$G(x) = \frac{1 - (1 - 4x)^{1/2}}{2x} = 2 \sum_{n \geq 0} \binom{1/2}{n+1} (-4x)^n$$

$$= \sum_{n \geq 0} 2 \binom{1/2}{n+1} (-4)^n x^n$$

$$G(x) = \sum_{n \geq 0} 2 \binom{1/2}{n+1} (-4)^n x^n = \sum_{n \geq 0} C_n x^n$$

$$C_n = 2 \binom{1/2}{n+1} (-4)^n$$

$$= 2 \cdot \left(\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots \frac{-(2n-1)}{2} \right) \cdot \frac{1}{(n+1)!} \cdot (-4)^n$$

$$= \frac{2^n}{(n+1)!} \prod_{k=1}^n (2k-1) = \frac{2^n}{(n+1)!} \prod_{k=1}^n \frac{(2k-1)2k}{2k}$$

$$= \frac{1}{n!(n+1)!} \prod_{k=1}^n (2k-1)2k = \frac{(2n)!}{n!(n+1)!}$$

$$= \frac{1}{n+1} \binom{2n}{n}$$

Catalan Number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

Average-Case Analysis of *QuickSort*

Quicksort

input: an array A of n numbers

Qsort(A):

choose a **pivot** $x = A[1]$;

partition A into L with all $L[i] < x$,

R with all $R[i] > x$;

Qsort(L) and Qsort(R);

Complexity: number of comparisons

worst-case: $\Theta(n^2)$

average-case: ?

Qsort(A):

choose a **pivot** $x = A[1]$;
partition A into L with all $L[i] < x$,
 R with all $R[i] > x$;
Qsort(L) and Qsort(R);

T_n :

average # of comparisons
used by Qsort

taken over all $n!$

total orders of A

pivot: the k -th smallest number in A

$$|L| = k-1 \quad |R| = n-k$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$nT_n = \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$\sum_{n \geq 0} nT_n x^n = \sum_{n \geq 0} \left(\sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k}) \right) x^n$$

$$= \sum_{n \geq 0} n(n-1)x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$\sum_{n \geq 0} n T_n x^n = \sum_{n \geq 0} n(n-1)x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$


$$= x^2 \sum_{n \geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$


$$= 2x \sum_{n \geq 0} \left(\sum_{k=0}^n T_k \right) x^n$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \quad F(x) = \sum_{n \geq 0} f_n x^n$$

shift:

$$x^k G(x) = \sum_{n \geq k} g_{n-k} x^n$$

addition:

$$F(x) + G(x) = \sum_{n \geq 0} (f_n + g_n) x^n$$

convolution:

$$F(x)G(x) = \sum_{n \geq 0} \sum_{k=0}^n f_k g_{n-k} x^n$$

differentiation:

$$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$\sum_{n \geq 0} n T_n x^n = \sum_{n \geq 0} n(n-1)x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$

$$\text{Blue square} = x^2 \sum_{n \geq 0} n(n-1)x^{n-2} = \frac{2x^2}{(1-x)^3}$$

$$\text{Orange square} = 2x \sum_{n \geq 0} \left(\sum_{k=0}^n T_k \right) x^n = 2x \sum_{n \geq 0} x^n \sum_{n \geq 0} T_n x^n = \frac{2xG(x)}{1-x}$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$\sum_{n \geq 0} n T_n x^n = \sum_{n \geq 0} n(n-1)x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$

$$\text{blue square} = \frac{2x^2}{(1-x)^3}$$

$$\text{orange square} = \frac{2xG(x)}{1-x}$$

Generating Function Algebra

$$G(x) = \sum_{n \geq 0} g_n x^n \quad F(x) = \sum_{n \geq 0} f_n x^n$$

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differentiation:

$$G'(x) = \sum_{n \geq 0} (n+1) g_{n+1} x^n$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$\sum_{n \geq 0} n T_n x^n = \sum_{n \geq 0} n(n-1) x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$

$$\text{blue square} = \frac{2x^2}{(1-x)^3}$$

$$\text{orange square} = \frac{2xG(x)}{1-x}$$

$$\text{pink square} = x \sum_{n \geq 0} (n+1) T_{n+1} x^n = xG'(x)$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$\sum_{n \geq 0} n T_n x^n = \sum_{n \geq 0} n(n-1)x^n + 2 \sum_{n \geq 0} \left(\sum_{k=0}^{n-1} T_k \right) x^n$$

$$\text{Pink square} = xG'(x) \quad \text{Blue square} = \frac{2x^2}{(1-x)^3} \quad \text{Orange square} = \frac{2xG(x)}{1-x}$$

$$xG'(x) = \frac{2x^2}{(1-x)^3} + \frac{2x}{1-x} G(x)$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x} G(x)$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x} G(x)$$

Solving: first-order linear differential equation

$$y' + P(x)y = Q(x)$$

$$y(x) = \frac{1}{u(x)} \int u(x) Q(x) \, dx \quad \text{with} \quad u(x) = e^{\int P(x) \, dx}$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x} G(x)$$

Solving: first-order linear differential equation

$$G(x) = e^{\int \frac{2}{1-x} dx} \int \frac{2x}{(1-x)^3} e^{-\int \frac{2}{1-x} dx} dx$$

$$= \frac{2}{(1-x)^2} \ln \frac{1}{1-x}$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x} G(x)$$

Solving: $G(x) = \frac{2}{(1-x)^2} \ln \frac{1}{1-x}$

Expanding: $\frac{2}{(1-x)^2} = 2 \sum_{n \geq 0} (n+1)x^n \quad \ln \frac{1}{1-x} = \sum_{n \geq 1} \frac{x^n}{n}$

Taylor

$$G(x) = 2 \sum_{n \geq 0} (n+1)x^n \sum_{n \geq 1} \frac{x^n}{n} = 2 \sum_{n \geq 1} \left(\sum_{k=1}^n (n-k+1) \frac{1}{k} \right) x^n$$

Recursion:

$$T_n = \frac{1}{n} \sum_{k=1}^n (n-1 + T_{k-1} + T_{n-k})$$

$$T_0 = T_1 = 0$$

Manipulation:

$$G(x) = \sum_{n \geq 0} T_n x^n$$

$$G'(x) = \frac{2x}{(1-x)^3} + \frac{2}{1-x} G(x)$$

Solving: $G(x) = \frac{2}{(1-x)^2} \ln \frac{1}{1-x}$

Expanding: $G(x) = 2 \sum_{n \geq 1} \left(\sum_{k=1}^n (n-k+1) \frac{1}{k} \right) x^n$

$$T_n = 2 \sum_{k=1}^n (n-k+1) \frac{1}{k} = 2(n+1) \sum_{k=1}^n \frac{1}{k} - 2 \sum_{k=1}^n k \cdot \frac{1}{k}$$

$$= 2(n+1)H(n) - 2n = 2n \ln n + O(n)$$