Combinatorics

The Sieve Methods

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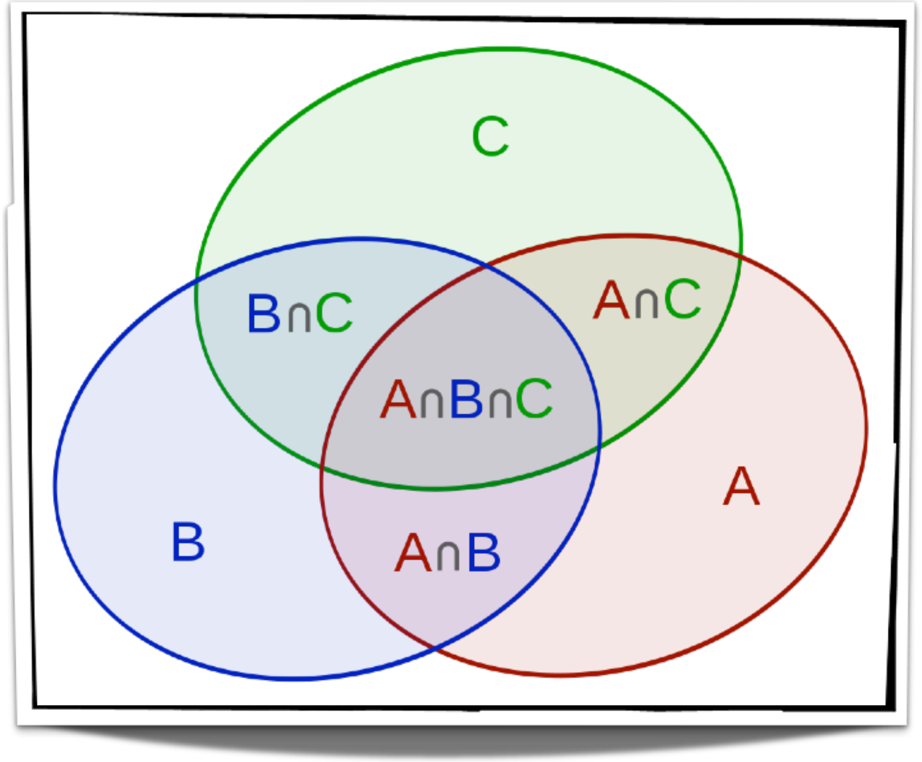
PIE (Principle of Inclusion-Exclusion)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$-|A \cap B| - |A \cap C| - |B \cap C|$$
$$+|A \cap B \cap C|$$



PIE (Principle of Inclusion-Exclusion)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \cdots + (-1)^{n-1}|A_1 \cap \cdots \cap A_n|$$

$$= \sum_{\substack{I\subseteq\{1,\ldots,n\} \\ I\neq\emptyset}} (-1)^{|I|-1} \left|\bigcap_{i\in I} A_i\right|$$

PIE (Principle of Inclusion-Exclusion)

$$A_1, A_2, \ldots, A_n \subseteq U$$ ← universe

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}\right| = \left|U - \bigcup_{i=1}^{n} A_i\right|$$

$$= |U| - \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left|\bigcap_{i \in I} A_i\right|$$

$$A_I = \bigcap_{i \in I} A_i \qquad A_\emptyset = U$$

PIE (Principle of Inclusion-Exclusion)

$$A_1, A_2, \ldots, A_n \subseteq U$$ ← universe

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}\right| = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \, |A_I|$$

where $A_I = \bigcap_{i \in I} A_i$ $\quad A_\emptyset = U$

PIE (Principle of Inclusion-Exclusion)

$$A_1, A_2, \ldots, A_n \subseteq U$$ ← universe

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}\right| = S_0 - S_1 + S_2 + \cdots + (-1)^n S_n$$

where $S_k = \sum_{|I|=k} |A_I|$ $\qquad A_I = \bigcap_{i \in I} A_i$

$$S_0 = |A_\emptyset| = |U| \qquad A_\emptyset = U$$

Surjections

of

$$f : [n] \xrightarrow{\text{onto}} [m]$$

$$U = [n] \to [m] \qquad A_i = [n] \to ([m] \setminus \{i\})$$

$$\left| \bigcap_{i \in [m]} \overline{A_i} \right| = \sum_{I \subseteq [m]} (-1)^{|I|} |A_I|$$

$$A_I = \bigcap_{i \in I} A_i \qquad A_\emptyset = U$$

Surjections

$$U = [n] \to [m] \qquad A_i = [n] \to ([m] \setminus \{i\})$$

$$A_\emptyset = U \qquad A_I = \bigcap_{i \in I} A_i \quad = [n] \to ([m] \setminus I)$$

$$|A_I| = (m - |I|)^n$$

$$\left| \bigcap_{i \in [m]} \overline{A_i} \right| = \sum_{I \subseteq [m]} (-1)^{|I|} \, |A_I|$$

$$= \sum_{I \subseteq [m]} (-1)^{|I|} (m - |I|)^n \; = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

$$= \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} k^n$$

Surjections

$$\left|[n] \xrightarrow{\text{onto}} [m]\right| = \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} k^n$$

$$(f^{-1}(0), f^{-1}(1), \ldots, f^{-1}(m-1))$$

ordered *m*-partition of [*n*]

$$\left|[n] \xrightarrow{\text{onto}} [m]\right| = m! \left\{ {n \atop m} \right\}$$

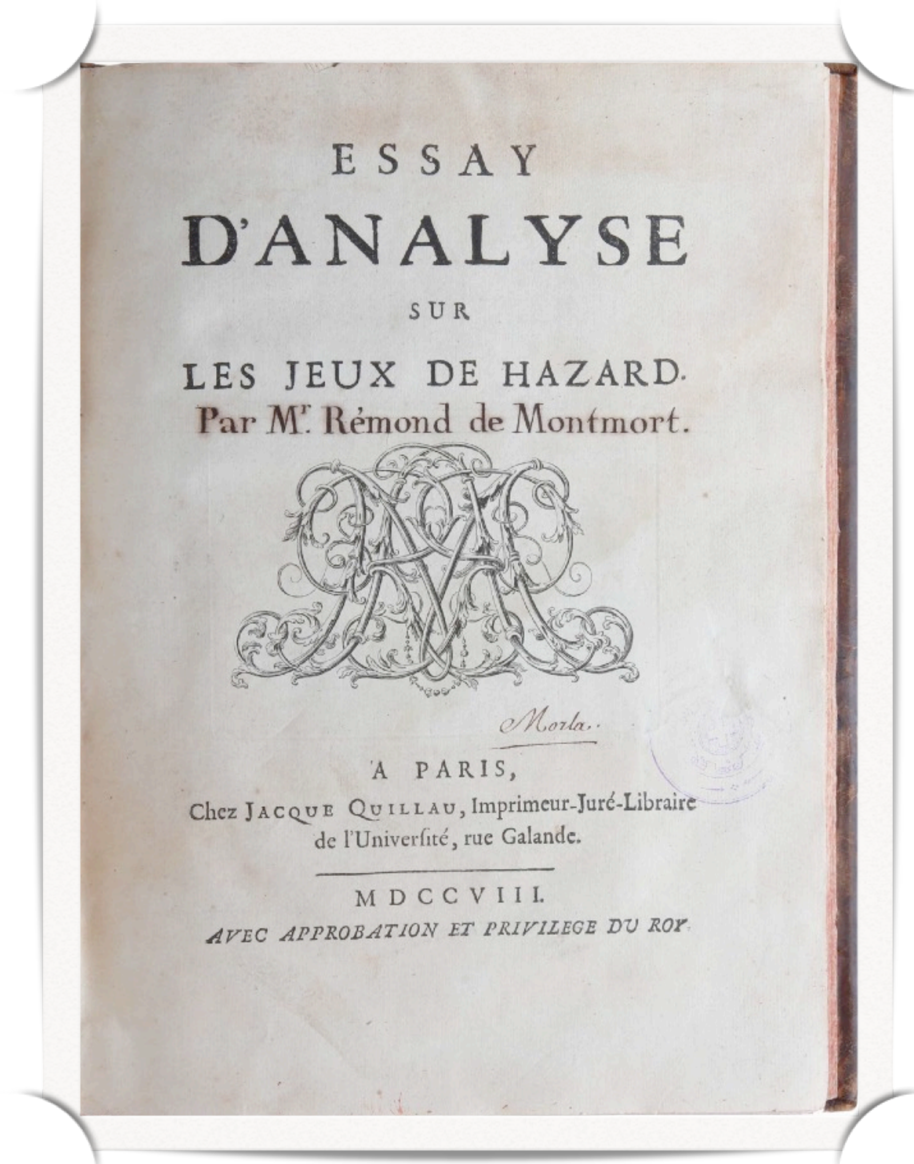
$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} k^n$$

Derangement

les problèmes des rencontrés:

Two decks, *A* and *B*, of cards:
The cards of *A* are laid out in a row,
and those of *B* are placed at random,
one at the top on each card of *A*.

What is the probability that
no 2 cards are the same in each pair?



Derangement

permutation π of $[n]$

$$\forall i \in [n], \quad \pi(i) \neq i$$

"permutations with no fixed point" $!n$

U : permutations of $[n]$

Derangement

permutation π of $[n]$

$$\forall i \in [n], \quad \pi(i) \neq i$$

"permutations with no fixed point" $!n$

$U = S_n$ symmetric group $A_i = \{\pi \mid \pi(i) = i\}$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

$A_I = \{\pi \mid \forall i \in I, \pi(i) = i\}$ $|A_I| = (n - |I|)!$

Derangement

$$U = S_n \qquad A_i = \{\pi \mid \pi(i) = i\}$$

$$A_I = \{\pi \mid \forall i \in I, \pi(i) = i\} \qquad |A_I| = (n - |I|)!$$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

$$= \sum_{I \subseteq [n]} (-1)^{|I|} (n - |I|)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}$$

Permutations with restricted positions

$\text{permutation } \pi \text{ of } [n]$

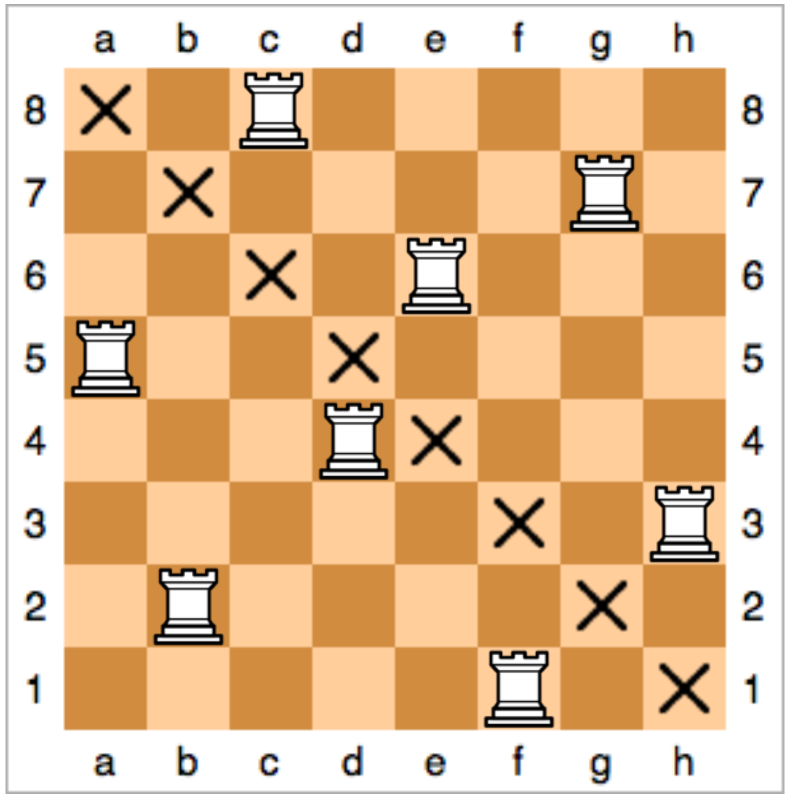
derangement: $\forall i \in [n], \quad \pi(i) \neq i$

generally: $\pi(i_1) \neq j_1, \pi(i_2) \neq j_2, \ldots$

forbidden positions $B \subseteq [n] \times [n]$

$$\forall i \in [n], \quad (i, \pi(i)) \not\in B$$

Chess board



permutation π of $[n]$

$$\{(i, \pi(i)) \mid i \in [n]\}$$

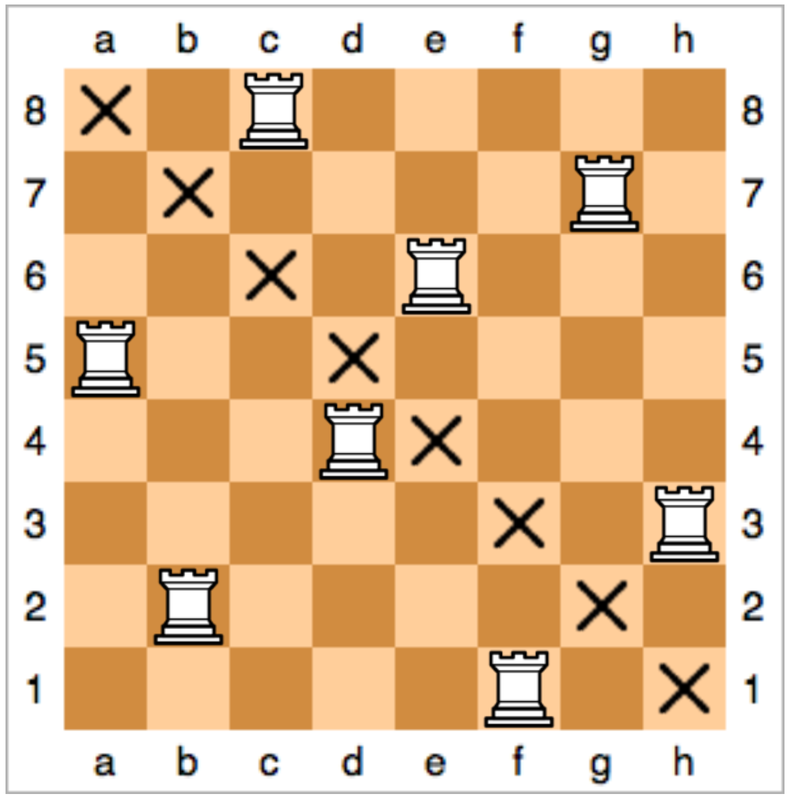
"A placement of non-attacking rooks"

forbidden positions $B \subseteq [n] \times [n]$

derangement:

$$B = \{(i, i) \mid i \in [n]\}$$

Chess board



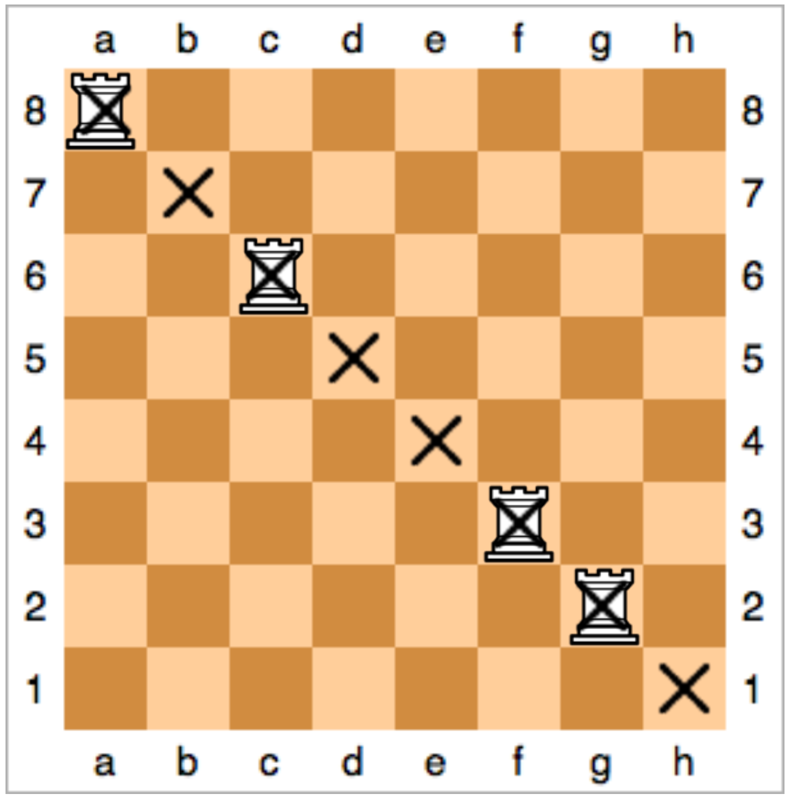
For a particular set of forbidden positions

$$B \subseteq [n] \times [n]$$

N_0 :

the # of placements of *n* **non-attacking rooks**?

Chess board



For a particular set of forbidden positions

$$B \subseteq [n] \times [n]$$

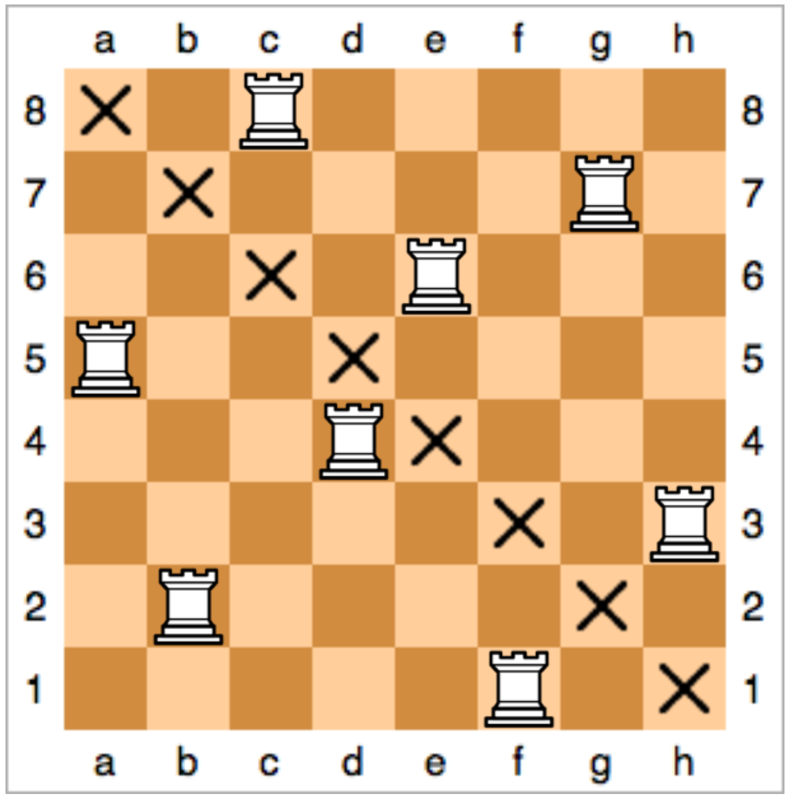
r_k :

# of ways of placing *k* **non-attacking rooks** in *B*

N_0 :

the # of placements of *n* **non-attacking rooks**?

Chess board



$B \subseteq [n] \times [n]$

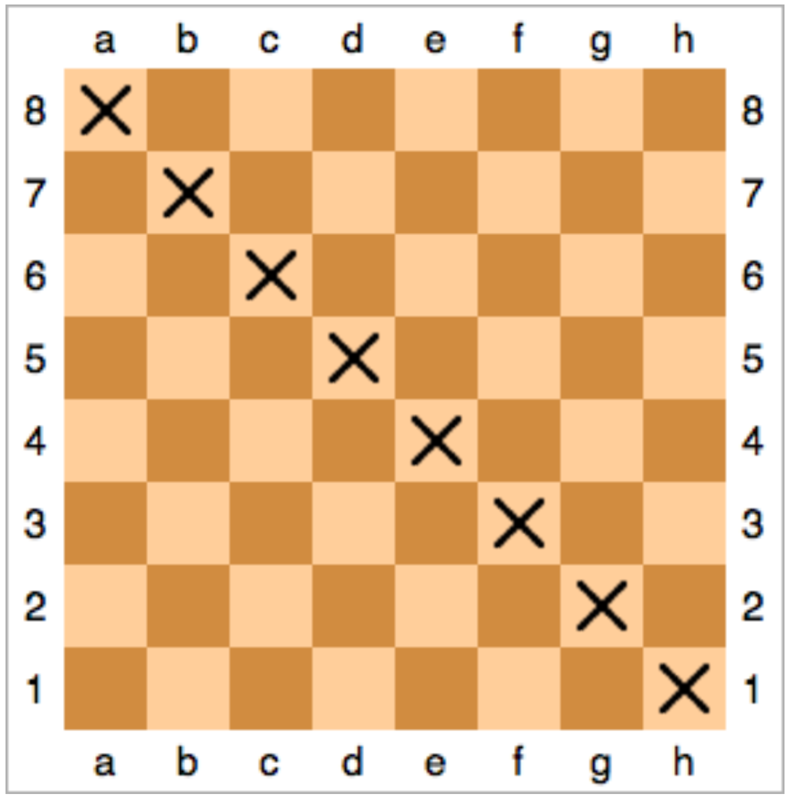
r_k :

# of ways of placing *k* **non-attacking rooks** in *B*

N_0 : # of placements of *n* **non-attacking rooks**

$$N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$$

Derangement again



$$B = \{(i,i) \mid i \in [n]\}$$

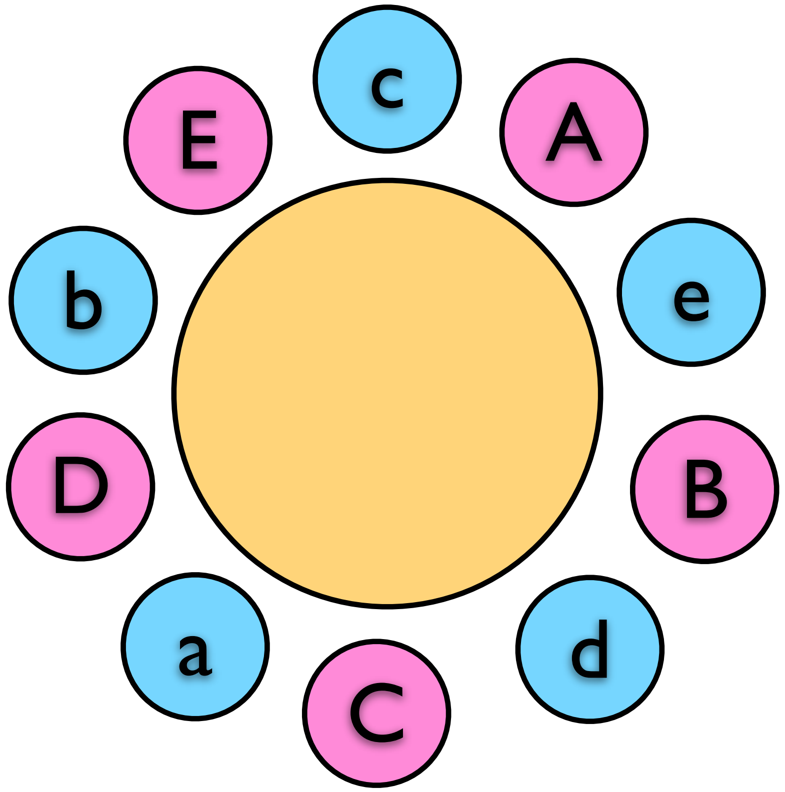
r_k : # of ways of placing *k* non-attacking rooks in *B*

$$\binom{n}{k}$$

$$N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)! \quad = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \quad = n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} \quad \approx \frac{n!}{e}$$

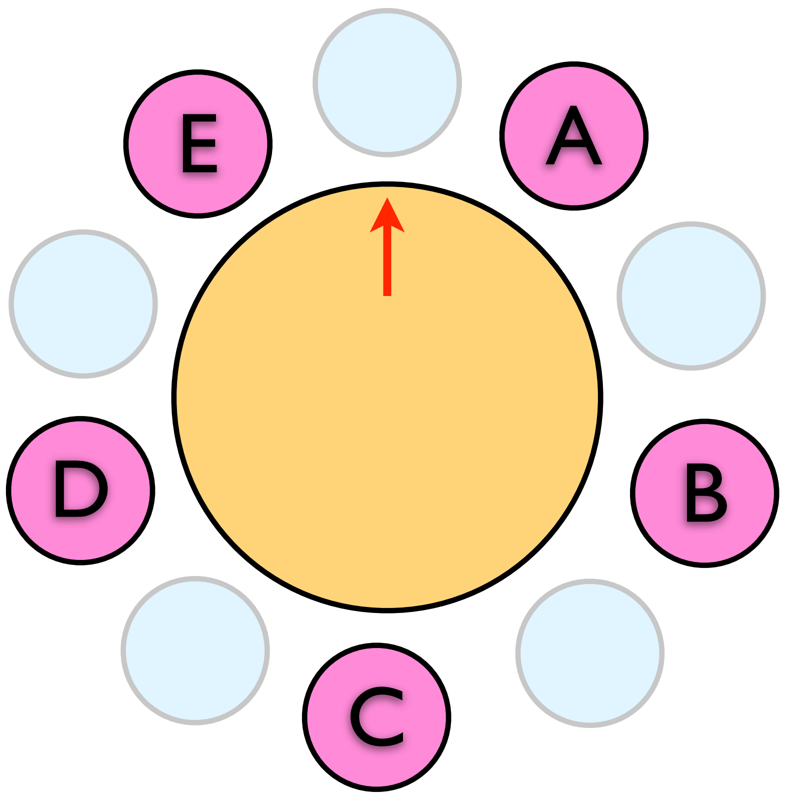
Problème des ménages



n couples sit around a table

- male-female alternative
- no one sit next to spouse

Problème des ménages



"Lady first!"

2(n!) ways

"Gentlemen, please sit."

permutation π of $[n]$

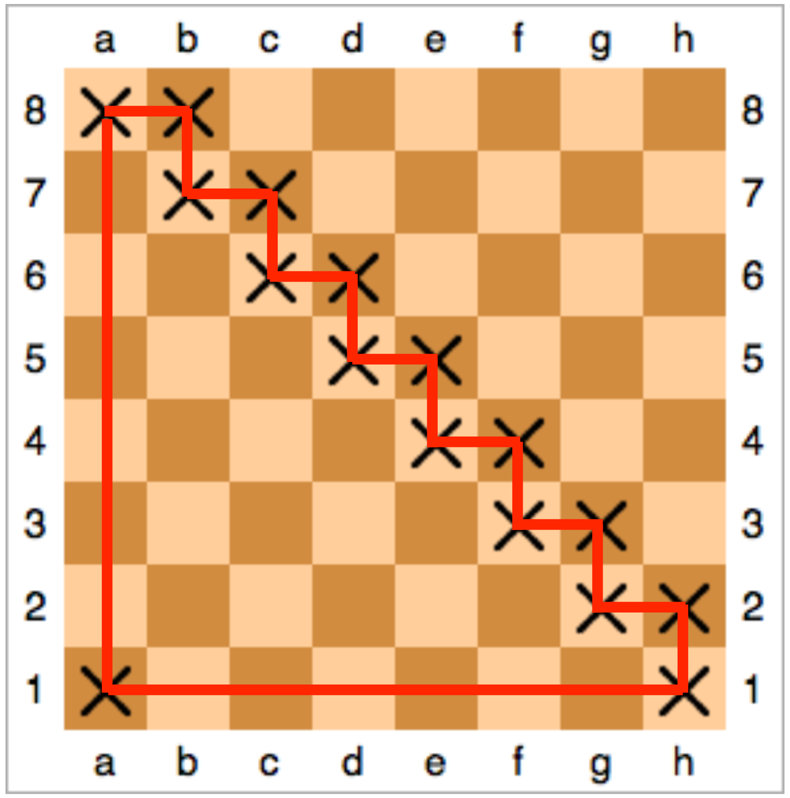
i : husband of the lady at the *i*-th position

$\pi(i)$: his seat

$$\pi(i) \neq i$$

$$\pi(i) \neq i+1 \pmod{n}$$

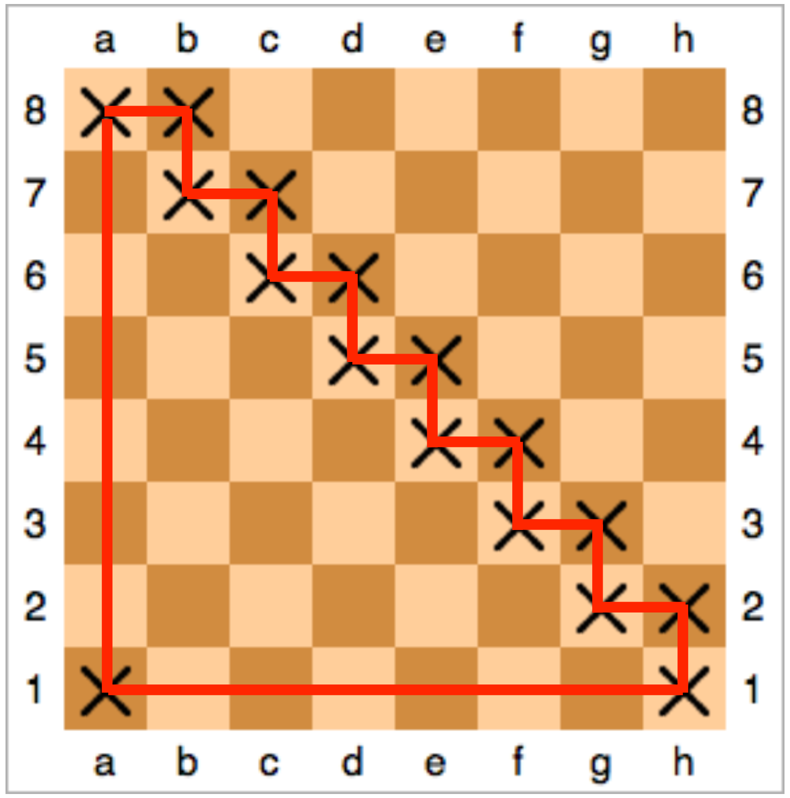
Problème des ménages



$$B = \{(i,i),(i,(i+1) \bmod n)\}$$

r_k : # of ways of placing *k* non-attacking rooks in *B*

Problème des ménages

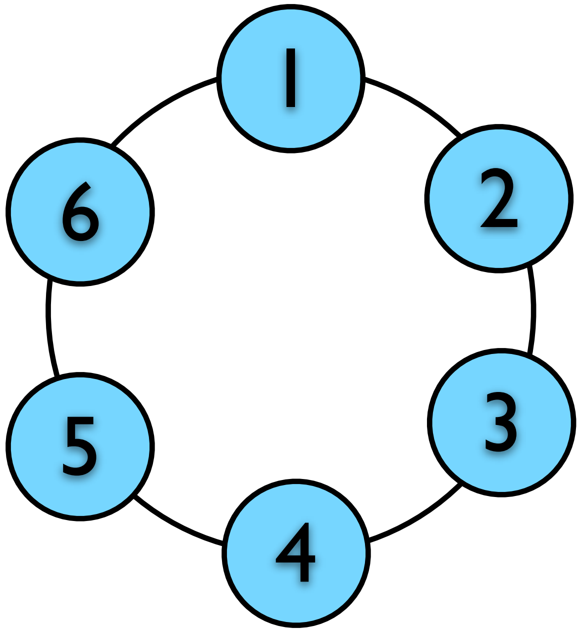


$$B = \{(i,i),(i,(i+1) \bmod n)\}$$

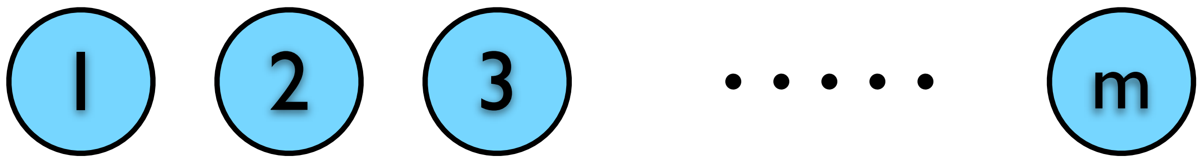
r_k : # of ways of choosing *k* non-consecutive points from a circle of 2*n* points

2*n* objects in a circle

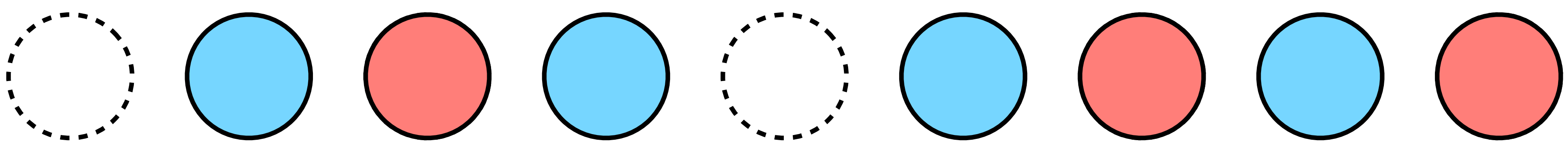
choose *k* non-consecutive objects



m objects in a line



L(*m*,*k*): choose *k* non-consecutive objects



m-*k* objects, *m*-*k*+1 space

choose *k* from *m*-*k*+1 space

$$L(m,k) = \binom{m-k+1}{k}$$

m objects in a circle

$C(m,k)$: choose *k* non-consecutive objects

$(m-k)C(m,k)$:

1. choose *k* non-consecutive objects from a circle
2. mark one of the remaining objects

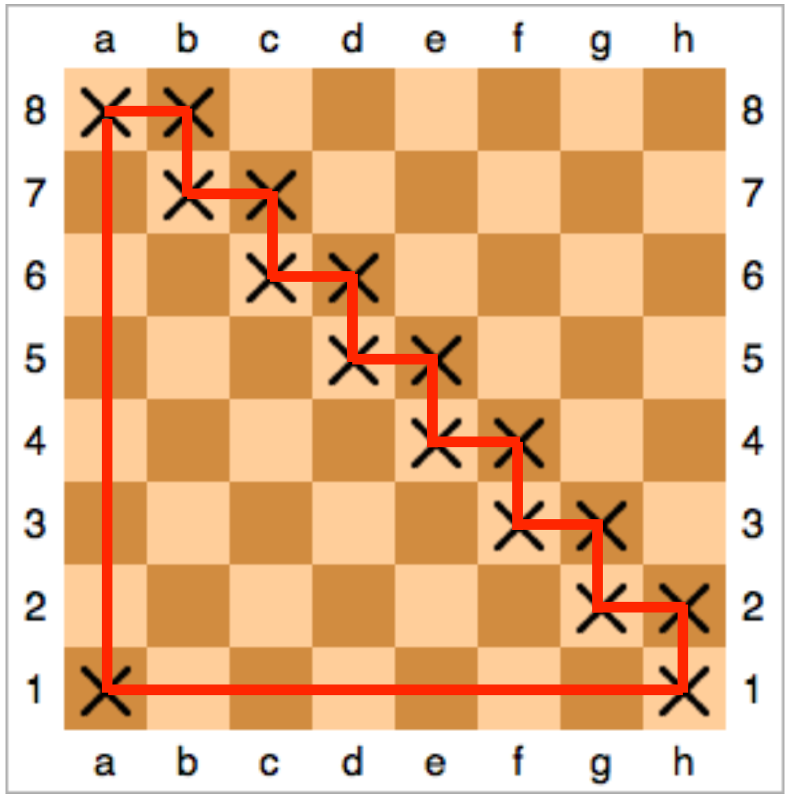
||

$m\,L(m-1,k)$:

1. mark one object in the circle, cut the circle by removing the object
2. choose *k* non-consecutive objects from the *m*-1 objects in a line

$$C(m,k) = \frac{m}{m-k}\binom{m-k}{k}$$

Problème des ménages



$$B = \{(i,i), (i, (i+1) \bmod n)\}$$

r_k : # of ways of choosing *k* non-consecutive points from a circle of *2n* points

$$\frac{2n}{2n-k}\binom{2n-k}{k}$$

$$N_0 = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!$$

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$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}\right| = \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} \, |A_I|$$

$$A_I = \bigcap_{i \in I} A_i \qquad A_\emptyset = U$$

Inversion

V: 2^n-dimensional vector space of all mappings

$$f : 2^{[n]} \to \mathbb{N}$$

linear transformation $\phi : V \to V$

$$\forall S \subseteq [n], \qquad \phi f(S) \triangleq \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} f(T)$$

then its inverse:

$$\forall S \subseteq [n], \qquad \phi^{-1} f(S) = \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} (-1)^{|T \setminus S|} f(T)$$

$$\phi f(S) \triangleq \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} f(T) \qquad \phi^{-1} f(S) = \sum_{\substack{T \supseteq S \\ T \subseteq [n]}} (-1)^{|T \setminus S|} f(T)$$

$$A_1, A_2, \ldots, A_n \subseteq U \qquad I \subseteq [n]$$

$$f_=(I) = |\{x \in U \mid \forall i \in I, x \in A_i, \forall j \notin I, x \notin A_j\}|$$

$$= \left| \left(\bigcap_{i \in I} A_i \right) \setminus \left(\bigcup_{j \notin I} A_j \right) \right|$$

$$f_\geq(I) = \sum_{\substack{J \supseteq I \\ J \subseteq [n]}} f_=(J) = \left| \bigcap_{i \in I} A_i \right| = |A_I|$$

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = f_=(\emptyset) = \sum_{\substack{I \supseteq \emptyset \\ I \subseteq [n]}} (-1)^{|I \setminus \emptyset|} f_\geq(I) = \sum_{I \subseteq [n]} (-1)^{|I|} |A_I|$$

PIE (Principle of Inclusion-Exclusion)

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $T \subseteq S$

$$\sum_{T \subseteq I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = T \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$A_1 = A_2 = \cdots = A_n = \{1\}$$

$$1 = \left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \, |A_I|$$

$$A_I = \bigcap_{i \in I} A_i \; = \{1\}$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \ldots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{n-|I|} = 0$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \ldots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{-|I|} = 0$$

$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1}$$

when $\{1, 2, \ldots, n\} \neq \emptyset$

$$\sum_{I \subseteq \{1,\ldots,n\}} (-1)^{|I|} = 0$$

PIE

(Principle of Inclusion-Exclusion)

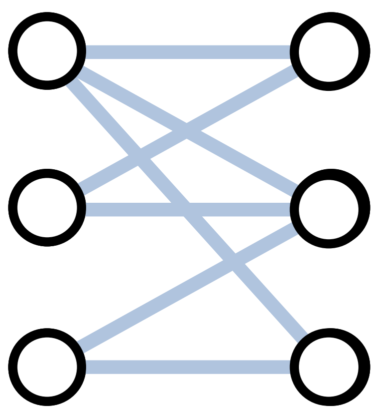
$$\sum_{I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for $T \subseteq S$

$$\sum_{T \subseteq I \subseteq S} (-1)^{|S|-|I|} = \begin{cases} 1 & S = T \\ 0 & \text{otherwise} \end{cases}$$

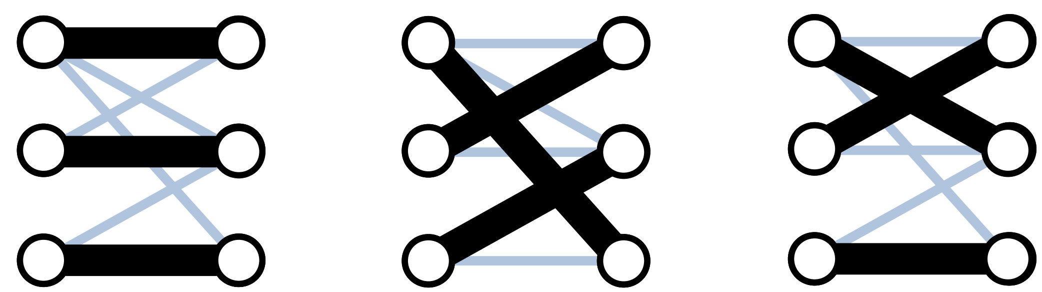
Bipartite Perfect

bipartite graph



$G([n],[n],E)$

perfect matchings



$\text{permutation } \pi \text{ of } [n]$ **s.t.** $(i, \pi(i)) \in E$

$n \times n \text{ matrix } A:$

$$A_{i,j} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

\# of P.M. in G

$$= \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)}$$

Permanent

$n \times n$ matrix A :

$$\text{perm}(A) = \sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)}$$

#P-hard

determinant:

$$\det(A) = \sum_{\pi \in S_n} (-1)^{r(\pi)} \prod_{i \in [n]} A_{i,\pi(i)}$$

poly-time by Gaussian elimination

Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

term in : $\prod_{i \in [n]} A_{i,f(i)}$ for some $f : [n] \to [n]$

$$T = f([n]) \subseteq I$$

coefficient of $\prod_{i \in [n]} A_{i,f(i)}$ in :

$$\sum_{T \subseteq I \subseteq [n]} (-1)^{n-|I|} = \begin{cases} 1 & T = [n] \\ 0 & o.w. \end{cases}$$

← f is a permutation

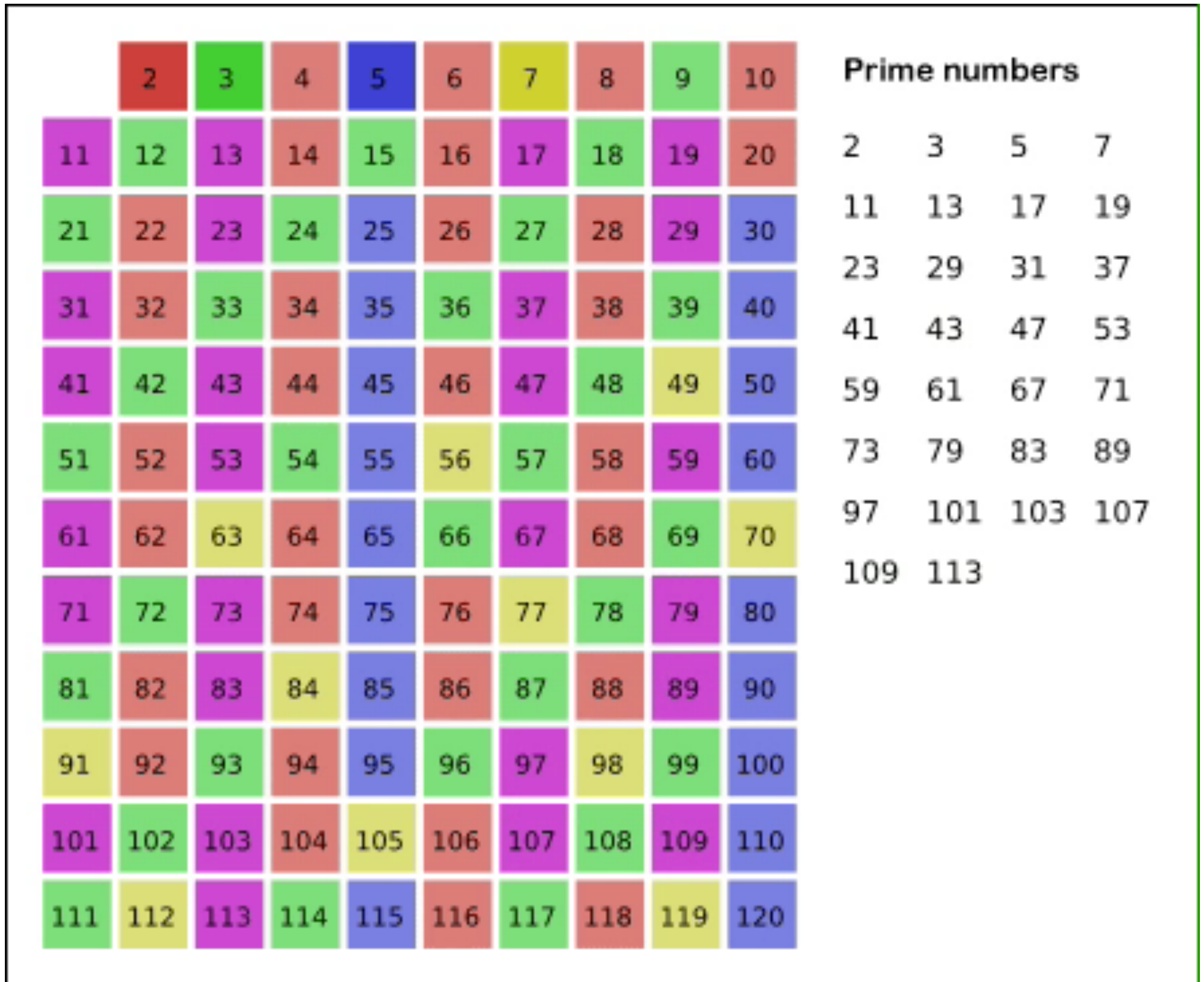
Ryser's formula

$$\sum_{\pi \in S_n} \prod_{i \in [n]} A_{i,\pi(i)} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \prod_{i \in [n]} \sum_{j \in I} A_{i,j}$$

O($n!$) time

O($n2^n$) time

Sieve of Eratosthenes



Euler Totient Function

$\phi(n)$ = $|\{1 \leq a \leq n \mid \gcd(a,n)=1\}|$

\# of $a \in \{1,2,\ldots,n\}$ relative prime to n

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$$\phi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

Euler Totient Function

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prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

Universe: $U = \{1, 2, ..., n\}$

$i = 1, 2, \ldots, r$ $\qquad A_i = \{1 \leq a \leq n \mid p_i | a\}$

$I \subseteq \{1, 2, \ldots, r\}$ $\qquad A_I = \{1 \leq a \leq n \mid \forall i \in I, p_i | a\}$

$|A_i| = \frac{n}{p_i}$ $\qquad |A_I| = \frac{n}{\prod_{i \in I} p_i}$

$$\phi(n) = \left| \bigcap_{i \in \{1,\ldots,r\}} \overline{A_i} \right| = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} |A_I|$$

Euler Totient Function

$\phi(n)$ $= |\{1 \leq a \leq n \mid \gcd(a,n)=1\}|$

prime decomposition: $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$I \subseteq \{1, 2, \ldots, r\}$ $\quad A_I = \{1 \leq a \leq n \mid \forall i \in I, p_i | a\}$

$$|A_I| = \frac{n}{\prod_{i \in I} p_i}$$

$$\phi(n) = \sum_{I \subseteq \{1,\ldots,r\}} (-1)^{|I|} |A_I|$$

$$= n \sum_{k=0}^{r} \sum_{I \in \binom{\{1,\ldots,r\}}{k}} \frac{(-1)^{|I|}}{\prod_{i \in I} p_i} = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right)$$

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