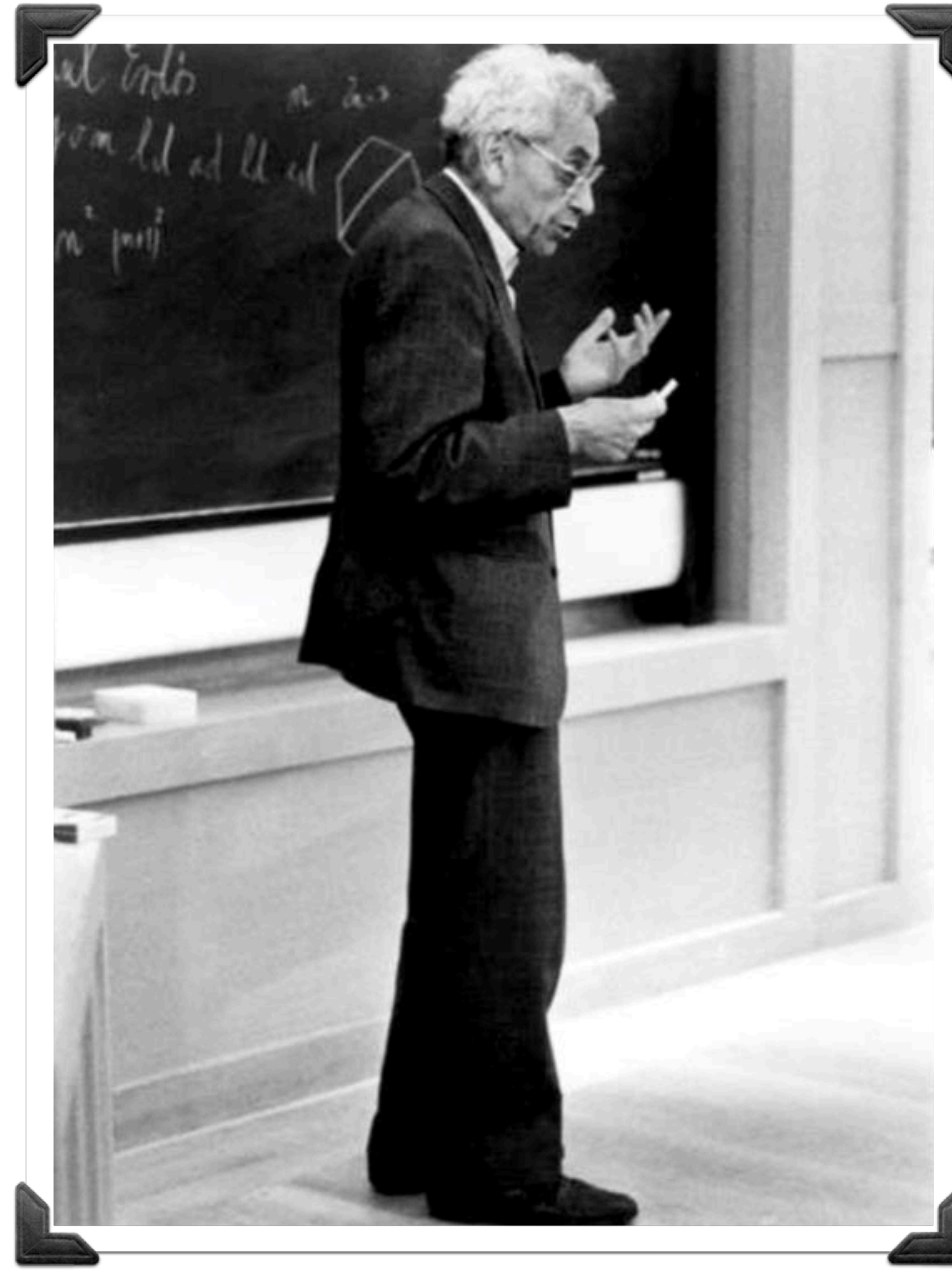
Combinatorics

The Probabilistic Method

尹一通 Nanjing University, 2025 Spring

The Probabilistic Method

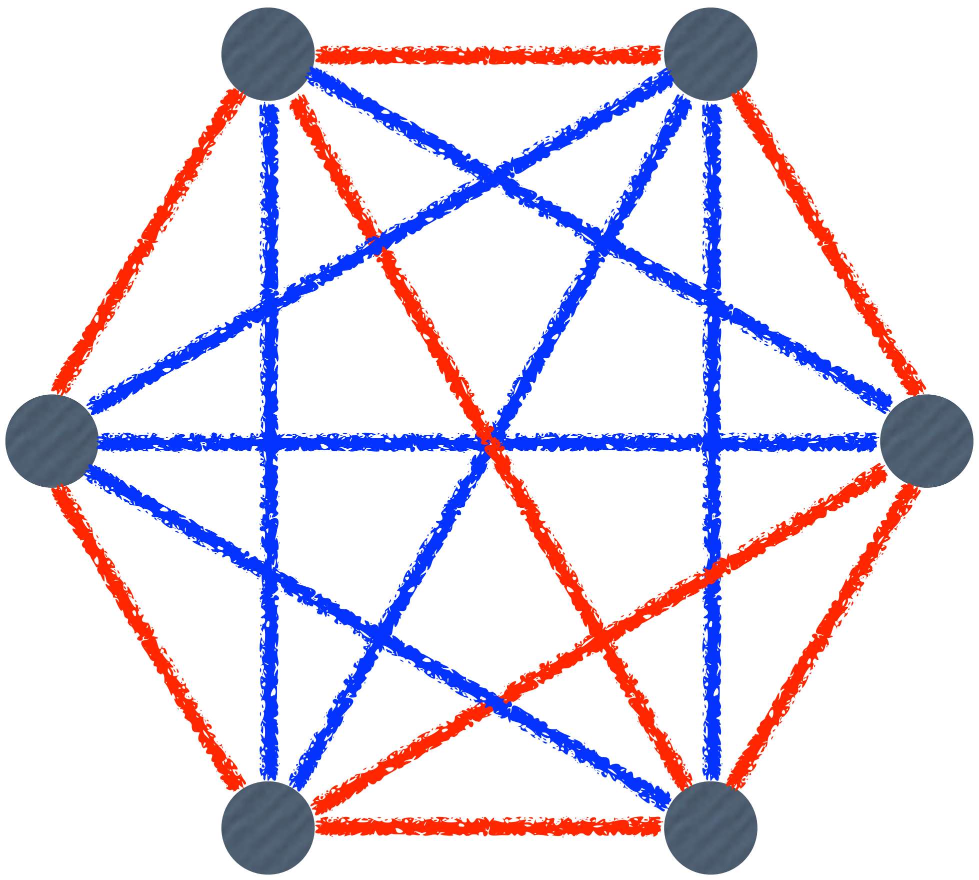


Paul Erdős
(1913-1996)

Ramsey Number

"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

- For any edge-2-coloring of K_6, there is a ***monochromatic*** K_3.



Ramsey Theorem

If $n \geq R(k,k)$, for any edge-2-coloring of K_n, there is a monochromatic K_k.

Ramsey number: $R(k,k)$

Theorem (Erdős 1947)

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with 2 colors so that there is no monochromatic K_k subgraph.

Each edge $e \in K_n$ is colored

- with prob 1/2
- with prob 1/2

For any K_k subgraph:

$$\Pr[\text{the } K_k \text{ is monochromatic}] = \Pr[\textcolor{red}{K_k} \text{ or } \textcolor{blue}{K_k}]$$
$$= 2^{1-\binom{k}{2}}$$

Theorem (Erdős 1947)

If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$ then it is possible to color the edges of K_n with 2 colors so that there is no monochromatic K_k subgraph.

Each edge $e \in K_n$ is colored
- with prob 1/2
- with prob 1/2

$$\Pr[\,\exists K_k \text{ is monochromatic}] \leq \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

$$\implies \Pr[\text{no } K_k \text{ is monochromatic}] > 0$$

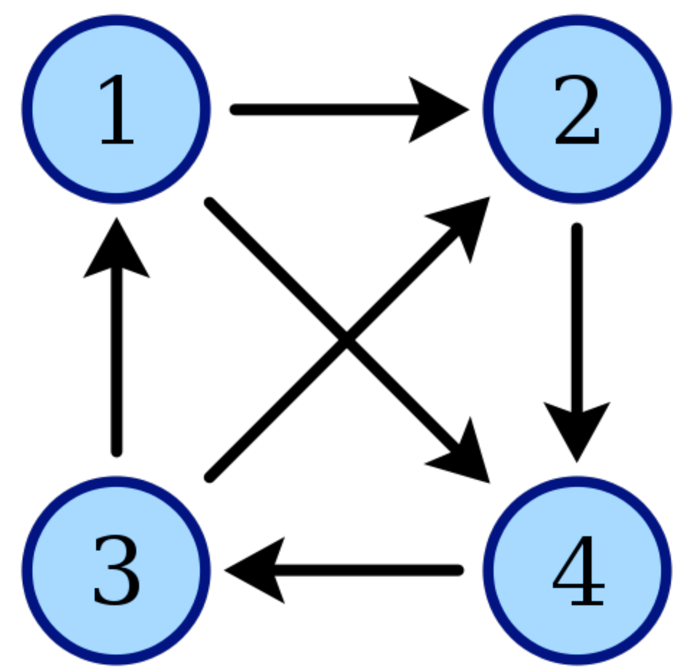
$\implies \exists$ a 2-coloring of edges of K_n without monochromatic K_k

Tournament

$T(V, E)$

n players, each pair has a match.

$u \to v$ iff u beats v.



k-paradoxical:

For every k-subset S of V, there is a player in $V \backslash S$ who beats all players in S.

"*Does there exist a k-paradoxical tournament for every finite k?*"

Theorem (Erdős 1963)
If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament T on n players $[n]$.

Fixed any $S \in \binom{[n]}{k}$

Event A_S: no player in $V \backslash S$ beat all players in S.

$$\Pr[A_S] = \left(1-2^{-k}\right)^{n-k}$$

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament T on n players $[n]$.

Event A_S: no player in $V \setminus S$ beat all players in S.

$$\forall S \in \binom{[n]}{k}: \quad \Pr[A_S] = \left(1-2^{-k}\right)^{n-k}$$

$$\Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] \le \sum_{S\in\binom{[n]}{k}} (1-2^{-k})^{n-k} \quad < 1$$

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament T on n players $[n]$.

Event A_S: no player in $V \backslash S$ beat all players in S.

$$\Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] < 1$$

$$\Pr[T \text{ is } k\text{-paradoxical}] = 1 - \Pr\left[\bigvee_{S\in\binom{[n]}{k}} A_S\right] > 0$$

Theorem (Erdős 1963)

If $\binom{n}{k}\left(1-2^{-k}\right)^{n-k} < 1$ then there is a k-paradoxical tournament of n players.

Pick a random tournament T on n players $[n]$.

$$\Pr[T \text{ is } k\text{-paradoxical}] > 0$$

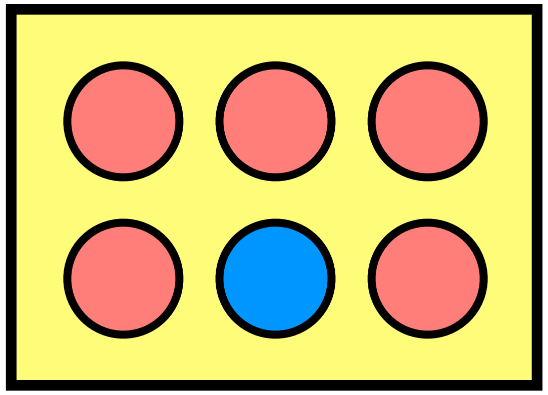
There is a k-paradoxical tournament on n players.

The Probabilistic Method

- Pick random ball from a box,

  Pr[the ball is blue]>0.

  ⇒ There is a blue ball.



- Define a probability space Ω, and a property P:

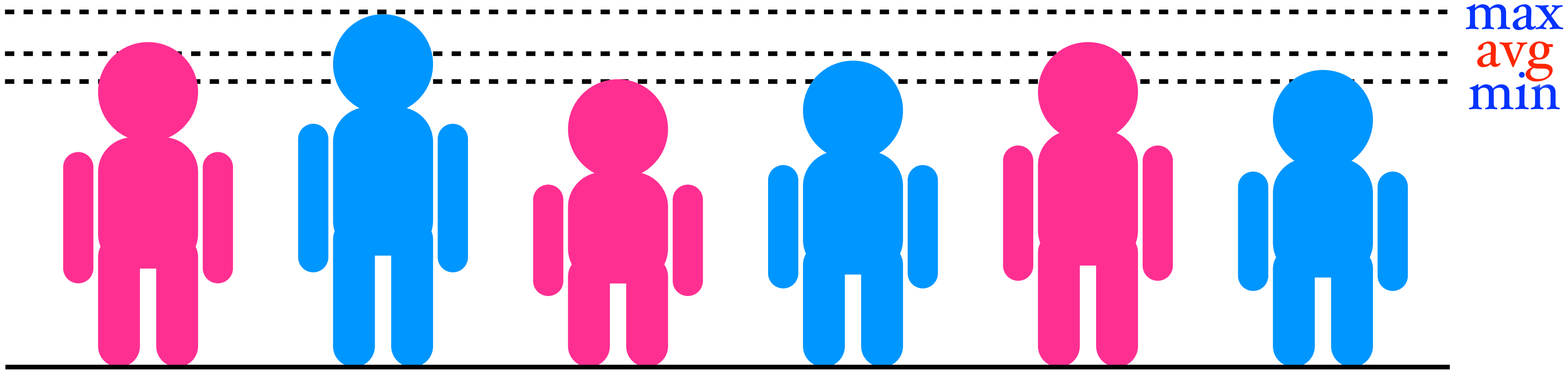
$$\Pr_x[P(x)] > 0$$

$\implies \exists$ a sample $x \in \Omega$ with property P.

Averaging Principle

- Average height of the students in class is l.

  $\Rightarrow$ There is a student of height $\geq l$ ($\leq l$)

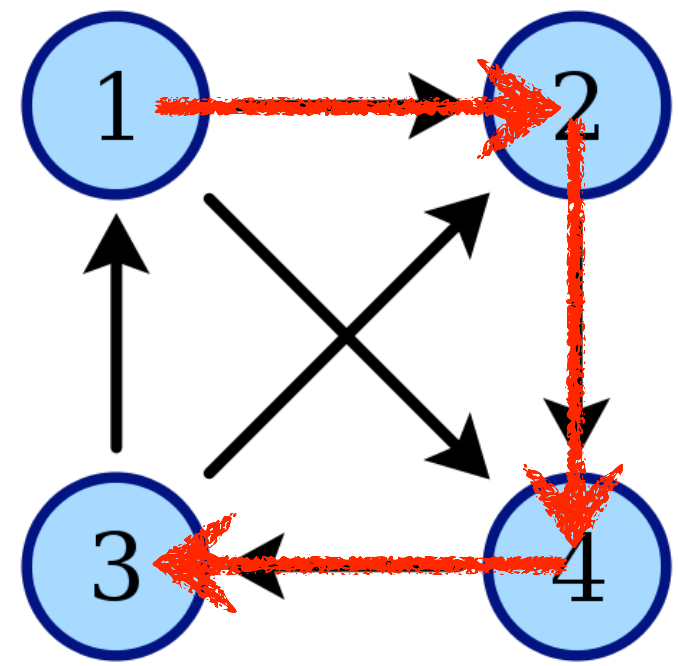


- For a random variable X,
  - $\exists\, x \leq \mathrm{E}[X]$, such that $X = x$ is possible;
  - $\exists\, x \geq \mathrm{E}[X]$, such that $X = x$ is possible.

Hamiltonian Paths in Tournament

Hamiltonian path:

a path visiting every vertex *exactly* once.



Theorem (Szele 1943)

There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

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There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

Pick a random tournament T on n players $[n]$.

For every permutation π of $[n]$,

$$X_\pi = \begin{cases} 1 & \pi \text{ is a Hamiltonian path} \\ 0 & \pi \text{ is } \textit{not} \text{ a Hamiltonian path} \end{cases}$$

Hamiltonian paths: $X = \sum_{\pi} X_\pi$

$$\mathrm{E}[X_\pi] = \Pr[X_\pi = 1] = 2^{-(n-1)}$$

Theorem (Szele 1943)

There is a tournament on n players with at least $n!2^{-(n-1)}$ Hamiltonian paths.

Pick a random tournament T on n players $[n]$.

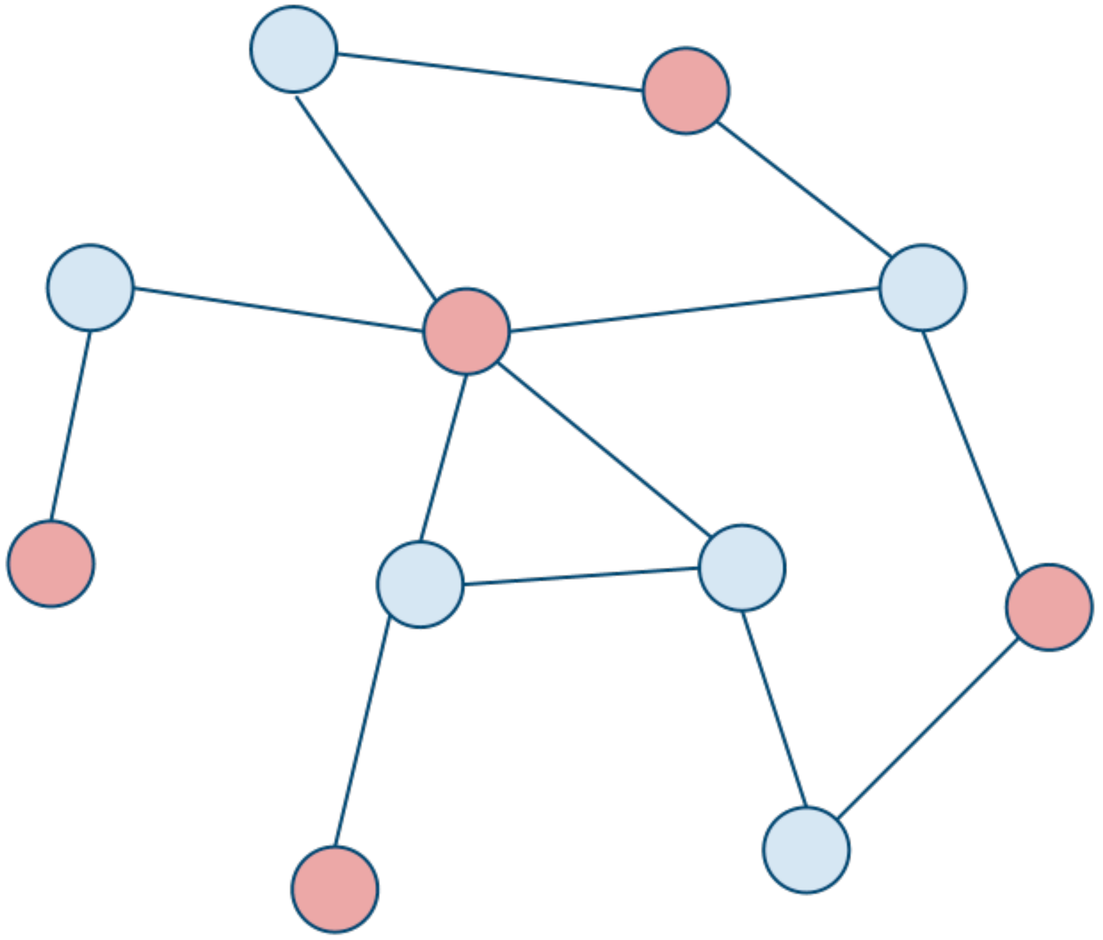
Hamiltonian paths: $X = \sum_{\pi} X_\pi$

$$\mathrm{E}[X_\pi] = \Pr[X_\pi = 1] = 2^{-(n-1)}$$

$$\mathrm{E}[X] = \sum_{\pi} \mathrm{E}[X_\pi] = n!2^{-(n-1)}$$

Large Independent Set

- Graph $G(V, E)$
- independent set $S \subseteq V$
  - no adjacent vertices in S
- max independent set is **NP**-hard



Theorem: G has n vertices and m edges

⇨ $\exists$ an independent set S of size $\dfrac{n^2}{4m}$

- Draw a random independent set $S \subseteq V$ (How?)
  - each $v \in V$ is selected into a random set R independently with probability p (to be fixed later)
  - for every $uv \in E$: delete one of u, v from R if $u, v \in R$
  - the resulting set is an independent set S
- Show that $\mathbf{E}[\,|S|\,] \geq \dfrac{n^2}{4m}$

$G(V,E)$: n vertices, m edges

1. sample a random R : each vertex is chosen *independently* with probability p

2. modify R to S : independent set!

$\forall uv \in E$ if $u, v \in R$

delete one of u, v from R

Y: # of edges in R $Y = \sum_{uv \in E} Y_{uv}$ $Y_{uv} = \begin{cases} 1 & u, v \in S \\ 0 & \text{o.w.} \end{cases}$

$$\mathbf{E}[\,|S|\,] \geq \mathbf{E}[\,|R| - Y] = \mathbf{E}[\,|R|\,] - \mathbf{E}[Y]$$

$$\mathbf{E}[\,|R|\,] = np \qquad \mathbf{E}[Y] = \sum_{uv \in E} \mathbf{E}[Y_{uv}] = mp^2$$

$G(V,E)$: n vertices, m edges

1. sample a random R : each vertex is chosen *independently* with probability p

2. modify R to S : independent set!

$\forall uv \in E$ if $u, v \in R$

delete one of u, v from R

$$\mathbf{E}[\,|S|\,] \geq np - mp^2 = \frac{n^2}{4m}$$

when $p = \dfrac{n}{2m}$

$G(V,E)$: n vertices, m edges

average degree $d = \frac{2m}{n}$

random independent set S:

$$\mathbf{E}[\,|S|\,] \geq \frac{n^2}{4m} = \frac{n}{2d}$$

Theorem: G has n vertices and m edges

⇨ $\exists$ an independent set S of size $\frac{n^2}{4m}$

Theorem: G has n vertices and m edges

$\Rightarrow$ $\exists$ an independent set S of size $\dfrac{n^2}{2m+n}$

- Draw a random independent set $S \subseteq V$
  - each $v \in V$ draws a real number $r_v \in [0,1]$ uniform and independent at random
  - each $v \in V$ joins S iff r_v is local maximal within the neighborhood of v
  - S must be an independent set
- $$\forall v \in V\colon \Pr[v \in S] = \frac{1}{d_v + 1} \implies \mathbf{E}[\,|S|\,] = \sum_{v \in V} \frac{1}{d_v + 1}$$

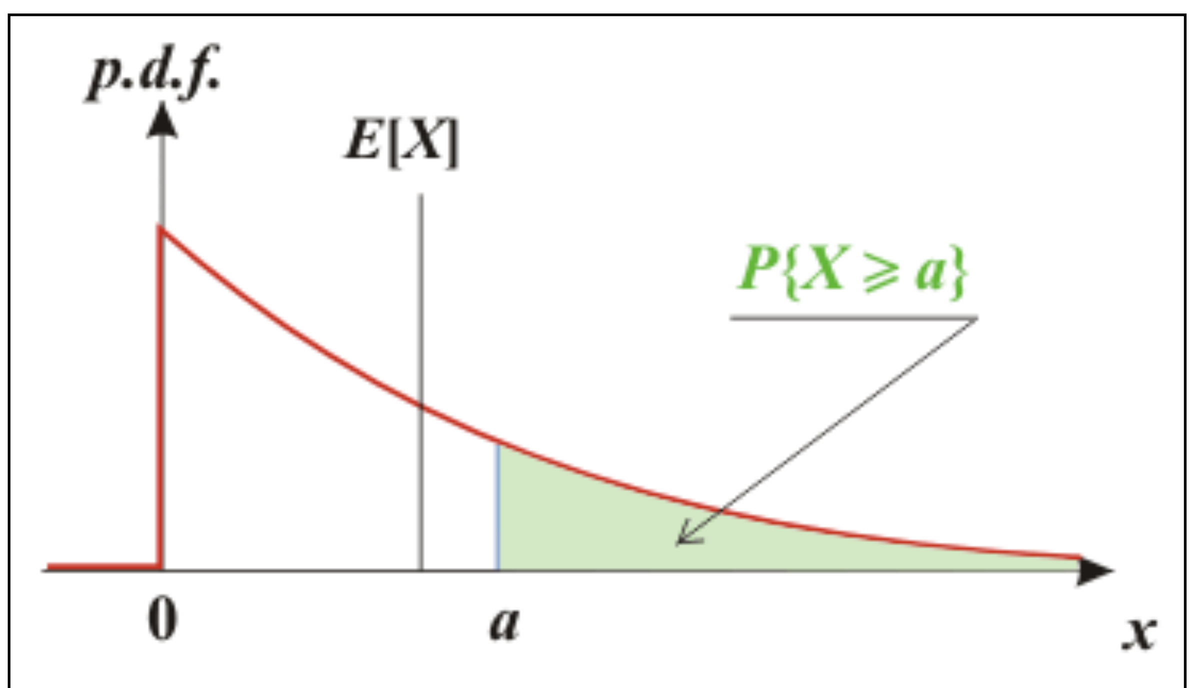
  (Cauchy-Schwarz) $\geq \dfrac{n^2}{2m+n}$

Markov's Inequality

Markov's Inequality:

For *nonnegative* X, for any $t > 0$,

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$



Markov's Inequality

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For *nonnegative* X, for any $t > 0$,

$$\Pr[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

Proof:

Let $Y = \begin{cases} 1 & \text{if } X \geq t, \\ 0 & \text{otherwise.} \end{cases} \Rightarrow Y \leq \left\lfloor \frac{X}{t} \right\rfloor \leq \frac{X}{t},$

$$\Pr[X \geq t] = \mathbf{E}[Y] \leq \mathbf{E}\left[\frac{X}{t}\right] = \frac{\mathbf{E}[X]}{t}.$$

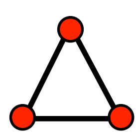
QED

Graph $G(V, E)$

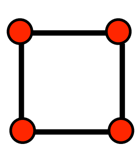
girth $g(G)$: length of the shortest cycle

chromatic number $\chi(G)$:

minimum number of color to properly color the vertices of G.



$g(G) = 3 \quad \chi(G)=3$



$g(G) = 4 \quad \chi(G)=2$

Intuition: Large cycles are easy to color!

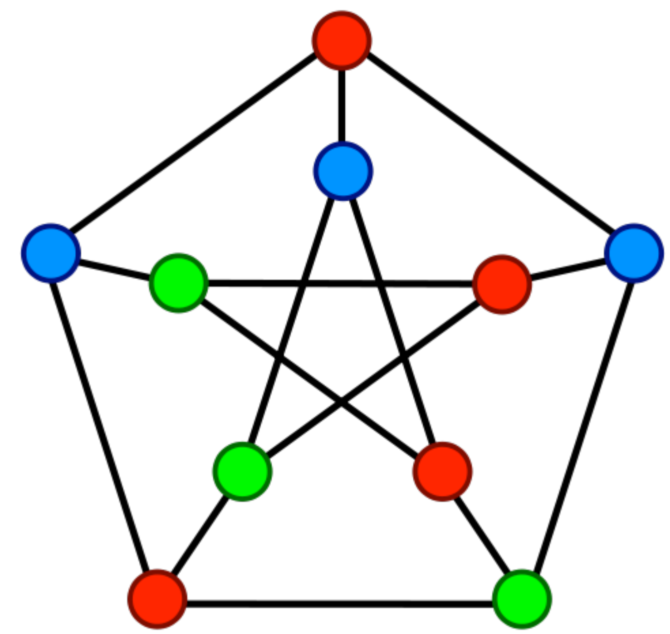
Theorem (Erdős 1959)

For all k, ℓ, there exists a finite graph G with $\chi(G) \geq k$ and $g(G) \geq \ell$.

coloring classes:

equivalence classes of vertices

"Independent sets!"



independence number $\alpha(G)$:

size of the largest independent set in G.

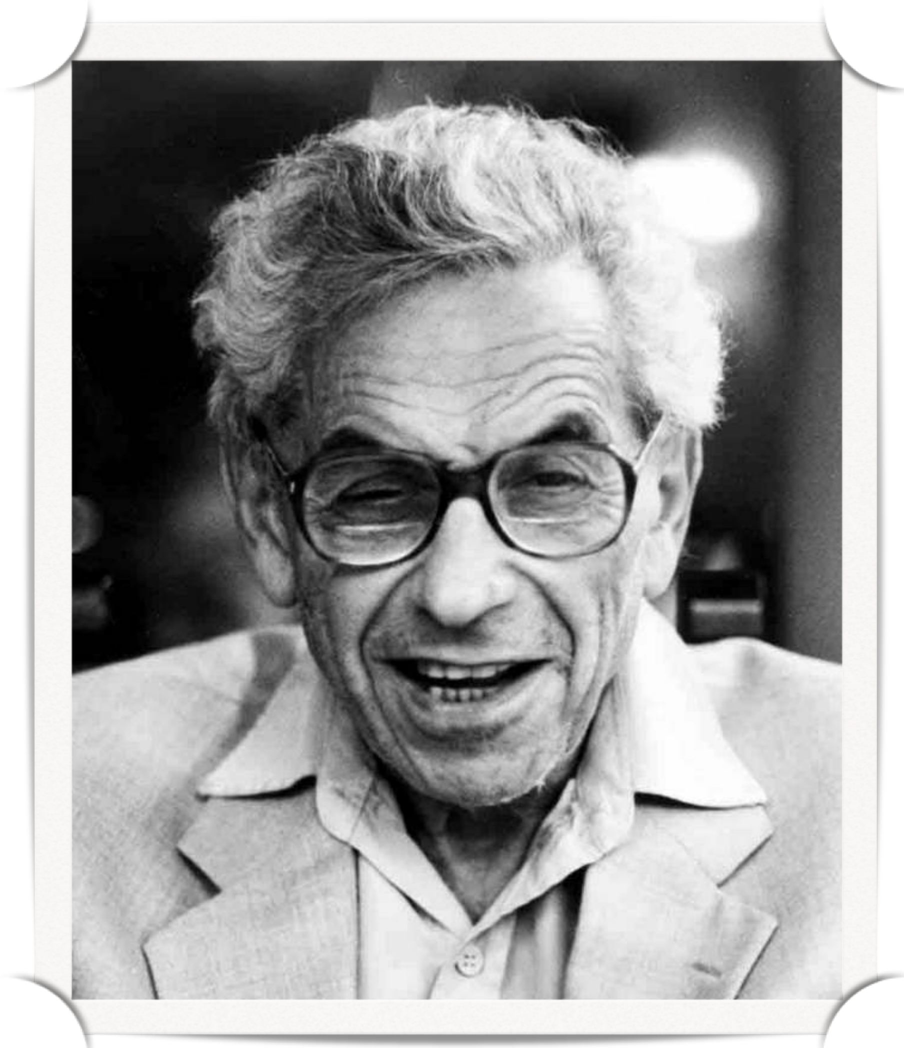
n vertices $\quad \chi(G) \geq \dfrac{n}{\alpha(G)} \geq k$, with $\alpha(G) \leq \dfrac{n}{k}$

For all k, ℓ, there exists a graph G on n vertices with $\alpha(G) \leq \frac{n}{k}$ and $g(G) \geq \ell$.

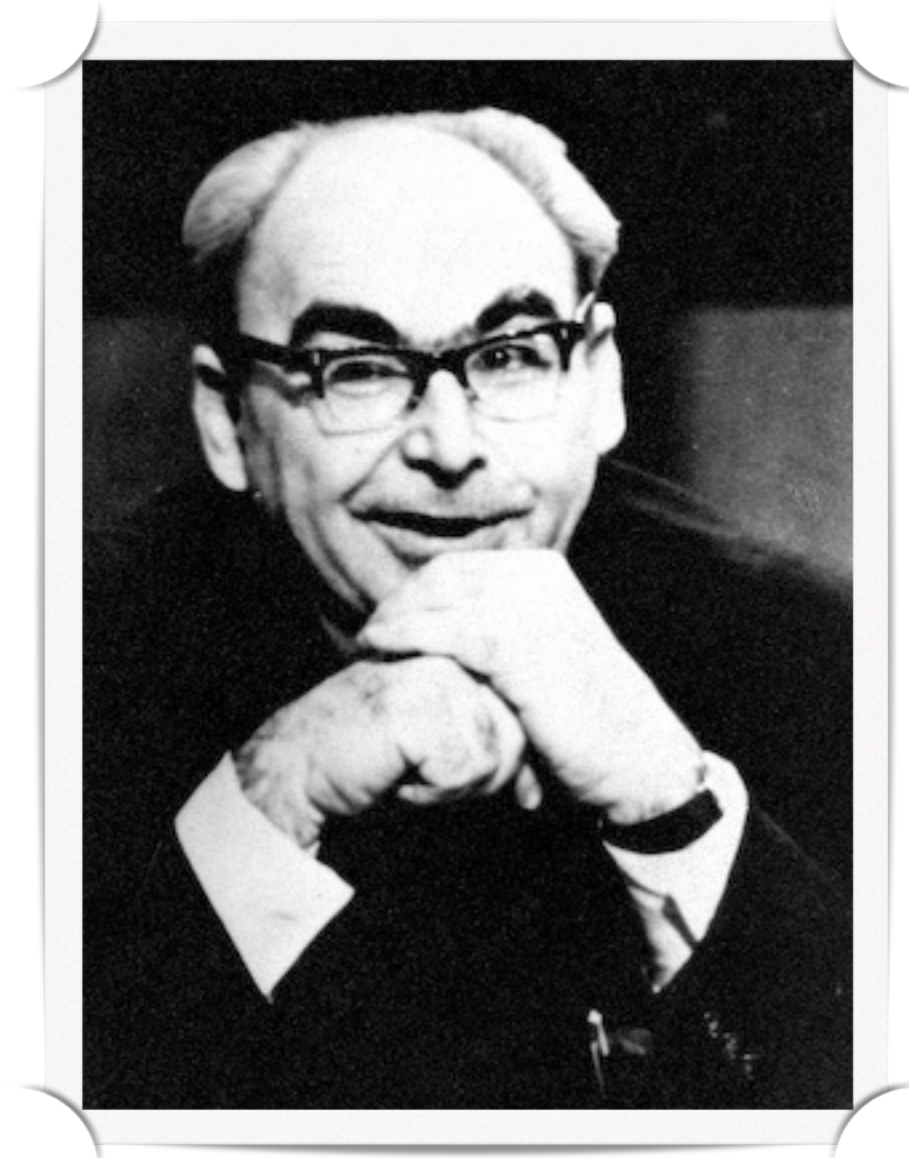
$$|V| = n \qquad \forall \{u, v\} \in \binom{V}{2}$$

independently $\Pr[\{u, v\} \in E] = p$

Random Graphs



Paul Erdős
(1913 - 1996)



Alfréd Rényi
(1921 - 1970)

Erdős-Rényi 1960 paper:

ON THE EVOLUTION OF RANDOM GRAPHS

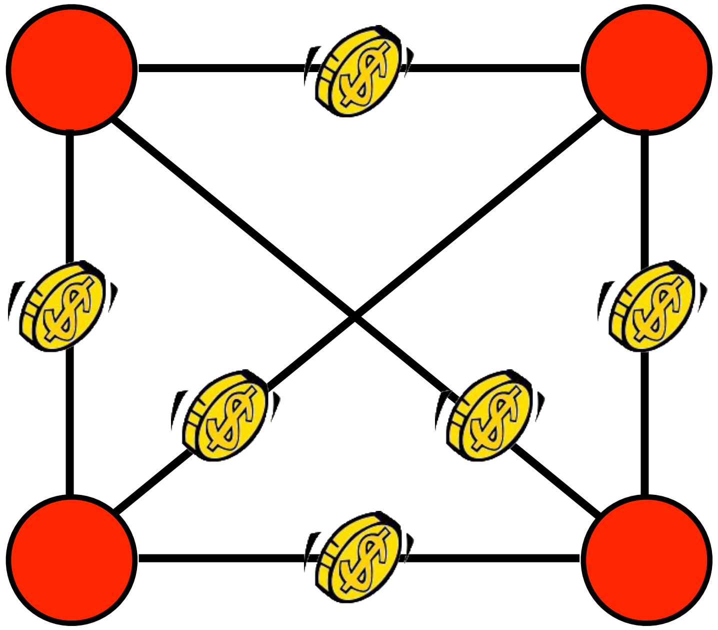
by

P. ERDÖS and A. RÉNYI
Institute of Mathematics
Hungarian Academy of Sciences, Hungary

1. Definition of a random graph

Let $E_{n,N}$ denote the set of all graphs having n given labelled vertices $V_1, V_2, \cdots, V_n$ and N edges. The graphs considered are supposed to be not oriented, without parallel edges and without slings (such graphs are sometimes called linear graphs). Thus a graph belonging to the set $E_{n,N}$ is obtained by choosing N out of the possible $\binom{n}{2}$ edges between the points $V_1, V_2, \cdots, V_n$, and therefore the number of elements of $E_{n,N}$ is equal to $\binom{\binom{n}{2}}{N}$. A random graph $\Gamma_{n,N}$ can be defined as an element of $E_{n,N}$ chosen at random, so that each of the elements of $E_{n,N}$ have the same probability to be chosen, namely $1\Big/\binom{\binom{n}{2}}{N}$. There is however an other slightly different point of view, which has some advantages. *We may consider the formation of a random graph as a stochastic process* defined as follows: At time $t=1$ we choose one out of the $\binom{n}{2}$ possible edges connecting the points $V_1, V_2, \cdots, V_n$,

$G(n,p)$



$|V| = n \qquad \forall u, v \in V$

independently $\Pr[\,\{u,v\} \in E\,] = p$

uniform random graph: $G(n, \frac{1}{2})$

For all k, ℓ, there exists a graph G on n vertices with $\alpha(G) \leq \frac{n}{k}$ and $g(G) \geq \ell$.

fix any large k, l exists n

$G \sim G(n,p)$

Plan:

$\Pr[\ \alpha(G) > n/k\ ] < 1/2$

$\Pr[\ g(G) < l\ ] < 1/2$

} union bound ⇒

$\Pr[\ \alpha(G) > n/k \vee g(G) < l\ ] < 1$

$\Pr[\ \alpha(G) \leq n/k \wedge g(G) \geq l\ ] > 0$

$G \sim G(n,p)$

$$\Pr[\alpha(G) \geq n/k] \quad \leq \Pr[\exists \text{ind. set of size } n/k]$$

$$\leq \Pr\left[\exists S \in \binom{[n]}{n/k} \forall \{u,v\} \in \binom{S}{2}, uv \notin G\right]$$

$$\leq \sum_{S \in \binom{[n]}{n/k}} \Pr\left[\forall \{u,v\} \in \binom{S}{2}, uv \notin G\right]$$

union bound

$$= \sum_{S \in \binom{[n]}{n/k}} \prod_{\{u,v\} \in \binom{S}{2}} \Pr[uv \notin G] \quad = \binom{n}{n/k}(1-p)^{\binom{n/k}{2}}$$

$$\leq n^{n/k}(1-p)^{\binom{n/k}{2}}$$

$G \sim G(n,p)$ $\Pr[\alpha(G) \geq n/k] \leq n^{n/k}(1-p)^{\binom{n/k}{2}}$

$\Pr[\ g(G) > l\ ] < ?$

for each *i*-cycle $\sigma : u_1 \to u_2 \to \ldots \to u_i \to u_1$

$\Pr[\sigma \text{ is a cycle in } G] = p^i$

$$X_\sigma = \begin{cases} 1 & \sigma \text{ is a cycle in } G \\ 0 & \text{otherwise} \end{cases}$$

# of length≤*l* cycles in *G* $X = \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} X_\sigma$

$$\mathbb{E}[X] = \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} \mathbb{E}[X_\sigma] = \sum_{i=3}^{\ell} \sum_{\sigma:|\sigma|=i} p^i$$

$$= \sum_{i=3}^{\ell} \frac{n(n-1)\cdots(n-i+1)}{2i} p^i \leq \sum_{i=3}^{\ell} \frac{n^i}{2i} p^i$$

$G \sim G(n,p)$ $\quad k = \frac{np}{3\ln n} \quad n/k = \frac{3\ln n}{p}$

$$\begin{aligned}\Pr[\alpha(G) \geq n/k] &\leq n^{n/k}(1-p)^{\binom{n/k}{2}} \\ &\leq n^{n/k}\mathrm{e}^{-p\binom{n/k}{2}} \\ &= (n\mathrm{e}^{-p(n/k-1)/2})^{n/k} = o(1)\end{aligned}$$

$X:$ # of length≤l cycles in G

$$\mathbb{E}[X] \leq \sum_{i=3}^{\ell}\frac{n^i}{2i}p^i = \sum_{i=3}^{\ell}\frac{n^{\theta i}}{2i} = o(n)$$

$$p = n^{\theta-1} \qquad \theta < \frac{1}{2\ell}$$

$$\Pr[X \geq \frac{n}{2}] \leq \frac{2\mathbb{E}[X]}{n} = o(1)$$

Markov

$G \sim G(n,p)$

$$p = n^{\theta-1} \qquad \theta < \frac{1}{2\ell} \qquad k = \frac{np}{3\ln n} = \frac{n^{1/2\ell}}{3\ln n}$$

$$\Pr[\alpha(G) \geq n/k] = o(1)$$

X : # of length≤l cycles in G

$$\Pr[X \geq \frac{n}{2}] = o(1)$$

∃ G: $\alpha(G) < n/k$

# of length≤l cycles in $G < n/2$

delete 1 vertex per each length≤l cycle in G ⇨ G'

$g(G') > l \qquad \alpha(G') \leq \alpha(G) < n/k$

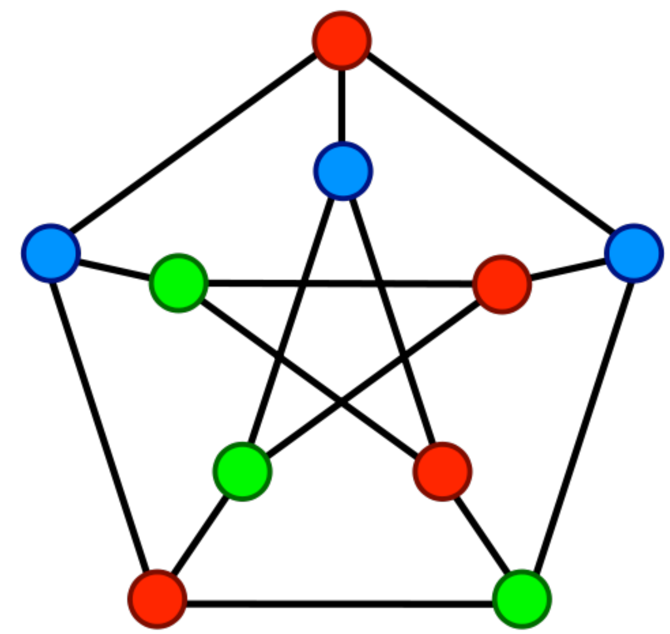
Theorem (Erdős 1959)

For all k, ℓ, there exists a finite graph G with $\chi(G) \geq k$ and $g(G) \geq \ell$.

coloring classes:

equivalence classes of vertices

"Independent sets!"



independence number $\alpha(G)$:

size of the largest independent set in G.

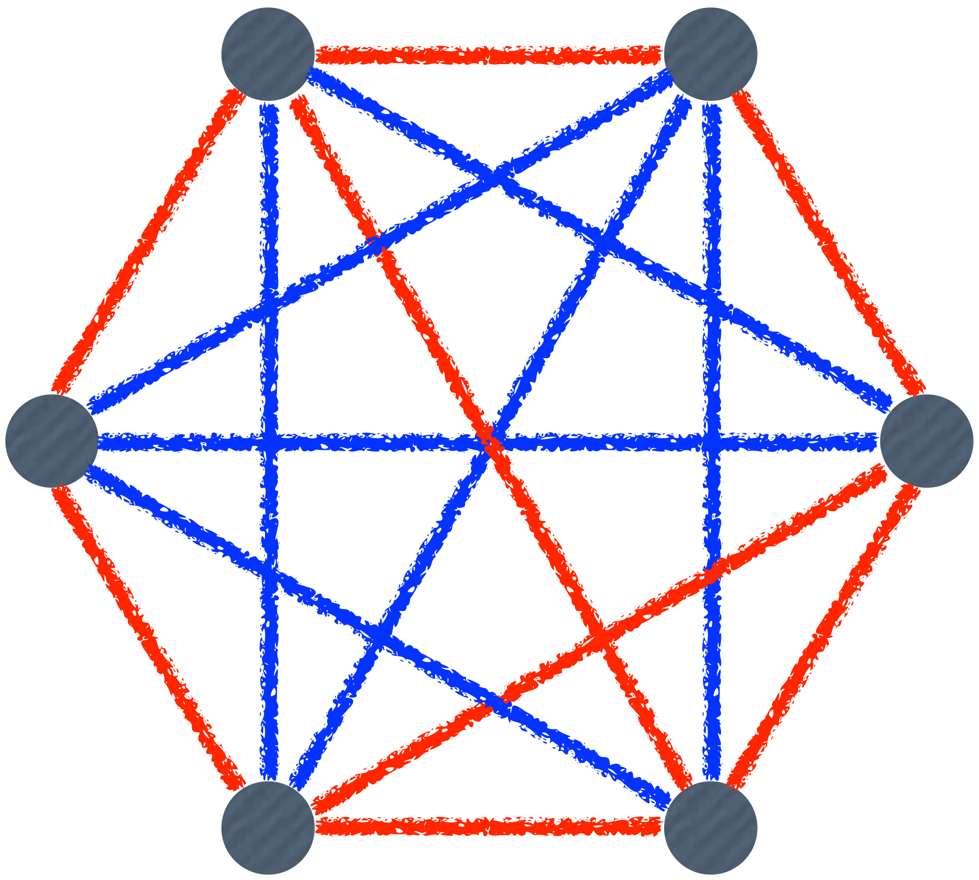
n vertices $\quad \chi(G) \geq \dfrac{n}{\alpha(G)} \geq k$, with $\alpha(G) \leq \dfrac{n}{k}$

Lovász Local Lemma

Ramsey Number

"In any party of six people, either at least three of them are mutual strangers or at least three of them are mutual acquaintances"

- For any edge-2-coloring of K_6, there is a ***monochromatic*** K_3.



Ramsey Theorem

If $n \geq R(k,k)$, for any edge-2-coloring of K_n, there is a monochromatic K_k.

Ramsey number: $R(k,k)$

$$R(k,k) > ?$$

"∃ a 2-coloring of K_n with no monochromatic K_k."

The Probabilistic Method:

a random 2-coloring of K_n

$\forall S \in \binom{[n]}{k}$

event A_S: S is a monochromatic K_k

To prove:

$$\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$$

Dependency!

Lovász Sieve

- Bad events: $A_1, A_2, \ldots, A_n$
- None of the bad events occurs:

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right]$$

- The probabilistic method: being good is possible

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

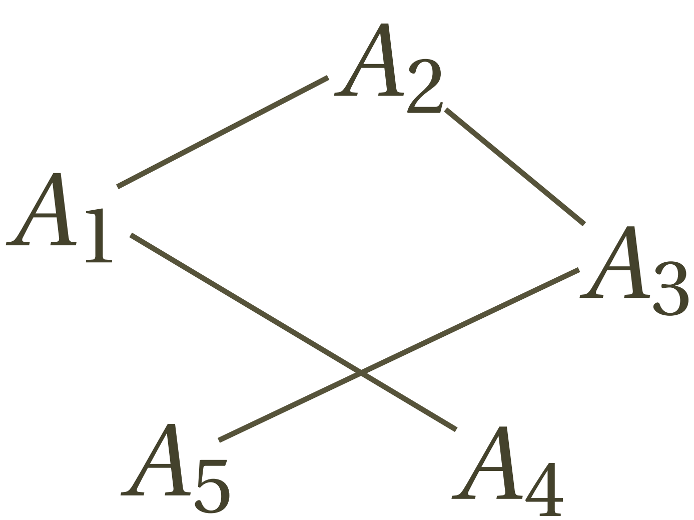
events: $A_1, A_2, \ldots, A_n$

dependency graph: $D(V, E)$

$$V = \{1, 2, \ldots, n\}$$

$ij \in E$ ⟺ A_i and A_j are dependent

d : max degree of dependency graph



$A_1(X_1, X_4)$ $A_4(X_4)$

$A_2(X_1, X_2)$ $A_5(X_3)$

$A_3(X_2, X_3)$

$X_1, \ldots, X_4$ mutually independent

events: $A_1, A_2, \ldots, A_n$

d : max degree of dependency graph

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \le p$
- $ep(d+1) \le 1$

$\Rightarrow$ $$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow$ $$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$$

$R(k,k) \geq n$

"∃ a 2-coloring of K_n with no monochromatic K_k."

a random 2-coloring of K_n:

$\forall \{u,v\} \in K_n$, *uniformly* and *independently* $\begin{cases} uv \\ uv \end{cases}$

$\forall S \in \binom{[n]}{k}$ event A_S: S is a monochromatic K_k

$$\Pr[A_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

A_S, A_T dependent ⟺ $|S \cap T| \geq 2$

max degree of dependency graph $d \leq \binom{k}{2}\binom{n}{k-2}$

To prove: $\Pr\left[\bigwedge_{S \in \binom{[n]}{k}} \overline{A_S}\right] > 0$

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \leq p$
- $\mathrm{e}p(d+1) \leq 1$

$\Rightarrow$ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

$$\Pr[A_S] = 2^{1-\binom{k}{2}}$$

$$d \leq \binom{k}{2}\binom{n}{k-2}$$

$\Rightarrow$ for some $n = ck2^{k/2}$ with constant c

$$\mathrm{e}2^{1-\binom{k}{2}}(d+1) \leq 1$$

To prove: $\Pr\left[\bigwedge_{S\in\binom{[n]}{k}} \overline{A_S}\right] > 0$

$$R(k,k) \geq n = \Omega(k2^{k/2})$$

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n} \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right] = \prod_{i=1}^{n} \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right]\right)$$

Lemma For any $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$,

$$\Pr\left[\bigwedge_{i=1}^{n} \mathcal{E}_i\right] = \prod_{k=1}^{n} \Pr\left[\mathcal{E}_k \mid \bigwedge_{i<k} \mathcal{E}_i\right].$$

proof:

$$\Pr\left[\mathcal{E}_n \,\middle|\, \bigwedge_{i=1}^{n-1} \mathcal{E}_i\right] = \frac{\Pr\left[\bigwedge_{i=1}^{n} \mathcal{E}_i\right]}{\Pr\left[\bigwedge_{i=1}^{n-1} \mathcal{E}_i\right]}$$

recursion!

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

I.H.

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1} \text{ for any } \{i_1, \ldots, i_m\}$$

induction on m:

$m = 1$, trivial

events: $A_1, A_2, \ldots, A_n$

$$\exists x_1, \ldots, x_n \in [0,1)$$
$$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$$

I.H. $\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1}$ for any $\{i_1, \ldots, i_m\}$

suppose i_1 adjacent to $i_2, \ldots, i_k$

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] = \frac{\Pr\left[A_{i_1} \overline{A_{i_2}} \cdots \overline{A_{i_k}} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right]}{\Pr\left[\overline{A_{i_2}} \cdots \overline{A_{i_k}} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right]}$$

$$\le \Pr\left[A_{i_1} \mid \overline{A_{i_{k+1}}} \cdots \overline{A_{i_m}}\right] = \Pr\left[A_{i_1}\right] \le x_{i_1} \prod_{j=2}^{k} (1 - x_{i_j})$$

$$= \prod_{j=2}^{k} \Pr\left[\overline{A_{i_j}} \mid \overline{A_{i_{j+1}}} \cdots \overline{A_{i_m}}\right] = \prod_{j=2}^{k} \left(1 - \Pr\left[A_{i_j} \mid \overline{A_{i_{j+1}}} \cdots \overline{A_{i_m}}\right]\right)$$

I.H. $\ge \prod_{j=2}^{k} (1 - x_{i_j})$

events: $A_1, A_2, \ldots, A_n$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i} (1 - x_j)$

$\Rightarrow \Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n} (1 - x_i)$

$$\Pr\left[A_{i_1} \mid \overline{A_{i_2}} \cdots \overline{A_{i_m}}\right] \le x_{i_1} \quad \text{for any} \quad \{i_1, \ldots, i_m\}$$

$$\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] = \prod_{i=1}^{n} \Pr\left[\overline{A_i} \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right] = \prod_{i=1}^{n} \left(1 - \Pr\left[A_i \,\middle|\, \bigwedge_{j=1}^{i-1} \overline{A_j}\right]\right)$$

$$\ge \prod_{i=1}^{n} (1 - x_i) > 0$$

events: $A_1, A_2, \ldots, A_n$

d : max degree of dependency graph

Lovász Local Lemma

- $\forall i,\ \Pr[A_i] \le p$
- $\mathrm{e}p(d+1) \le 1$

$\Rightarrow$ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$

General Lovász Local Lemma

$\exists x_1, \ldots, x_n \in [0,1)$

$\forall i, \Pr[A_i] \le x_i \prod_{j \sim i}(1 - x_j)$

$\Rightarrow$ $\Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] \ge \prod_{i=1}^{n}(1 - x_i)$

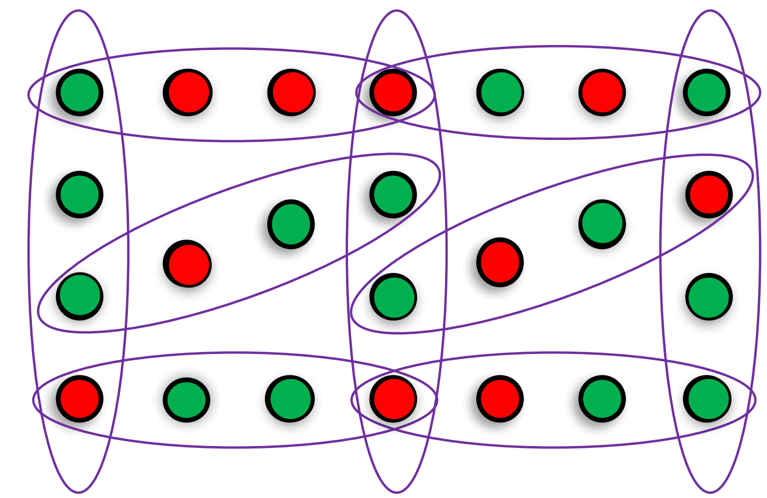
Constraint Satisfaction Problem (CSP)

- **Variables:** $x_1, \ldots, x_n \in [q]$
- (local) **Constraints:** $C_1, \ldots, C_m$
  - each C_i is defined on a subset $\mathsf{vbl}(C_i)$ of variables

$$C_i : [q]^{\mathsf{vbl}(C_i)} \to \{\mathtt{True}, \mathtt{False}\}$$

- Any $\boldsymbol{x} \in [q]^n$ is a **CSP solution** if it satisfies all $C_1, \ldots, C_m$
- **Examples:**
  - k-CNF, (hyper)graph coloring, set cover, unique games…
  - vertex cover, independent set, matching, perfect matching, …

Hypergraph Coloring



- k-uniform hypergraph $H = (V, E)$:
  - V is vertex set, $E \subseteq \binom{V}{k}$ is set of hyperedges
- degree of vertex $v \in V$: # of hyperedges $e \ni v$
- proper q-coloring of H:
  - $f : V \to [q]$ such that no hyperedge is *monochromatic*

$$\forall e \in E, \quad |f(e)| > 1$$

Theorem: For any k-uniform hypergraph H of max-degree Δ,

$$\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

$k \geq \log_q \Delta + \log_q \log_q \Delta + O(1)$

Hypergraph Coloring

Theorem: For any k-uniform hypergraph H of max-degree Δ,

$$\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies H \text{ is } q\text{-colorable}$$

- Uniformly and independently color each $v \in V$ a random color $\in [q]$
- Bad event A_e for each hyperedge $e \in E \subseteq \binom{V}{k}$: e is monochromatic
  - $\Pr[A_e] \leq p = q^{1-k}$
- Dependency degree for bad events $d \leq k(\Delta - 1)$
  - $\Delta \leq \frac{q^{k-1}}{\mathrm{e}k} \implies \mathrm{e}p(d+1) \leq 1$ Apply LLL