Probability Theory & Mathematical Statistics

Limit Theorems

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Limit Theorems

Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mu = \mathbb{E}[X_1]$ and $\mathbf{Var}[X_1] = \sigma^2$.

And let $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ be the <u>sample mean</u>.

- <u>Law of large numbers</u> (**LLN**): sample mean → expectation

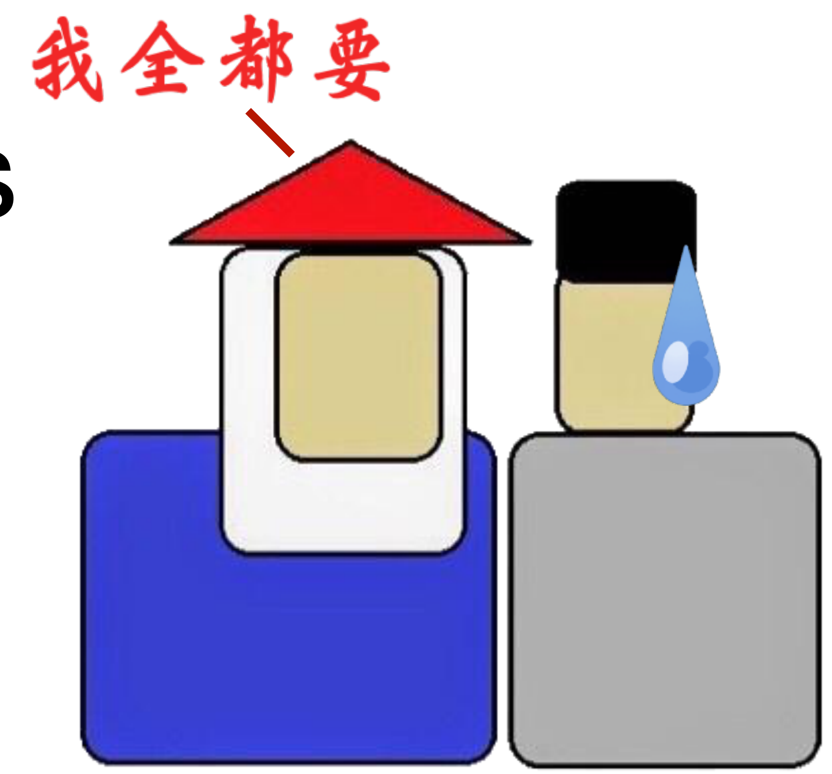
$$\overline{X}_n \longrightarrow \mu \quad \text{as } n \to \infty$$

- <u>Central limit theorem</u> (**CLT**): standardized sample mean → standard normal

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \longrightarrow N(0,1) \quad \text{as } n \to \infty$$

Convergence

- A real sequence $\{a_n\}$ converges to $a \in \mathbb{R}$, denoted $\lim_{n\to\infty} a_n = a$ or $a_n \to a$, if for all $\epsilon > 0$, there is N such that $|a_n - a| < \epsilon$ for all $n > N$
- A sequence $f_1, f_2, \ldots : \Omega \to \mathbb{R}$ is said to <u>**converge pointwise**</u> to $f : \Omega \to \mathbb{R}$, if and only if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$
- For random variables $X_1, X_2, \ldots$ and X on probability space $(\Omega, \Sigma, \Pr)$:
  - random variables $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ and $X : \Omega \to \mathbb{R}$ are functions
  - CDFs $F_{X_1}, F_{X_2}, \ldots : \mathbb{R} \to [0,1]$ and $F_X : \mathbb{R} \to [0,1]$ are functions
- Should $X_n \to X$ be: $X_n \to X$ pointwise or $F_{X_n} \to F_X$ pointwise?



Convergence of Random Variables

0.  $\cdots \to U_{[0,1]}$

Modes of Convergence

- Let $X, X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be random variables on prob. space $(\Omega, \Sigma, \Pr)$.

- $\{X_n\}$ converges in distribution (依分布收敛) to X, denoted $X_n \xrightarrow{D} X$, if

$$F_{X_n}(x) = \Pr(X_n \leq x) \to F_X(x) = \Pr(X \leq x) \quad \text{as} \quad n \to \infty$$

  for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- $\{X_n\}$ converges in probability (依概率收敛) to X, denoted $X_n \xrightarrow{P} X$, if

$$\Pr(|X_n - X| > \epsilon) = 0 \quad \text{as} \quad n \to \infty \quad \text{for all } \epsilon > 0$$

- $\{X_n\}$ converges almost surely to X, denoted $X_n \xrightarrow{a.s.} X$, if $\exists A \in \Sigma$ such that

$$\lim_{n \to \infty} X_n(\omega) = X(\omega) \quad \text{for all } \omega \in A, \quad \text{and} \quad \Pr(A) = 1$$

Modes of Convergence

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.

- $X_n \xrightarrow{D} X$ (convergence in distribution / in law / weak convergence of measure) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$F_{X_n} \to F_X$ pointwise on continuous set

for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- $X_n \xrightarrow{P} X$ (convergence in probability / in measure) if

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

$X_n \to X$ in measure

- $X_n \xrightarrow{a.s.} X$ (convergence almost surely / almost everywhere / w.p. 1) if

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$X_n \to X$ pointwise on a set of measure 1

Convergence in Distribution

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{D} X$ (convergence in distribution / in law / weak convergence of measure) if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$F_{X_n} \to F_X$ pointwise on continuous set

for all $x \in \mathbb{R}$ at which $F_X(x)$ is continuous

- The restriction on continuity set is necessary, consider:

uniform X_n on $(0, 1/n)$, which satisfies $X_n \xrightarrow{D} X$, where $\Pr(X = 0) = 1$

- $X_n \xrightarrow{D} X$ and $F_X = F_Y \implies X_n \xrightarrow{D} Y$ (convergence in distribution depends only on distribution)
- $X_n \xrightarrow{D} X$ is a weak convergence of measures

Convergence in Probability

- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{P} X$ (convergence <u>in probability</u>) if

$$\lim_{n\to\infty} \Pr(|X_n - X| > \epsilon) = 0 \quad \text{for all } \epsilon > 0$$

$X_n \to X$ in measure

- Functions $X_n : \Omega \to \mathbb{R}$ converges to $X : \Omega \to \mathbb{R}$ <u>in measure</u> $\Pr$
- $X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$
  - **Counterexample for converse**: X is uniform on $[0,1]$ and $X_n = 1 - X$
- If $X_n \xrightarrow{D} c$, where $c \in \mathbb{R}$ is a constant, then $X_n \xrightarrow{P} c$
  - **Proof**: $\Pr(|X_n - c| > \epsilon) = \Pr(X_n < c - \epsilon) + \Pr(X_n > \epsilon + c) \to 0$ if $X_n \xrightarrow{D} c$

Almost Sure Convergence

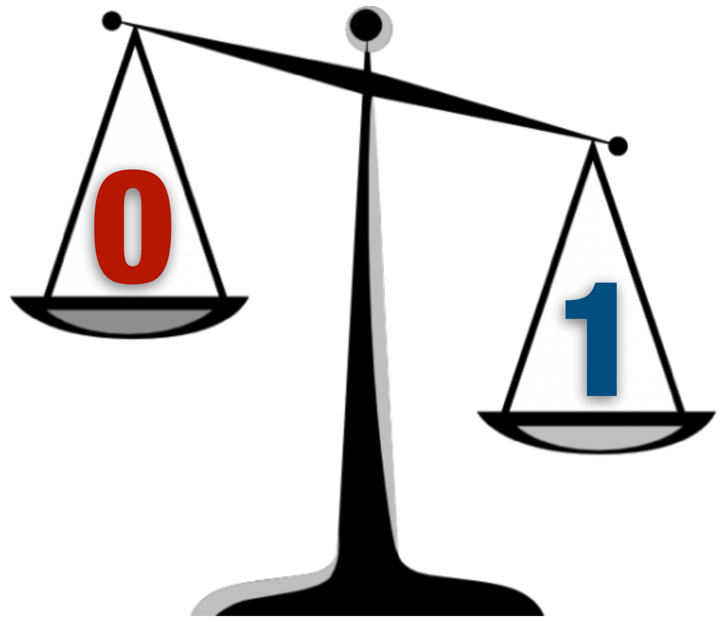
- Let $X_1, X_2, \ldots$ and X be random variables on probability space $(\Omega, \Sigma, \Pr)$.
- $X_n \xrightarrow{a.s.} X$ (convergence <u>almost surely</u> / <u>almost everywhere</u> / **w.p. 1**) if

$$\Pr\left(\lim_{n\to\infty} X_n = X\right) = 1$$

$X_n \to X$ pointwise on a set of measure 1

- $X_n : \Omega \to \mathbb{R}$ converges to $X : \Omega \to \mathbb{R}$ **almost everywhere** except a null set
- The event $\lim_{n\to\infty} X_n = X$ is: $\bigcap_{m=1}^{\infty} \bigcup_{n_0=1}^{\infty} \bigcap_{n=n_0}^{\infty} \{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| \leq 1/m\}$
- $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$
    - **Counterexample for converse**: $\{X_n\}$ are independent $\text{Bernoulli}(1/n)$. We have $X_n \xrightarrow{P} 0$, but we do not have $X_n = 0$ almost everywhere as $n \to \infty$.

Borel–Cantelli Lemmas*



(博雷尔–坎特利引理 / 波莱尔–坎泰利引理 / zero-one law)

- Let $A_1, A_2, \ldots$ be a sequence of events from a probability space $(\Omega, \Sigma, \Pr)$.
  Let A be the event that infinitely many of the A_n occurs:
  $$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$
  denoted A_n ***infinitely often***, or A_n ***i.o.***
- (1st lemma) $\sum_{n=1}^{\infty} \Pr(A_n) < \infty \implies \Pr(A) = 0$
- (2nd lemma) $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent $\implies \Pr(A) = 1$

Continuity of Probability Measures*

- Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an increasing sequence of events, and write A for their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i \,.$$

Then $\Pr(A) = \lim_{i \to \infty} \Pr(A_i)$.

- **Proof**: Express A as a disjoint union $A = A_1 \uplus (A_2 \backslash A_1) \uplus (A_3 \backslash A_2) \uplus \cdots$. Then

$$\begin{aligned}\Pr(A) &= \Pr(A_1) + \sum_{i=1}^{\infty} \Pr(A_{i+1} \backslash A_i) \\ &= \Pr(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n-1} [\Pr(A_{i+1}) - \Pr(A_i)] \\ &= \lim_{n \to \infty} \Pr(A_n)\end{aligned}$$

Continuity of Probability Measures*

- Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an increasing sequence of events, and write A for their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i \,.$$

  Then $\Pr(A) = \lim_{i \to \infty} \Pr(A_i)$.

- Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ be an decreasing sequence of events, and write B for their limit

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i \,.$$

  Then $\Pr(B) = \lim_{i \to \infty} \Pr(B_i)$.

- **Proof**: Consider the complements $B_1^c \subseteq B_2^c \subseteq B_3^c \subseteq \ldots$ which is an increasing sequence.

Borel–Cantelli Lemmas*

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

(1st lemma) $\sum_{n=1}^{\infty} \Pr(A_n) < \infty \implies \Pr(A) = 0$

Proof: By union bound, $\Pr\left(\bigcup_{m=n}^{\infty} A_m\right) \leq \sum_{m=n}^{\infty} \Pr(A_m)$, which $\rightarrow 0$ as $n \rightarrow \infty$,

assuming that $\sum_{n=1}^{\infty} \Pr(A_n) < \infty$ converges.

And by **continuity** of $\Pr$, we have $\Pr(A) = \lim_{n\to\infty} \Pr\left(\bigcup_{m=n}^{\infty} A_m\right) = 0$

Borel–Cantelli Lemmas*

$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

(2nd lemma) $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent $\implies \Pr(A) = 1$

Proof: By independence, $\Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = \prod_{m=n}^{\infty} \big(1 - \Pr(A_m)\big) \leq \exp\left(-\sum_{m=n}^{\infty} \Pr(A_m)\right) = 0,$

assuming the divergence of $\sum_{n=1}^{\infty} \Pr(A_n) = \infty$.

By **continuity** of $\Pr$, $\Pr(A^c) = \Pr\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c\right) = \lim_{n \to \infty} \Pr\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0 \implies \Pr(A) = 1$

Strength of Convergence

- $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{P} X) \implies (X_n \xrightarrow{D} X)$

Proof* $(X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X)$: Let $A_n(\epsilon) = \{\,|X_n - X| > \epsilon\}$. Then for any $\epsilon > 0$

$$\lim_{n\to\infty} X_n = X \implies \bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m^c(\epsilon)$$

Assume $X_n \xrightarrow{a.s.} X$. Then $1 = \Pr\left(\lim_{n\to\infty} X_n = X\right) = \Pr\left(\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m^c(\epsilon)\right)$

$$\implies 0 = \Pr\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m(\epsilon)\right) = \lim_{n\to\infty}\Pr\left(\bigcup_{m=n}^{\infty} A_m(\epsilon)\right)$$ (by continuity of probability measure)

$$\implies \Pr(|X_n - X| > \epsilon) = \Pr(A_n(\epsilon)) \le \Pr\left(\bigcup_{m=n}^{\infty} A_m(\epsilon)\right) \to 0 \text{ as } n \to \infty$$

$$\implies X_n \xrightarrow{P} X$$

Strength of Convergence

- $(X_n \xrightarrow{a.s.} X) \implies (X_n \xrightarrow{P} X) \implies (X_n \xrightarrow{D} X)$

Proof* $(X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X)$: Fix any $\epsilon > 0$. It holds that

$$\{X_n \le x\} \subseteq \{X \le x + \epsilon\} \cup \{|X_n - X| > \epsilon\} \implies F_{X_n}(x) \le F_X(x+\epsilon) + \Pr(|X_n - X| > \epsilon)$$

$$\{X \le x - \epsilon\} \subseteq \{X_n \le x\} \cup \{|X_n - X| > \epsilon\} \implies F_X(x-\epsilon) \le F_{X_n}(x) + \Pr(|X_n - X| > \epsilon)$$

$$\implies F_X(x-\epsilon) - \Pr(|X_n - X| > \epsilon) \le F_{X_n}(x) \le F_X(x+\epsilon) + \Pr(|X_n - X| > \epsilon)$$

Assume $X_n \xrightarrow{P} X$. Then $\Pr(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$ for all $\epsilon > 0$. Therefore,

$$F_X(x-\epsilon) \le \liminf_{n\to\infty} F_{X_n}(x) \le \limsup_{n\to\infty} F_{X_n}(x) \le F_X(x+\epsilon) \quad \text{for all } \epsilon > 0$$

Furthermore, if F_X is continuous at x, then

$$F_X(x-\epsilon) \uparrow F_X(x) \text{ and } F_X(x+\epsilon) \downarrow F_X(x) \quad \text{as } \epsilon \downarrow 0.$$

Condition for Almost Sure Convergence*

- If $\sum_{n=1}^{\infty} \Pr(|X_n - X| > \epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$

Proof: For any $\epsilon > 0$, let $A_n(\epsilon) = \{|X_n - X| > \epsilon\}$. Then due to **Borel–Cantelli**: $\forall \epsilon > 0$

$$\sum_{n=1}^{\infty} A_n(\epsilon) < \infty \implies \Pr\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m(\epsilon)\right) = \Pr\left(A_n(\epsilon) \text{ infinitely often}\right) = 0$$

$$\implies \Pr\left(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m(1/k)\right) = 0 \text{ by countable additivity}$$

$$\implies \Pr\left(\lim_{n\to\infty} X_n = X\right) = \Pr\left(\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty} A_m^c(1/k)\right) = 1$$

$$\implies X_n \xrightarrow{a.s.} X$$

Almost Sure vs. In Probability Convergence*

- Let $\{X_n\}$ be independent Bernoulli trials with parameter $1/n$. Then

$$X_n \xrightarrow{P} 0, \text{ but it does not hold } X_n \xrightarrow{a.s.} 0$$

Proof: For any $\epsilon > 0$, $\Pr(|X_n| > \epsilon) \leq \Pr(X_n = 1) = \frac{1}{n} \to 0$ as $n \to \infty$ $\implies X_n \xrightarrow{P} 0$

$\{X_n\}$ are independent and $\sum_{n=1}^{\infty} \Pr(X_n = 1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$, then by **Borel–Cantelli:**

$\Pr(X_n = 1 \text{ infinitely often}) = 1 \implies \Pr\left(\lim_{n\to\infty} X_n = 0\right) = 0 \implies X_n \xrightarrow{a.s.} 0$ does not hold

Coupling*

- <u>Skorokhod's representation theorem</u>:

  If $X_n \xrightarrow{D} X$, then there exist random variables $Y_1, Y_2, \ldots$ and Y on some $(\Omega', \mathscr{F}, \mathbb{P})$, satisfying $F_{X_n} = F_{Y_n}$ for all $n \geq 1$ and $F_X = F_Y$, such that $Y_n \xrightarrow{a.s.} Y$

Proof: Apply inverse transform sampling. Let $\Omega' = [0,1]$, $\mathscr{F}$ the Borel σ-field on $[0,1]$, and $\mathbb{P}$ the uniform law. For $u \in \Omega' = [0,1]$, let

$$Y_n(u) = \inf\{x \mid u \leq F_{X_n}(x)\} \text{ and } Y(u) = \inf\{x \mid u \leq F_X(x)\}$$

Due to inverse transform sampling, $F_{X_n} = F_{Y_n}$ for all $n \geq 1$ and $F_X = F_Y$.

It can also be verified that $Y_n(u) \to Y(u)$ for all points u of continuity of Y, meanwhile the set $D \subseteq [0,1]$ of discontinuities of Y is countable , thus $\mathbb{P}(D) = 0$, which implies

$$Y_n \xrightarrow{a.s.} Y$$

Continuous Mapping Theorem*

- Continuous mapping theorem: If $g : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$X_n \xrightarrow{D} X \implies g(X_n) \xrightarrow{D} g(X)$$

$$X_n \xrightarrow{P} X \implies g(X_n) \xrightarrow{P} g(X)$$

$$X_n \xrightarrow{a.s.} X \implies g(X_n) \xrightarrow{a.s.} g(X)$$

Proof (for convergence in distribution):

Construct $\{Y_n\}$ and Y as in Skorokhod's representation theorem. By continuity of g,

$$Y_n(u) \to Y(u) \implies g(Y_n(u)) \to g(Y(u)) \implies g(Y_n) \xrightarrow{a.s.} g(Y) \implies g(X_n) \xrightarrow{D} g(X)$$

Other Convergence Modes*

- $X_n \xrightarrow{1} X$ (convergence <u>in mean</u>) if

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|\right] = 0$$

- $X_n \xrightarrow{r} X$ (convergence <u>in rth mean</u> / <u>in the L^r-norm</u>) if

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^r\right] = 0$$

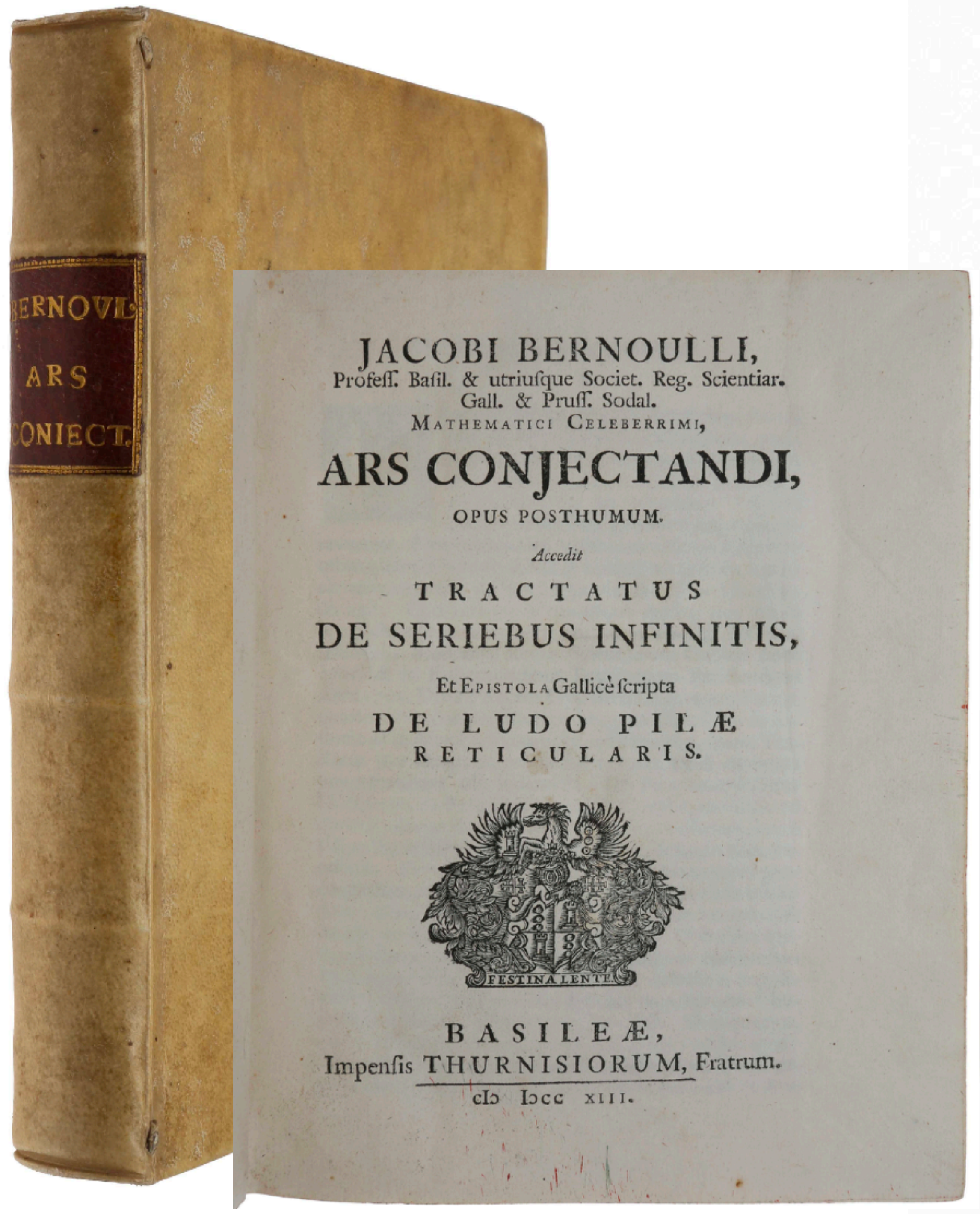
$$(X_n \xrightarrow{a.s.} X) \Longrightarrow (X_n \xrightarrow{P} X) \Longrightarrow (X_n \xrightarrow{D} X)$$

$$\Uparrow$$

$$(X_n \xrightarrow{s} X) \Longrightarrow (X_n \xrightarrow{r} X) \Longrightarrow (X_n \xrightarrow{1} X)$$

(for $s \geq r \geq 1$)

LLN and CLT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ARS CONJECTANDI,
OPUS POSTHUMUM.
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Et EPISTOLA Gallicè scripta
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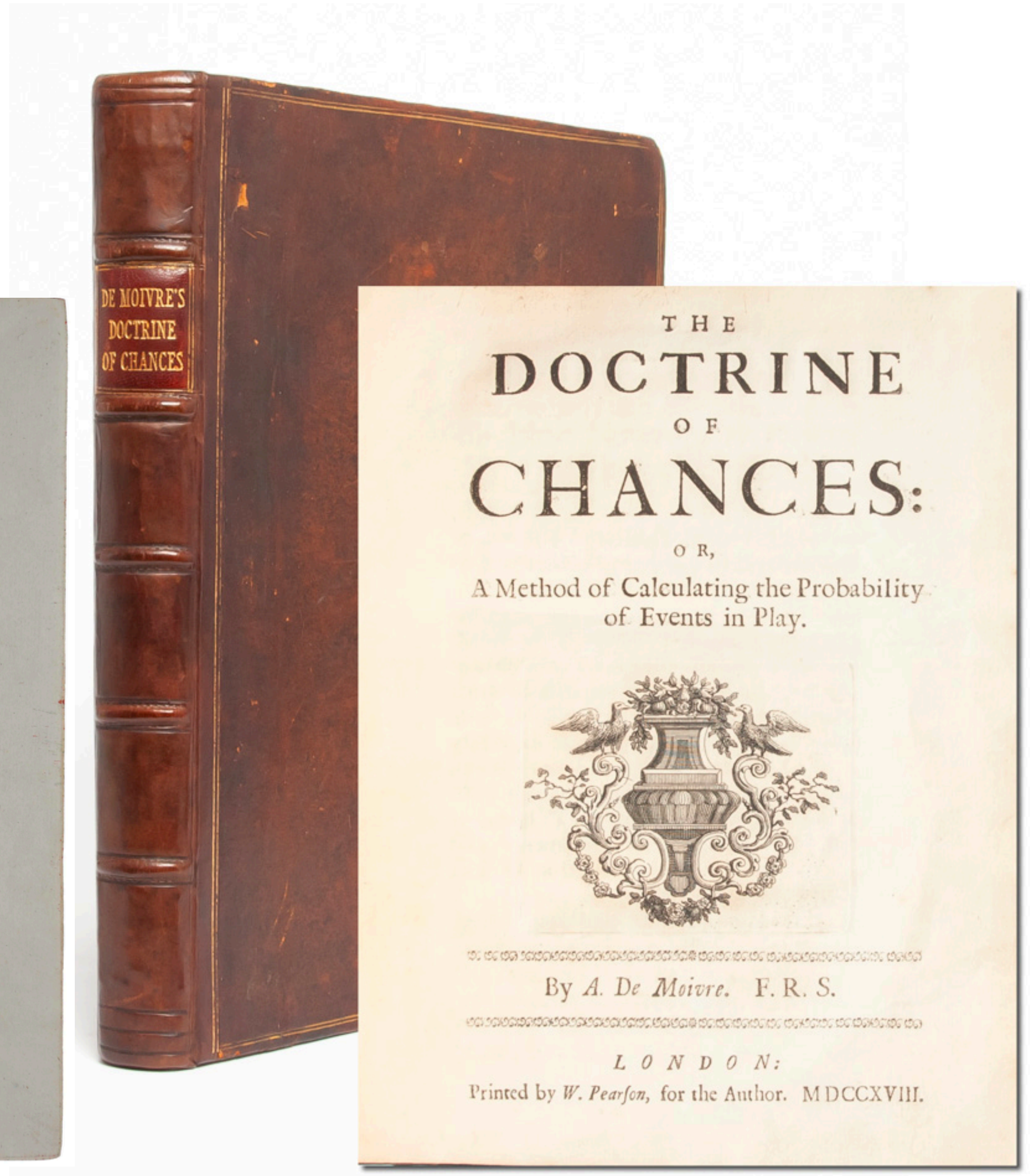

DE MOIVRE'S
DOCTRINE
OF CHANCES
THE
DOCTRINE
OF
CHANCES:
OR,
A Method of Calculating the Probability
of Events in Play.
By A. De Moivre. F. R. S.
LONDON:
Printed by W. Pearson, for the Author. MDCCXVIII.

Bernoulli's Law of Large Number

In *Ars Conjectandi* (1713)

- Let $X_1, X_2, \ldots$ be *i.i.d.* Bernoulli trials with $\mathbb{E}[X_1] = p \in [0,1]$. Then

$$\Pr\left(\left| \frac{X_1 + X_2 + \cdots + X_n}{n} - p \right| > \epsilon \right) \to 0 \quad \text{as } n \to \infty \quad \text{for all } \epsilon > 0$$

i.e. $\overline{X}_n \xrightarrow{P} p$, where $\overline{X}_n$ is the sample mean $\overline{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$

Proof: By Chebyshev's inequality, $\Pr(|\overline{X}_n - p| > \epsilon) \leq \dfrac{p(1-p)}{n\epsilon^2} \to 0$ as $n \to \infty$

(This is of course not the original proof of Bernoulli.)

Law of Large Numbers (LLN)

Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with finite mean $\mathbb{E}[X_1] = \mu$.

And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the **sample mean.**

- **Weak law** (Khinchin's law) **of large number**:

$$\overline{X}_n \xrightarrow{P} \mu \text{ as } n \to \infty$$

- **Strong law** (Kolmogorov's law) **of large number**:

$$\overline{X}_n \xrightarrow{a.s.} \mu \text{ as } n \to \infty$$

(*The deviation* $|\overline{X}_n - \mu|$ *is always small for all sufficiently large* n)

Weak LLN Assuming Bounded Variance

- Let $X_1, X_2, \ldots$ be independent random variables with finite mean $\mathbb{E}[X_i] = \mu$ and finitely bounded variance $\mathbf{Var}[X_i] \leq \sigma^2$.

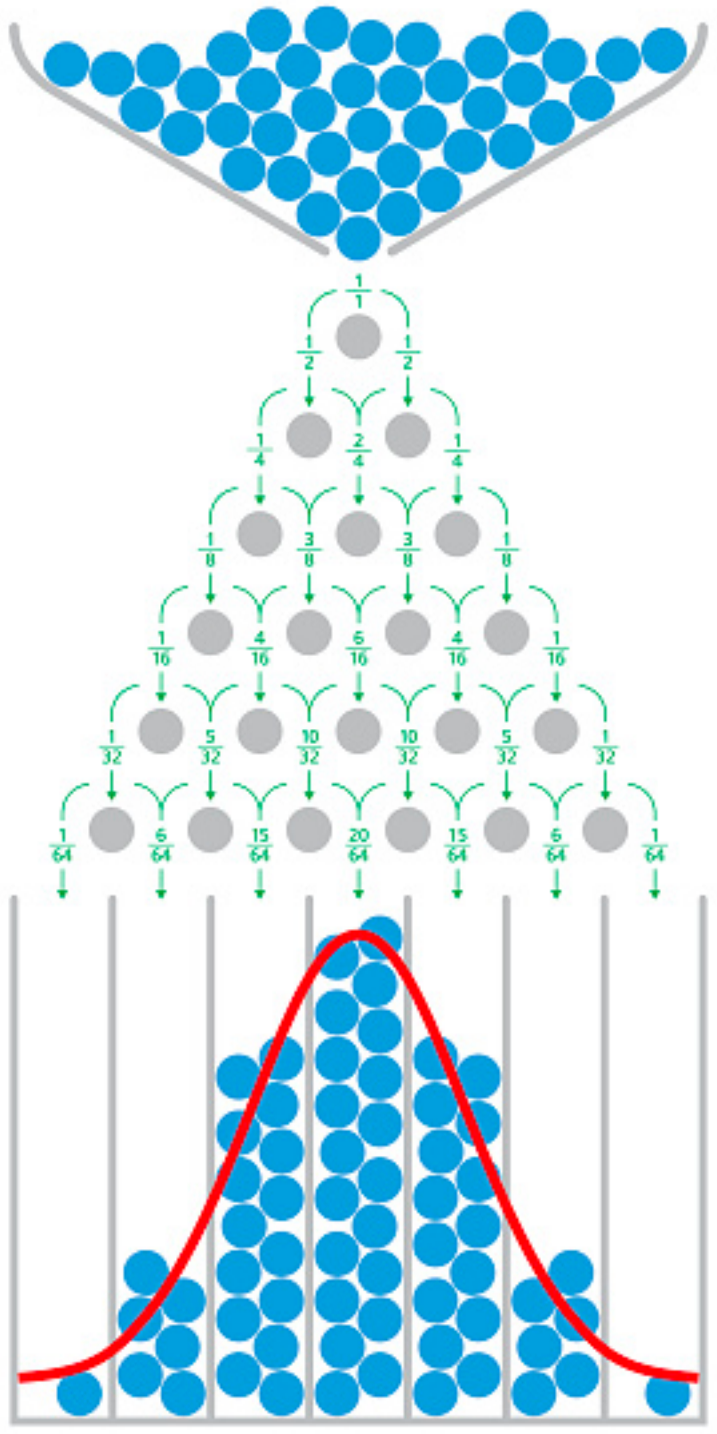
  Then the sample mean $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$$\overline{X}_n \xrightarrow{P} \mu \text{ as } n \to \infty$$

Proof: By **Chebysev's inequality**, $\Pr(|\overline{X}_n - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \to 0$ as $n \to \infty$

De Moivre—Laplace Theorem

(棣莫弗－拉普拉斯定理)



- Let $p \in (0,1)$ and $X_n \sim B(n,p)$. Then its standardization

$$\frac{X_n - np}{\sqrt{np(1-p)}} \xrightarrow{D} N(0,1) \quad \text{as} \quad n \to \infty$$

- For any $p \in (0,1)$, any radius $r > 0$, and any $\epsilon > 0$, there is an n_0 such that for all $n > n_0$ and all k such that $\left| (k - np)/\sqrt{np(1-p)} \right| < r$,

$$\binom{n}{k} p^k (1-p)^{n-k} \in (1 \pm \epsilon) \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{(k-np)^2}{2np(1-p)}}$$

Central Limit Theorem (CLT)

- Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mathbb{E}[X_1] = \mu$ and $\mathbf{Var}[X_1] = \sigma^2$.

  And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the **sample mean.**

- **<u>Classical</u>** (Lindeberg–Lévy) **<u>central limit theorem</u>**:

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty$$

Convergence Rate of CLT
(Berry–Esseen theorem)

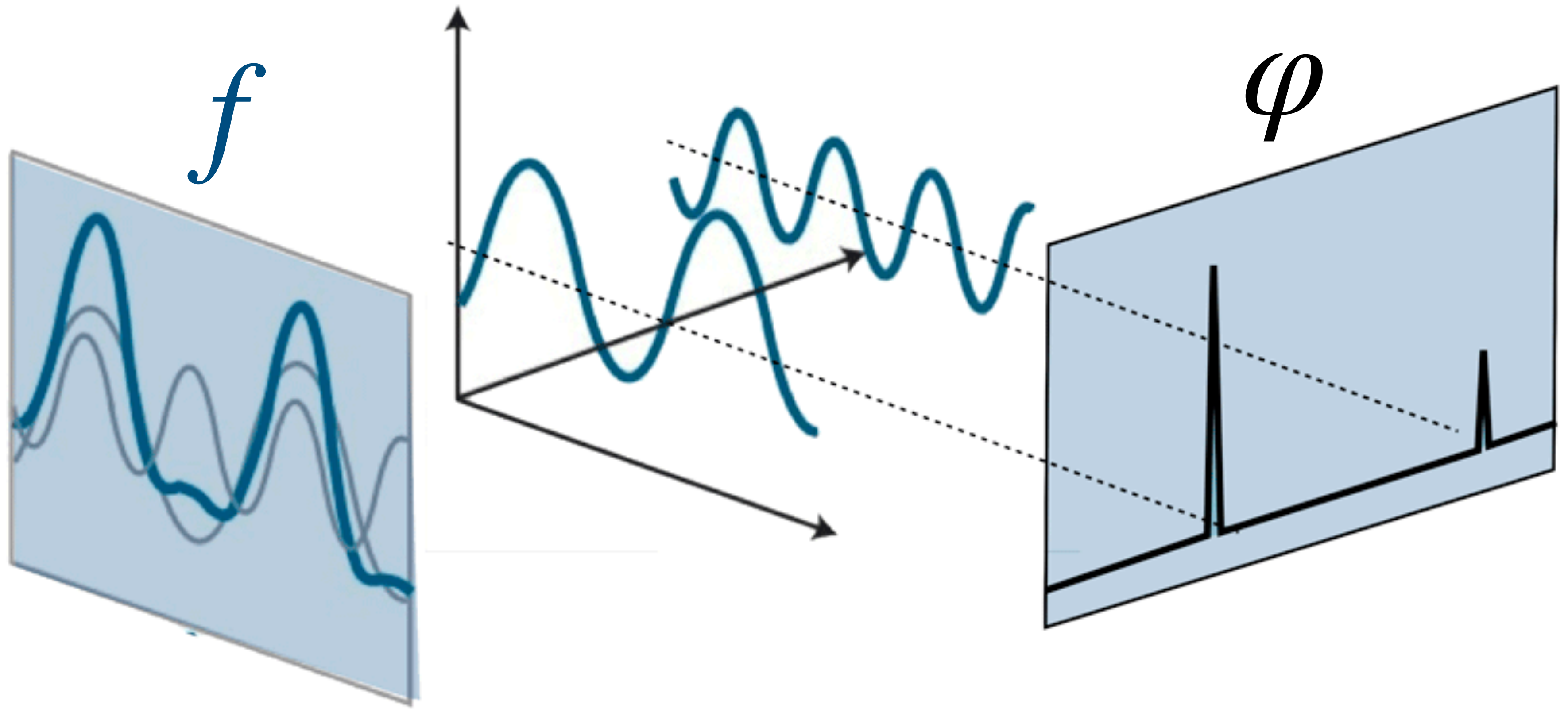
- Berry–Esseen theorem: Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mathbb{E}[X_1] = \mu$, $\mathbf{Var}[X_1] = \sigma^2$, and $\rho = \mathbb{E}[\,|X_1 - \mu|^3\,]$. And let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

  There is an absolute constant C, such that for any z

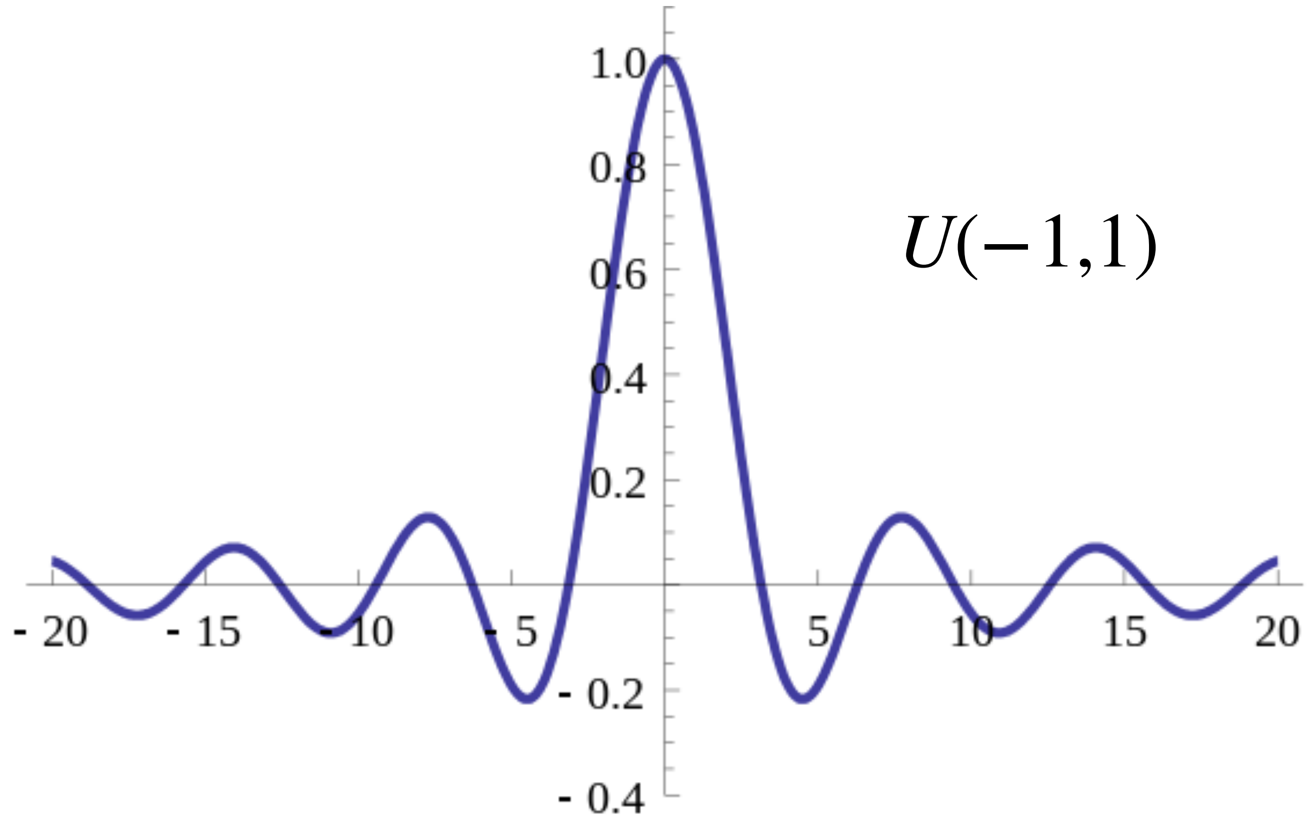
$$\left| \Pr\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le z \right) - \Phi(z) \right| \le \frac{C\rho}{\sigma^3\sqrt{n}}$$

  where Φ stands for the CDF for standard normal distribution $N(0,1)$

Characteristic Function



Characteristic Functions



- The moment generating function (MGF) of X is the function $M_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$$M_X(t) = \mathbb{E}[\mathrm{e}^{tX}]$$

- The characteristic function (特征函数) of X is the function $\varphi_X : \mathbb{R} \to \mathbb{C}$

$$\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}], \text{ where } i = \sqrt{-1}$$

- **Fourier transform:** $\varphi_X(t) = \int \mathrm{e}^{itx}\,\mathrm{d}F_X(x) = \mathbb{E}[\cos tX] + i\mathbb{E}[\sin tX]$
- Unlike MGF, φ_X always exists and is finite, because $|\mathrm{e}^{itx}| = 1$

Boundedness of Characteristic Function

$$\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}]$$

- $|\varphi_X(t)| \leq 1$ for all $t \in \mathbb{R}$
- If $\mathbb{E}[|X^k|] < \infty$, then

$$\varphi_X(t) = \sum_{j=0}^{k} \frac{\mathbb{E}[X^j]}{j!}(it)^j + o(t^k)$$

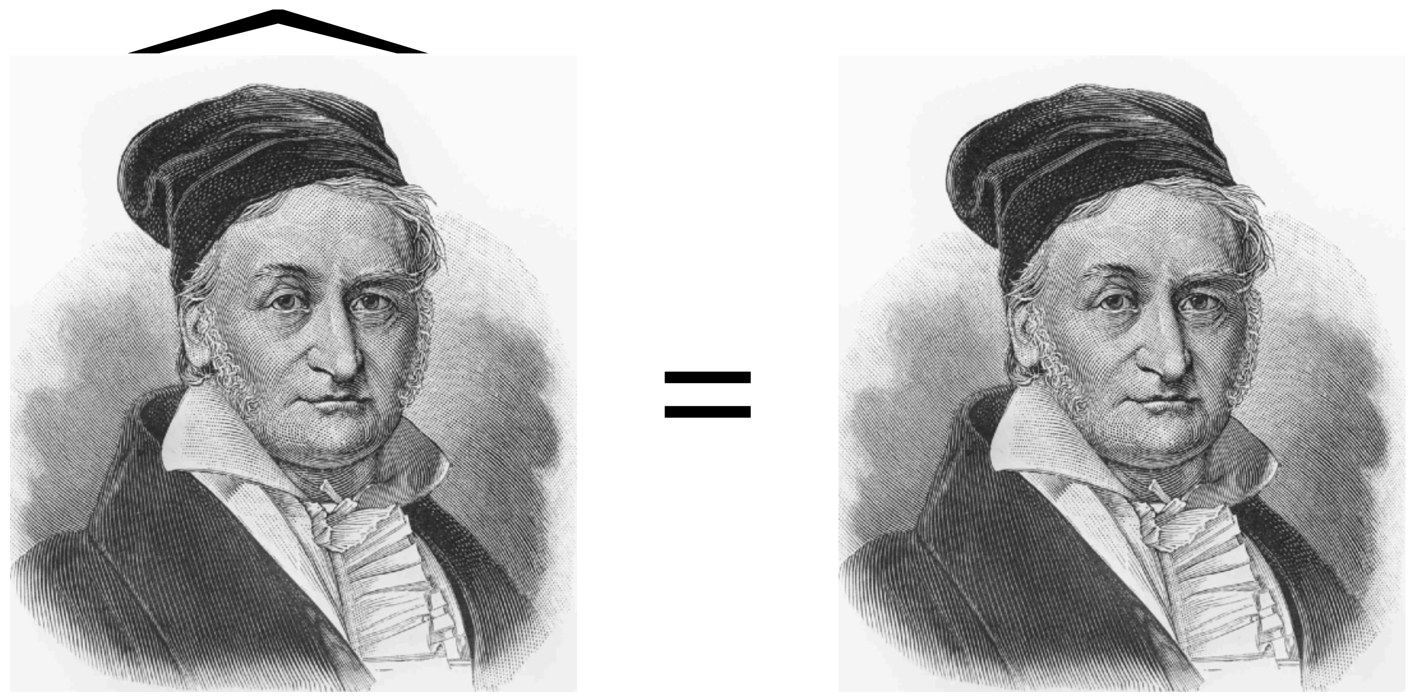
$(\varphi_X(t) = 1 + i\mu t + o(t)\ )$

$(\varphi_X(t) = 1 + i\mu t - \frac{(\sigma^2 + \mu^2)}{2}t^2 + o(t^2))$

Proof: $|\varphi_X(t)| \leq \int |\mathrm{e}^{itx}| \,\mathrm{d}F_X(x) = \int \mathrm{d}F_X(x) = 1$ (for Lebesgue-Stieltjes integral)

Taylor's expansion: $\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \mathbb{E}\left[\sum_{j=0}^{k} \frac{X^j}{j!}(it)^j + o(t^k)\right] = \sum_{j=0}^{k} \frac{\mathbb{E}[X^j]}{j!}(it)^j + o(t^k)$

Normal Characteristic Function



- If $X \sim N(0,1)$, then

$$\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \mathrm{e}^{-t^2/2}$$

Proof (using complex integration):

$$\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{itx - x^2/2}\,\mathrm{d}x = \mathrm{e}^{-t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}(x-it)^2}\,\mathrm{d}x = \mathrm{e}^{-t^2/2}$$

because $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}(x-it)^2}\,\mathrm{d}x = 1$ via contour integration

Normal Characteristic Function

- If $X \sim N(0,1)$, then

$$\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \mathrm{e}^{-t^2/2}$$

Proof (without using complex integration): $\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \mathbb{E}[\cos tX] + i\mathbb{E}[\sin tX]$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)\mathrm{e}^{-x^2/2}\,\mathrm{d}x + \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\sin(tx)\mathrm{e}^{-x^2/2}}_{\text{odd function}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)\mathrm{e}^{-x^2/2}\,\mathrm{d}x$$

$$\frac{\mathrm{d}\varphi_X(t)}{\mathrm{d}t} = \mathbb{E}\left[\frac{\mathrm{d}\mathrm{e}^{itX}}{\mathrm{d}t}\right] = \mathbb{E}[iX\mathrm{e}^{itX}] = i\mathbb{E}[\underbrace{X\cos tX}_{\text{odd function}}] - \mathbb{E}[X\sin tX] = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)\mathrm{e}^{-x^2/2}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(tx)\,\mathrm{d}\mathrm{e}^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\sin(tx)\mathrm{e}^{-x^2/2}\Big|_{-\infty}^{\infty} - \frac{t}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)\mathrm{e}^{-x^2/2}\,\mathrm{d}x = -\frac{t}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)\mathrm{e}^{-x^2/2}\,\mathrm{d}x$$

$$\Longrightarrow \quad \frac{\mathrm{d}\varphi_X(t)}{\mathrm{d}t} = -t\varphi_X(t) \quad \Longrightarrow \quad \varphi_X(t) = \mathrm{e}^{-t^2/2}$$ (solving the ODE subject to $\varphi_X(0) = \mathbb{E}[\mathrm{e}^{i\cdot 0\cdot X}] = 1$)

Linear Transformation

- If X and Y are independent, then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$
- If $Y = aX + b$ for $a, b \in \mathbb{R}$, then $\varphi_Y(t) = \mathrm{e}^{itb}\varphi_X(at)$

Proof: For independent X and Y,

$$\varphi_{X+Y}(t) = \mathbb{E}[\mathrm{e}^{it(X+Y)}] = \mathbb{E}[\mathrm{e}^{itX}]\mathbb{E}[\mathrm{e}^{itY}] = \varphi_X(t)\varphi_Y(t)$$

For $Y = aX + b$,

$$\varphi_Y(t) = \mathbb{E}[\mathrm{e}^{it(aX+b)}] = \mathrm{e}^{itb}\mathbb{E}[\mathrm{e}^{itaX}] = \mathrm{e}^{itb}\varphi_X(at)$$

Continuity Theorem

- If X is **continuous** with density function f_X and characteristic function φ_X, then (by **Fourier inversion theorem**)

$$\varphi_X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{itx} f_X(x)\,\mathrm{d}x \quad \text{and} \quad f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-itx}\varphi_X(t)\,\mathrm{d}t$$

  Hence, the distribution of continuous X is **uniquely identified** by φ_X.

- For **general** random variables: (*it's more complicated, but similarly*)

$$F_X = F_Y \ \text{ iff } \ \varphi_X = \varphi_Y$$

- <u>**Lévy's continuity theorem**</u>: Let $\{X_n\}$ and X be random variables.

$$X_n \xrightarrow{D} X \quad \text{iff} \quad \varphi_{X_n} \to \varphi_X \text{ pointwise on } \mathbb{R} \text{ as } n \to \infty$$

Convolution of Normal Distribution

- If $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\nu, \tau^2)$ are independent, then
  - $X + Y \sim N(\mu + \nu, \sigma^2 + \tau^2)$

Proof (by characteristic function): Let $Z \sim N(0,1)$.

$$X \sim N(\mu, \sigma^2) \implies X = \sigma Z + \mu \implies \varphi_X(t) = \mathrm{e}^{it\mu}\varphi_Z(\sigma t) = \mathrm{e}^{it\mu - \sigma^2 t^2/2}$$

By the same calculation: $Y \sim N(\nu, \tau^2) \implies \varphi_Y(t) = \mathrm{e}^{it\nu - \tau^2 t^2/2}$

$$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = \mathrm{e}^{it\mu - \sigma^2 t^2/2} \cdot \mathrm{e}^{it\nu - \tau^2 t^2/2} = \mathrm{e}^{it(\mu+\nu) - (\sigma^2+\tau^2)t^2/2}$$

which is the characteristic function of normal distribution $N(\mu + \nu, \sigma^2 + \tau^2)$.

Law of Large Numbers (LLN)

Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with finite mean $\mathbb{E}[X_1] = \mu$.

And let $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ be the **sample mean.**

- **<u>Weak law</u>** (Khinchin's law) **<u>of large number</u>:**

$$\overline{X}_n \xrightarrow{P} \mu \ \text{ as } \ n \to \infty$$

- **<u>Strong law</u>** (Kolmogorov's law) **<u>of large number</u>:**

$$\overline{X}_n \xrightarrow{a.s.} \mu \ \text{ as } \ n \to \infty$$

Proof of the Weak Law of Large Numbers

- Let $X_1, X_2, \ldots$ be *i.i.d.* with finite mean $\mathbb{E}[X_1] = \mu$. Let $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$
- The characteristic function $\varphi_{X_j}(t) = \mathbb{E}[\mathrm{e}^{itX_j}] = 1 + i\mu t + o(t)$

$$\implies \varphi_{\overline{X}_n}(t) = \varphi_{X_1+\cdots+X_n}(t/n) = \prod_{j=1}^{n} \varphi_{X_j}(t/n) = \left(1 + \frac{i\mu t}{n} + o\left(\frac{t}{n}\right)\right)^n$$

$$\to \mathrm{e}^{it\mu} \quad \text{for all } t \in \mathbb{R} \quad \text{as } n \to \infty$$

- Meanwhile, $\varphi_X(t) = \mathbb{E}[\mathrm{e}^{itX}] = \mathrm{e}^{it\mu}$ for constant $X = \mu$
- $\implies \overline{X}_n \xrightarrow{D} \mu$ by **Lévy's continuity theorem** $\implies \overline{X}_n \xrightarrow{P} \mu$ for constant μ

Central Limit Theorem (CLT)

- Let $X_1, X_2, \ldots$ be *i.i.d.* random variables with $\mathbb{E}[X_1] = \mu$ and $\mathbf{Var}[X_1] = \sigma^2$.

  Let $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ be the **standardized sample mean**, where $\overline{X}_n = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

- **Classical** (Lindeberg–Lévy) **central limit theorem**:

$$Z_n \xrightarrow{D} N(0,1) \quad \text{as } n \to \infty$$

Proof of the Central Limit Theorem

- Let $X_1, X_2, \ldots$ be *i.i.d.* with finite $\mathbb{E}[X_1] = \mu$ and $\mathbf{Var}[X_1] = \sigma^2$. Let $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$
- For standardized $Y_j = (X_j - \mu)/\sigma \implies \varphi_{Y_j}(t) = \mathbb{E}[\mathrm{e}^{itY_j}] = 1 - \frac{t^2}{2} + o(t^2)$
- The **standardized sample mean:** $Z_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{Y_1 + \cdots + Y_n}{\sqrt{n}}$

$$\implies \varphi_{Z_n}(t) = \varphi_{Y_1+\cdots+Y_n}\left(t/\sqrt{n}\right) = \prod_{j=1}^{n} \varphi_{Y_j}\left(t/\sqrt{n}\right) = \left(1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right)^n$$

$\to \mathrm{e}^{-t^2/2}$ for all $t \in \mathbb{R}$ as $n \to \infty$ (characteristic function of $N(0,1)$)

- $\implies \overline{Z}_n \xrightarrow{D} Z \sim N(0,1)$ by **Lévy's continuity theorem**