Probability Theory & Mathematical Statistics

Random Variable

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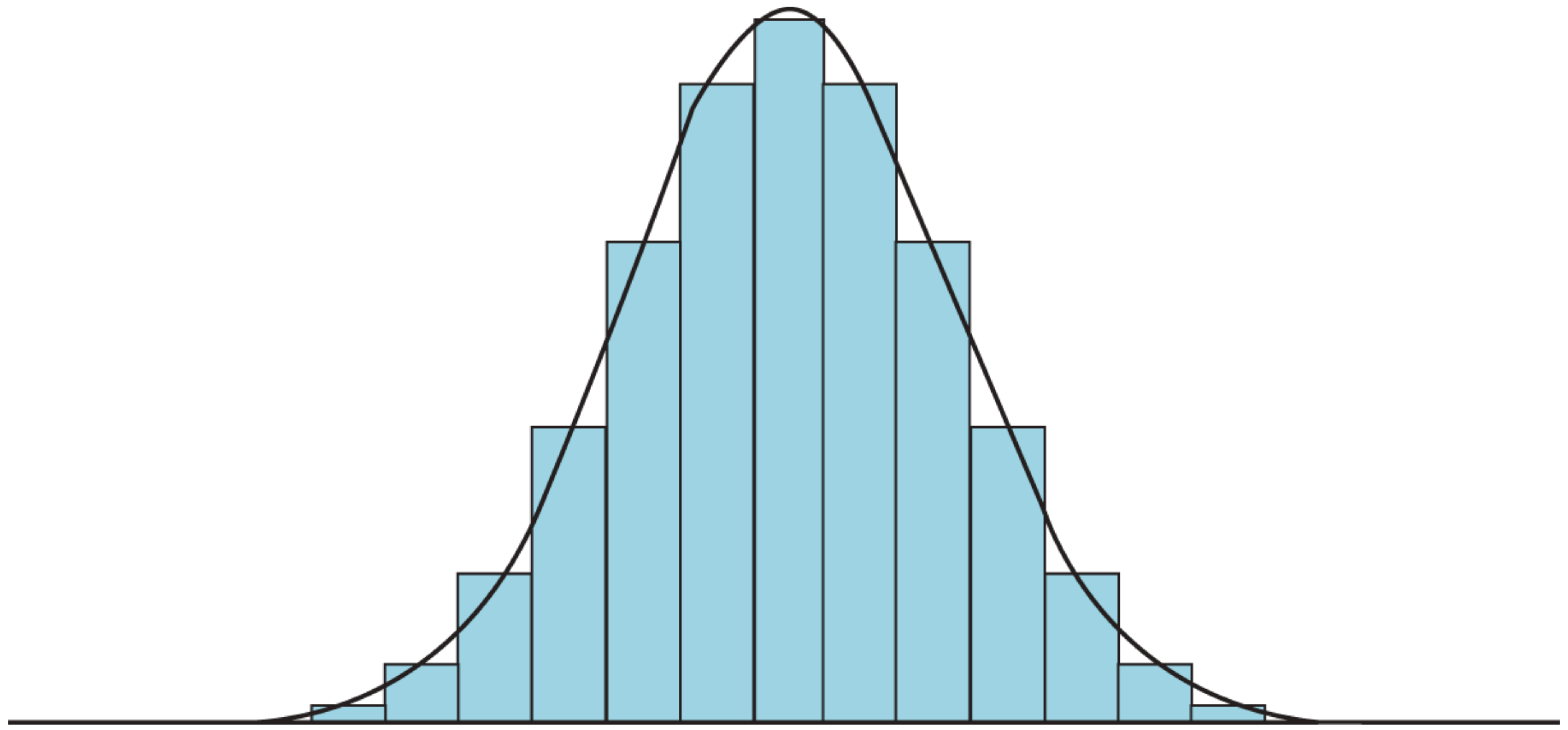
Random Variable



“*Variables*” that are Random

- 令X和Y分别为两次掷🎲的结果：
  - 考虑X^2和XY——它们是相同的随机量吗？
  - $2X$和$X+Y$呢？或者任意凸组合$\lambda X+(1-\lambda)Y$之间呢？
- 设🪙正面朝上概率为p：令X表示连续抛🪙直至正面朝上为止的抛🪙次数；
  令Y表示连抛n次🪙，其中正面朝上的次数；
- 令X表示从一个装有M个⚽$N-M$个🏀的🗑️中（有/无放回地）取出n个球中⚽的个数；
- n个顶点，任意两点间独立以概率p连一条边，产生随机图G，令$X=\chi(G)$为最小染色数；
- 令X为$[0,1]$中均匀分布的随机实数；令Y为$[0,\infty)$上满足$\Pr(Y\geq y)=\mathrm{e}^{-y}$的随机实数。

Random Variable

- Roll a 🎲, let X be the outcome of the roll, let $Y \in \{0,1\}$ indicate its oddness.

samples in Ω	values of X	values of Y
⚀	1	1
⚁	2	0
⚂	3	1
⚃	4	0
⚄	5	1
⚅	6	0

Random Variable

- Let X be the sum of two independent 🎲 rolls.

⚀⚀	2	⚀⚁	3	⚀⚂	4	⚀⚃	5	⚀⚄	6	⚀⚅	7
⚁⚀	3	⚁⚁	4	⚁⚂	5	⚁⚃	6	⚁⚄	7	⚁⚅	8
⚂⚀	4	⚂⚁	5	⚂⚂	6	⚂⚃	7	⚂⚄	8	⚂⚅	9
⚃⚀	5	⚃⚁	6	⚃⚂	7	⚃⚃	8	⚃⚄	9	⚃⚅	10
⚄⚀	6	⚄⚁	7	⚄⚂	8	⚄⚃	9	⚄⚄	10	⚄⚅	11
⚅⚀	7	⚅⚁	8	⚅⚂	9	⚅⚃	10	⚅⚄	11	⚅⚅	12

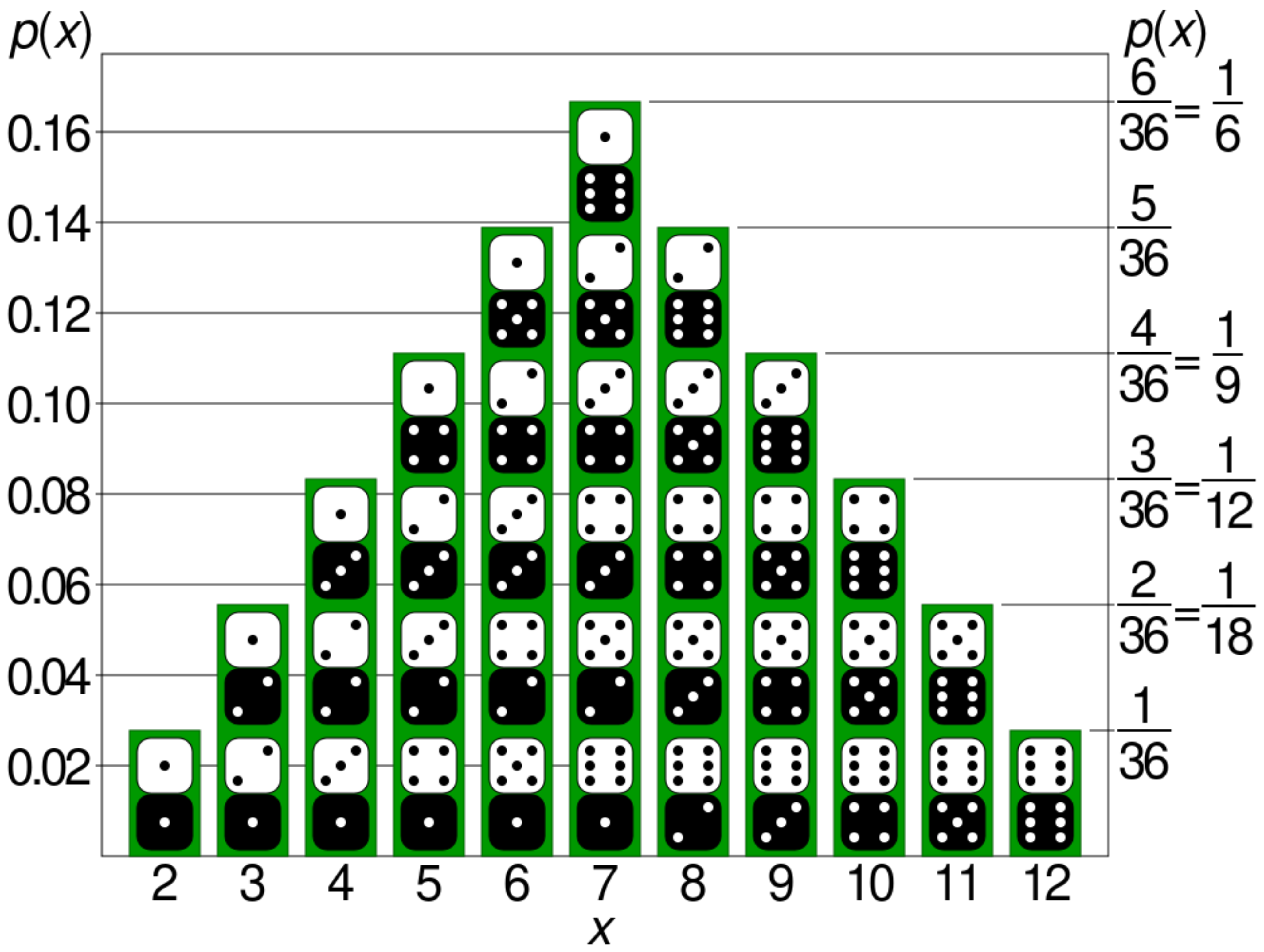
Random Variable (随机变量)

- Given $(\Omega, \Sigma, \Pr)$, a <u>random variable</u> is a function $X : \Omega \to \mathbb{R}$
  - satisfying that $\forall x \in \mathbb{R}, \{\omega \in \Omega \mid X(\omega) \leq x\} \in \Sigma$ (i.e. X is <u>Σ-measurable</u>)
- $X \leq x$ (where $x \in \mathbb{R}$) denotes the event $\{\omega \in \Omega \mid X(\omega) \leq x\}$
- $X > x$ (where $x \in \mathbb{R}$) denotes the event $\{\omega \in \Omega \mid X(\omega) > x\}$
- $X \in S$ (where $S \subseteq \mathbb{R}$ is countable ∩,∪ of intervals $(y, x]$) denotes the event $\{\omega \in \Omega \mid X(\omega) \in S\}$
- For <u>discrete random variable</u> $X : \Omega \to \mathbb{Z}$, this includes all subsets $S \subseteq \mathbb{Z}$

$$\Pr(X \in S)$$

Distribution of Random Variable

- Let X be the sum of two **independent** 🎲 rolls.



Distribution (分布)

- The cumulative distribution function (CDF) (累积分布函数) or just **distribution function** (分布函数) of a random variable X is the $F_X : \mathbb{R} \to [0,1]$ given by

$$F_X(x) = \Pr(X \leq x)$$

- All probabilities regarding X can be deduced from F_X. (Prob. space is no longer needed.)
- Two random variables X and Y are **identically distributed** if $F_X = F_Y$
- Monotone: $\forall x, y \in \mathbb{R}$, if $x \leq y$ then $F_X(x) \leq F_X(y)$
- Bounded: $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$

Discrete Random Variable

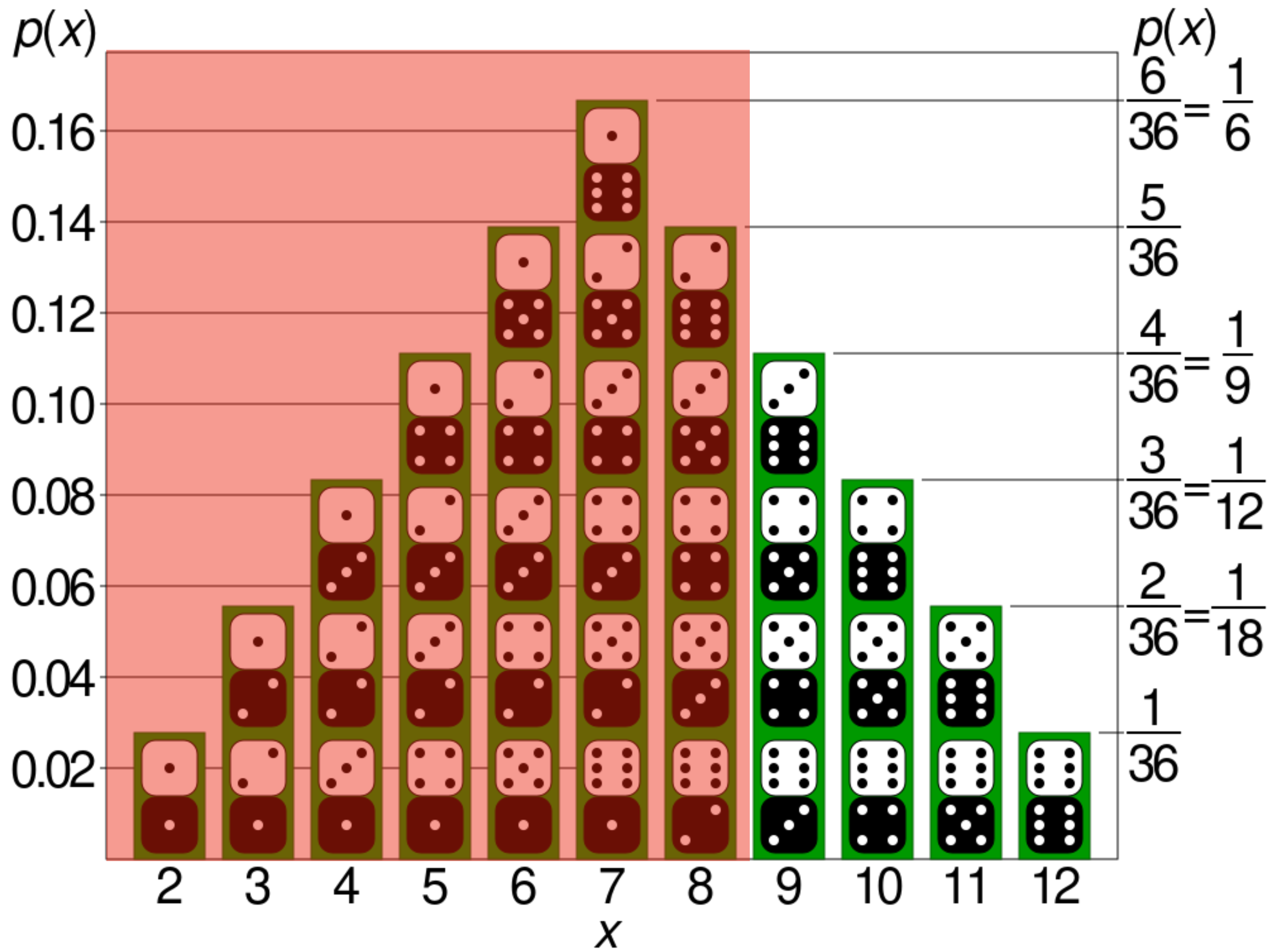
- A random variable $X : \Omega \to \mathbb{R}$ is called discrete if $X(\Omega)$ is countable.

- For a discrete random variable X, its probability mass function (*pmf*) (概率质量函数) $p_X : \mathbb{R} \to [0,1]$ is given by

$$p_X(x) = \Pr(X = x)$$

- The CDF F_X satisfies

$$F_X(y) = \sum_{x \le y} p_X(x)$$



Continuous Random Variable

- A random variable $X : \Omega \to \mathbb{R}$ is called continuous, if its CDF can be expressed as

$$F_X(y) = \Pr(X \le y) = \int_{-\infty}^{y} f_X(x)\,\mathrm{d}x$$

for some integrable probability density function (*pdf*) (概率密度函数) f_X.

- Never mind what type of integral for now. (Riemann integral? Lebesgue integral?)
- There are random variables that are neither discrete nor continuous.

Independence

- Two *discrete* random variables X and Y are **independent** if $X = x$ and $Y = y$ are independent events for all x and y.

- *Discrete* random variables $X_1, \ldots, X_n$ are (**mutually**) **independent** if $X_1 = x_1, \ldots, X_n = x_n$ are mutually independent events for all $x_1, \ldots, x_n$

$$p_{(X_1,\ldots,X_n)}(x_1, \ldots, x_n) = \Pr(X_1 = x_1 \cap \cdots \cap X_n = x_n) = p_{X_1}(x_1) \cdots p_{X_n}(x_n)$$

- The pairwise (and k-wise) independence are defined in the same way.
  - **Example:** The construction of $2^n - 1$ pairwise independent random bits out of n mutually independent random bits by XOR.

- For *general* random variables, the events $X_i = x_i$ are replaced by $X_i \leq x_i$.

Random Vector (随机向量)

- Given $(\Omega, \Sigma, \Pr)$, a <u>random vector</u> is an $\boldsymbol{X} = (X_1, \ldots, X_n)$ where each X_i is a random variable defined on the probability space $(\Omega, \Sigma, \Pr)$.

- The <u>joint CDF</u> (联合累积分布函数) $F_{\boldsymbol{X}} : \mathbb{R}^n \to [0,1]$ is given by

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \Pr(X_1 \le x_1 \cap \cdots \cap X_n \le x_n)$$

- For *discrete* random vector, the <u>joint mass function</u> (联合质量函数) is given by

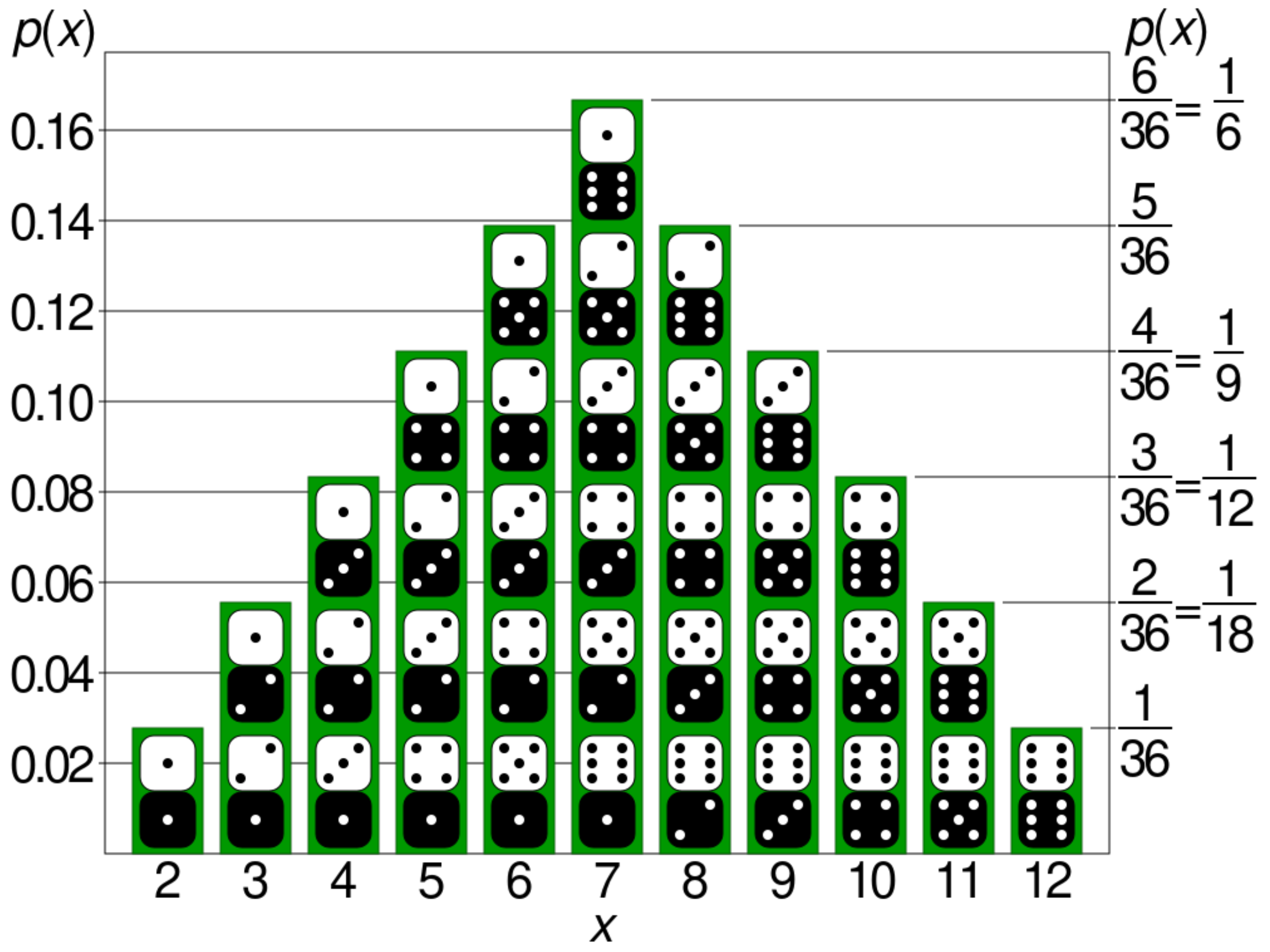
$$p_{\boldsymbol{X}}(x_1, \ldots, x_n) = \Pr(X_1 = x_1 \cap \cdots \cap X_n = x_n)$$

- The <u>marginal distribution</u> of X_i in $(X_1, \ldots, X_n)$ is given by

$$p_{X_i}(x_i) = \sum_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n} p_{(X_1, \ldots, X_n)}(x_1, \ldots, x_n)$$

Y \ X	x_1	x_2	x_3	x_4	$p_Y(y)$ ↓
y_1	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{8}{32}$
y_2	$\frac{3}{32}$	$\frac{6}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{15}{32}$
y_3	$\frac{9}{32}$	0	0	0	$\frac{9}{32}$
$p_X(x)$ →	$\frac{16}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{32}{32}$

Discrete Random Variable



Probability Mass Function (概率质量函数)

- Consider *integer-valued* discrete random variable $X : \Omega \to \mathbb{Z}$
- Its <u>probability mass function</u> (*pmf*) $p_X : \mathbb{Z} \to [0,1]$ is given by

$$p_X(k) = \Pr(X = k)$$

- <u>As histogram</u>: p_X gives the "histogram" of the probability distribution
- <u>As vector</u>: p_X can be seen as a vector $p_X \in [0,1]^R$ such that $\|p_X(x)\|_1 = 1$, where $R = X(\Omega)$ is the range of values of X
- Its function $Y = f(X)$ is also a discrete random variable, where $p_Y(y) = \sum_{x:f(x)=y} p_X(x)$

Discrete Random Variables

- Basic discrete probability distributions:
  - discrete uniform distribution (古典概型)
  - **Bernoulli trial** (coin flip)
  - **binomial distribution** (# of successes in n trials)
  - **geometric distribution** (# of trials to get a success)
  - negative binomial distribution
  - hypergeometric distribution
  - **Poisson distribution** (idealized binomial distribution)
  - … …
- Probability distributions of discrete objects:
  - multinomial distribution (balls into bins)
  - Erdős–Rényi model (random graph)
  - Galton-Watson process (random tree)
  - … …

Bernoulli Trial (伯努利试验)
(A coin flip)

- A **Bernoulli trial** is an experiment with two possible outcomes.

- A **Bernoulli random variable** X takes values in $\{0,1\}$, its *pmf* is

$$p_X(k) = \Pr(X = k) = \begin{cases} p & \text{if } k = 1 \\ 1 - p & \text{if } k = 0 \end{cases}$$

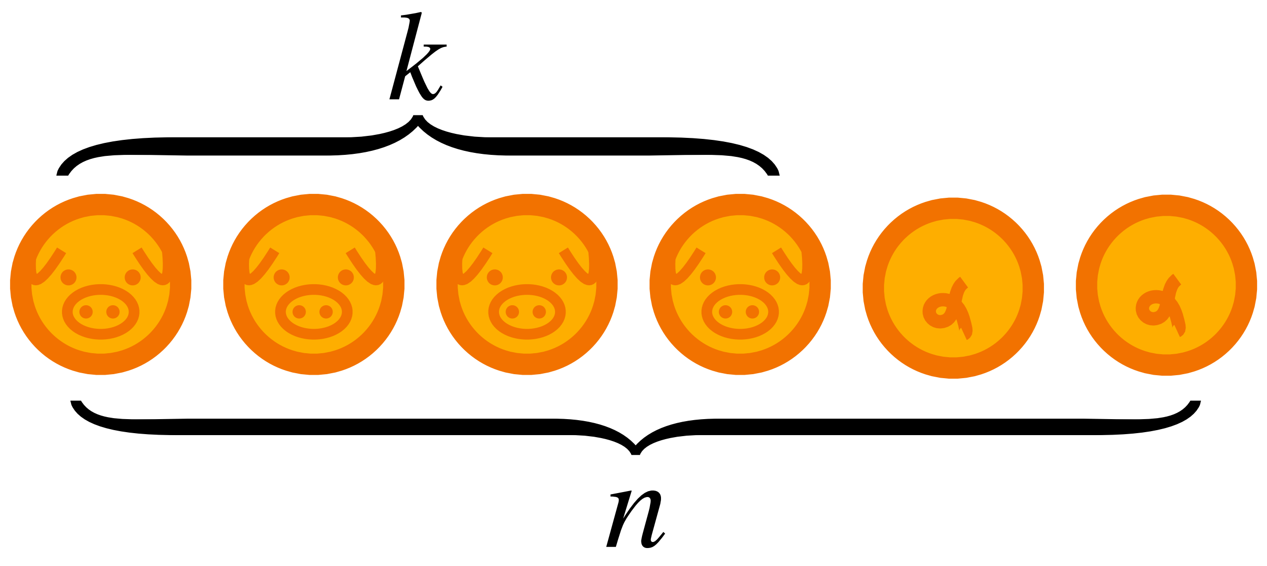
  where $p \in [0,1]$ is a parameter.

- **Indicator**: For event A, the indicator X of A is a random variable defined by

$$X = I(A) = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}, \quad \text{a Bernoulli R.V. with parameter } \Pr(A)$$

Binomial Distribution (二项分布)

(Number of HEADs in n coin flips)



- X: number of successes in n **<u>*i.i.d.*</u>** (*independent and identically distributed*) Bernoulli trials with parameter p

- A <u>binomial random variable</u> X takes values in $\{0,1,\ldots,n\}$, and

$$p_X(k) = \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0,1,\ldots,n$$

- We say that X follows the **<u>binomial distribution</u>** with parameters n and p

denoted $X \sim \mathrm{Bin}(n, p)$ or $\mathrm{B}(n, p)$

Geometric Distribution (几何分布)

(Number of coin flips to get a HEADs)

- X: number of i.i.d. Bernoulli trials needed to get one success
- A <u>geometric random variable</u> X takes values in $\{1,2,\ldots\}$, and

$$p_X(k) = \Pr(X = k) = (1-p)^{k-1}p, \quad k = 1,2,\ldots$$

- We say that X follows the <u>**geometric distribution**</u> with parameter $p \in [0,1]$

denoted $X \sim \text{Geo}(p)$ or $\text{Geometric}(p)$

Geometric Distribution (几何分布)

(Number of coin flips to get a HEADs)

- Geometric random variable $X \sim$ Geo(p) is **memoryless**: for $k \geq 1, n \geq 0$

$$\Pr(X = k + n \mid X > n) = \Pr(X = k)$$

Proof: $\Pr(X = k + n \mid X > n) = \dfrac{\Pr(X = k + n)}{\Pr(X > n)} = \dfrac{(1-p)^{n+k-1}p}{\sum_{k=n}^{\infty}(1-p)^k p}$

$$= \frac{(1-p)^{k-1}p}{\sum_{k=0}^{\infty}(1-p)^k p} = (1-p)^{k-1}p$$

- Geometric distribution is the ***only*** **discrete memoryless distribution** (with the range of values $\{1,2,\ldots\}$).

Two Ways of Constructing Random Variables

- As a <u>function of random variables</u> $Y = f(X_1, X_2, \ldots, X_n)$
  - Binomial Y: function f is sum, and $(X_1, \ldots, X_n)$ are i.i.d. Bernoulli trials
  - independent $Y_1 \sim \text{Bin}(n_1, p)$, $Y_2 \sim \text{Bin}(n_2, p) \Longrightarrow Y_1 + Y_2 \sim \text{Bin}(n_1 + n_2, p)$
- As a <u>stopping time</u> T of a sequence $X_1, X_2, \ldots, X_T$
  - A random variable T is a stopping time with respect to $X_1, X_2, \ldots$ if for all $t \geq 1$ the occurrence of $T = t$ is determined by the values of $X_1, X_2, \ldots, X_t$
  - Geometric T: time for the first success in i.i.d. Bernoulli trials $X_1, X_2, \ldots$

Sum of Independent Random Variables

- If discrete random variables X and Y are **independent**, then:

$$p_{X+Y}(z) = \Pr(X + Y = z) = \sum_x \Pr(X = x \cap Y = z - x)$$ (total probability)

(independence) $$= \sum_x p_X(x)p_Y(z - x) = \sum_y p_X(z - y)p_Y(y)$$

- This defines a **convolution** (卷积) between mass functions:

$$p_{X+Y} = p_X * p_Y$$

Sum of Independent Random Variables

- If discrete random variables X and Y are **independent**, then:

$$p_{X+Y}(z) = \sum_{x} p_X(x)p_Y(z-x) = \sum_{y} p_X(z-y)p_Y(y)$$

- For *i.i.d.* Bernoulli random variables $X_1, \ldots, X_n \in \{0,1\}$:

$$p_{X_1+\cdots+X_n}(k) = p \cdot p_{X_1+\cdots+X_{n-1}}(k-1) + (1-p) \cdot p_{X_1+\cdots+X_{n-1}}(k)$$

$$= \binom{n-1}{k-1} p^k (1-p)^{n-k} + \binom{n-1}{k} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

Negative Binomial Distribution (负二项分布)
("multiple successes" generalization of geometric distribution)

- X: number of failures in a sequence of i.i.d. Bernoulli trials before r successes
- A negative binomial random variable X takes values in $\{0,1,2,\ldots\}$, and

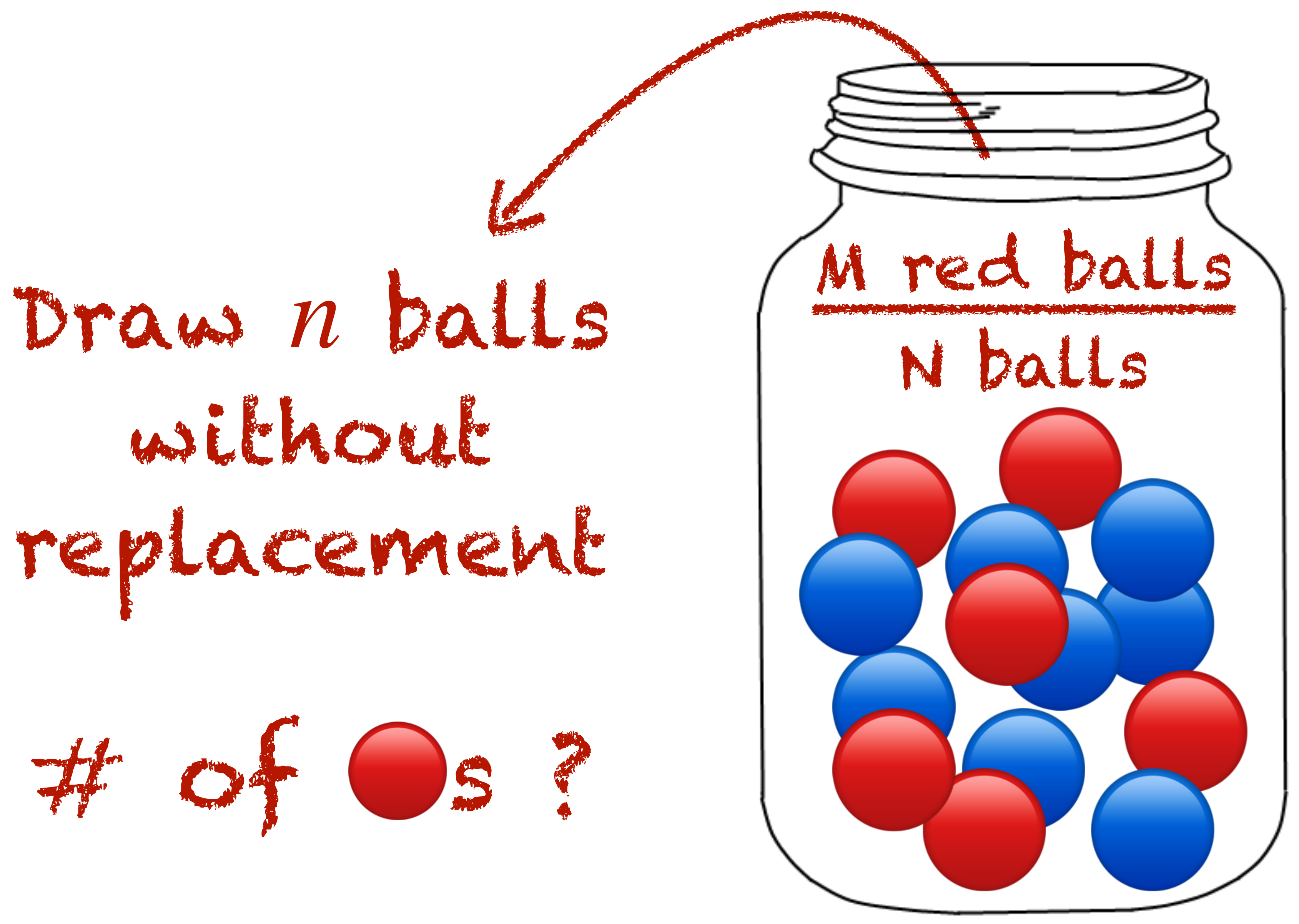
$$p_X(k) = \Pr(X = k) = \binom{k+r-1}{k}(1-p)^k p^r = (-1)^k \binom{-r}{k}(1-p)^k p^r$$

for $k = 0,1,2,\ldots$

- We say that X follows the negative binomial distribution with parameters r, p
- $X = (X_1 - 1) + (X_2 - 1) + \cdots + (X_r - 1)$ for i.i.d. $X_i \sim$ Geo(p)

Hypergeometric Distribution (超几何分布)
(“without replacement” variant of binomial distribution)

- X: number of successes in n draws, without replacement (无放回), from a *finite population* of N objects, including exactly M ones, drawings of whom are counted as successes



Hypergeometric Distribution (超几何分布)
(“without replacement” variant of binomial distribution)

- X: number of successes in n draws, without replacement (无放回), from a *finite population* of N objects, including exactly M ones, drawings of whom are counted as successes
- A hypergeometric random variable X takes values in $\{0,1,\ldots,n\}$, and

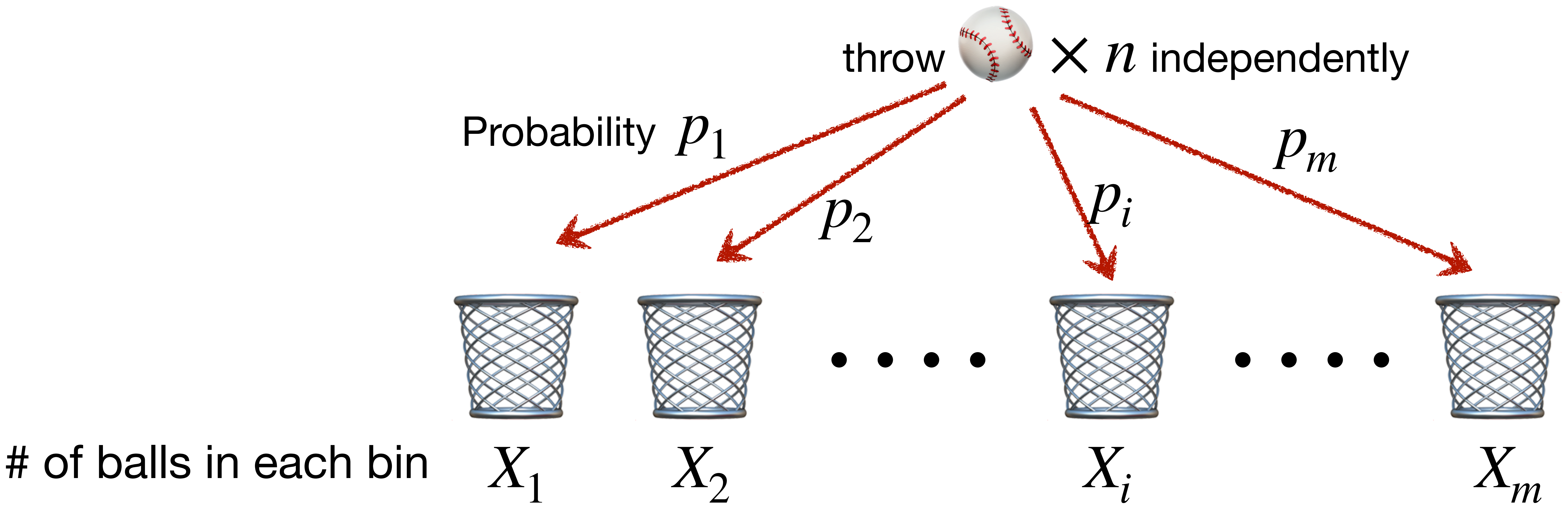
$$p_X(k) = \Pr(X = k) = \binom{M}{k}\binom{N-M}{n-k} \Big/ \binom{N}{n}, \qquad k = 0,1,\ldots,n$$

- We say that X follows the hypergeometric distribution with parameters N, M, n, where $N \geq 0$, $0 \leq M \leq N$, and $0 \leq n \leq N$ are integers.

Multinomial Distribution (多项式分布)
("multi-dimensional" generalization of binomial distribution)

- **Trials with multiple outcomes**: There are n *i.i.d.* trials, each having m possible outcomes, where the probability of the ith outcome is p_i. Let X_i be the # of ith outcomes.

- **Balls-into-bins model**: Throw n balls into m bins. Each ball is thrown independently such that the ith bin receives the ball with probability p_i. Let X_i be the # of balls in the ith bin.



Multinomial Distribution (多项式分布)
(“multi-dimensional” generalization of binomial distribution)

- Suppose that n balls are thrown into m bins, where each ball is thrown independently such that the ith bin receives the ball with probability p_i, where $p_1 + \cdots + p_m = 1$ is given.

- $(X_1, X_2, \ldots, X_m)$: the ith bin receives exactly X_i balls

- $(X_1, \ldots, X_m)$ takes values $(k_1, \ldots, k_m) \in \{0,1,\ldots,n\}^m$ that $k_1 + \cdots + k_m = n$, and

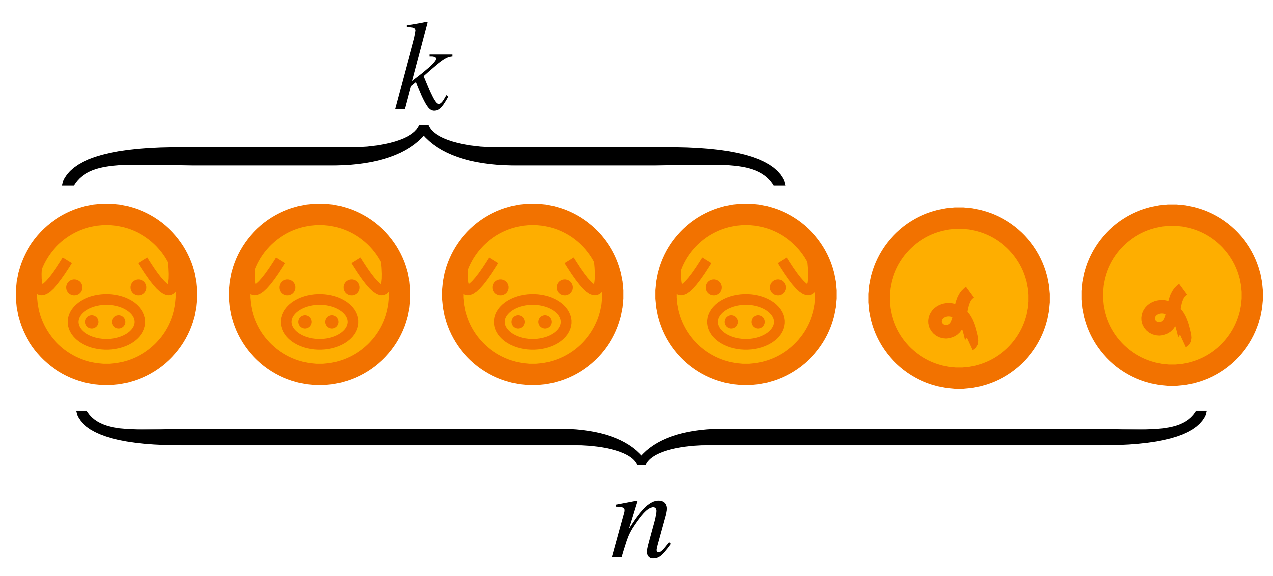
$$p_{(X_1,\ldots,X_m)}(k_1, \ldots, k_m) = \Pr\left(\bigcap_{i=1}^m (X_i = k_i)\right) = \frac{n!}{k_1!k_2!\cdots k_m!} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$

- We say that $(X_1, X_2, \ldots, X_m)$ follows the <u>**multinomial distribution**</u> with parameters m, n, and $p = (p_1, \ldots, p_m) \in [0,1]^m$ such that $p_1 + \cdots + p_m = 1$.

- $X_i \sim \text{Bin}(n, p_i)$ for each individual $1 \le i \le m$. (The <u>**marginal distribution**</u> of X_i is Bin(n, p_i))

Binomial Distribution (二项分布)

(Number of HEADs in n coin flips)



- X: number of successes in n ***i.i.d.*** Bernoulli trials with parameter p
- A binomial random variable X takes values in $\{0,1,\ldots,n\}$, and

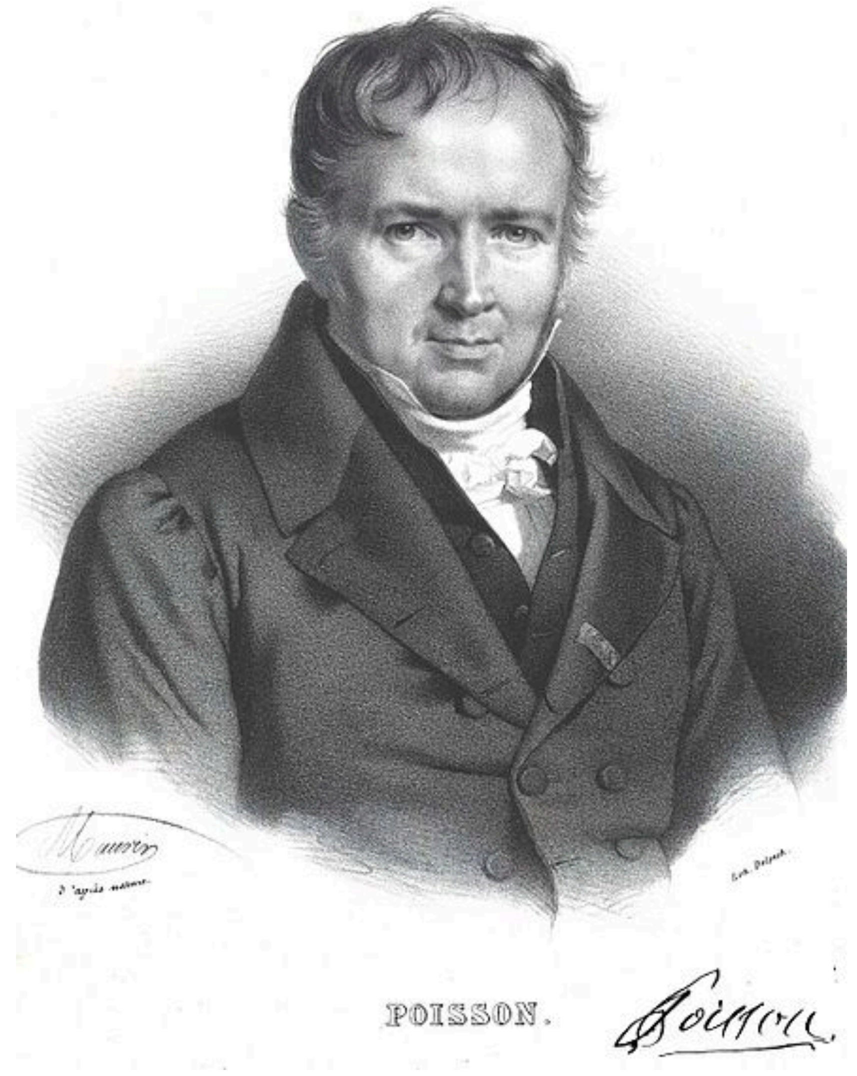
$$p_X(k) = \Pr(X = k) = \binom{n}{k} p^k (1-p)^k, \qquad k = 0,1,\ldots,n$$

- Typical in real life: large unknown population size $n \to \infty$ with known $np = \lambda$

$$p_{\mathrm{Bin}(n,\lambda/n)}(k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} = \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}\cdot\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} \approx \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}$$

A "universal" distribution for all sufficiently large n, knowing the mean $\lambda = np$?

Poisson Distribution (泊松分布)
(Idealized binomial distribution when $n \to \infty$)



- A Poisson random variable X takes values in $\{0,1,2,\ldots\}$, and

$$p_X(k) = \Pr(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0,1,2,\ldots$$

- It is a well-defined probability distribution over $\{0,1,2,\ldots\}$: $\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = 1$

- We say that X follows the **Poisson distribution** with parameter $\lambda > 0$

denoted $X \sim \text{Pois}(\lambda)$

Sum of Poisson Variables

- Independent $X \sim$ Bin(n_1, p), $Y \sim$ Bin$(n_2, p) \implies X + Y \sim$ Bin$(n_1 + n_2, p)$
- By the *heuristics* Bin$(n, p) \approx$ Pois(np), it seems that the following should hold:
  - independent $X \sim$ Pois(λ_1), $Y \sim$ Pois$(\lambda_2) \implies X + Y \sim$ Pois$(\lambda_1 + \lambda_2)$
- **Proof**: $p_{X+Y}(k) = \Pr(X + Y = k) = \sum_{i=0}^{k} \Pr(X = i \cap Y = k - i) = \sum_{i=0}^{k} p_X(i) p_Y(k - i)$

$$= \sum_{i=0}^{k} \frac{\mathrm{e}^{-\lambda_1}\lambda_1^i}{i!} \frac{\mathrm{e}^{-\lambda_2}\lambda_2^{k-i}}{(k-i)!} = \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} = \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^k}{k!}$$

Poisson Approximation

- $(X_1, \ldots, X_m)$ follows the **multinomial distribution** with parameters $m, n, p_1 + \cdots + p_m = 1$
  - n balls are thrown into m bins independently according to the distribution $(p_1, \ldots, p_m)$
  - after all n balls are thrown, the ith bin receives X_i balls
- $(Y_1, \ldots, Y_m)$: each $Y_i \sim$ **Pois**(λ_i) **independently**, where $\lambda_i = np_i$

Proposition: $(X_1, \ldots, X_m)$ is identically distributed as $(Y_1, \ldots, Y_m)$ given that $\sum_{i=1}^{m} Y_i = n$

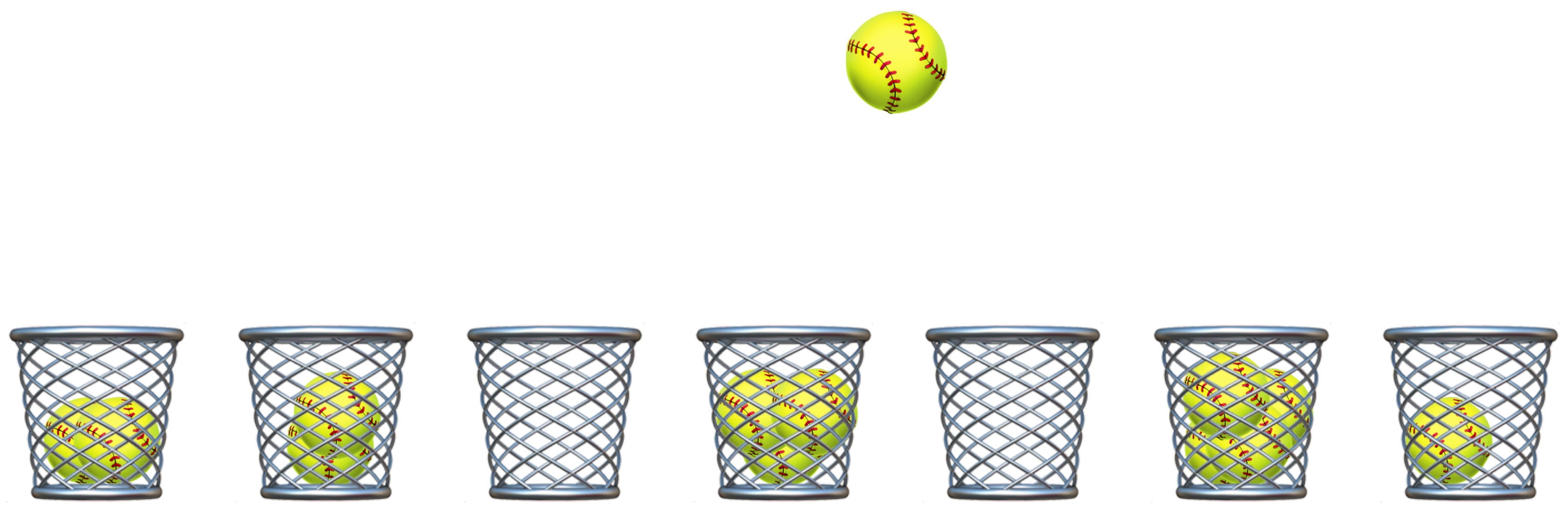
Proof: Observe that $Y_1 + \cdots + Y_m \sim$ Pois(n). For any $k_1, \ldots, k_m \geq 0$ that $k_1 + \cdots + k_m = n$:

$$\Pr[(Y_1, \ldots, Y_m) = (k_1, \ldots, k_m) \mid Y_1 + \cdots + Y_m = n] = \left(\prod_{i=1}^{m} \frac{e^{-np_i}(np_i)^{k_i}}{k_i!} \right) \Big/ \left(\frac{e^{-n} n^n}{n!} \right)$$

$$= \frac{n!}{k_1! \cdots k_m!} p_1^{k_1} \cdots p_m^{k_m} = \Pr[(X_1, \ldots, X_m) = (k_1, \ldots, k_m)]$$

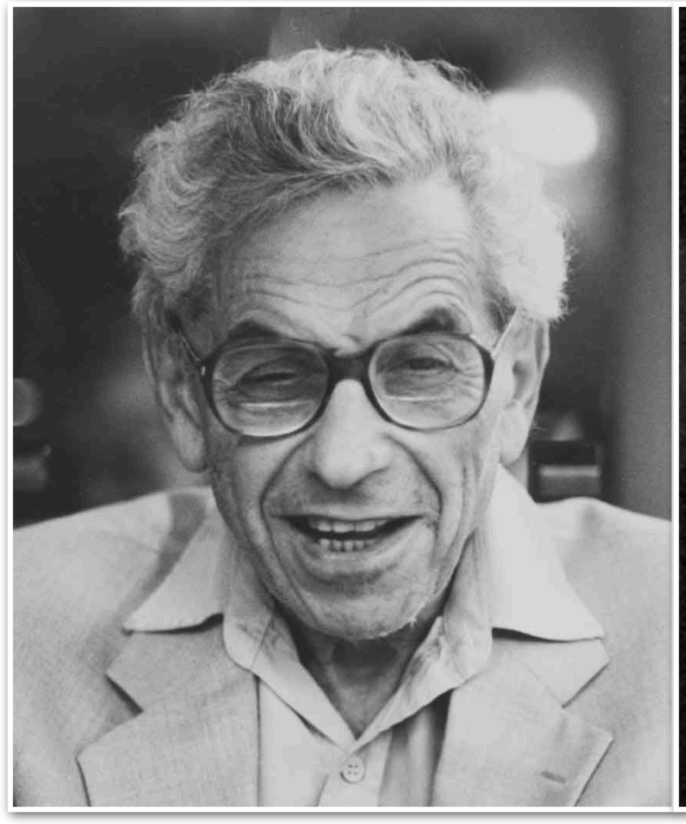
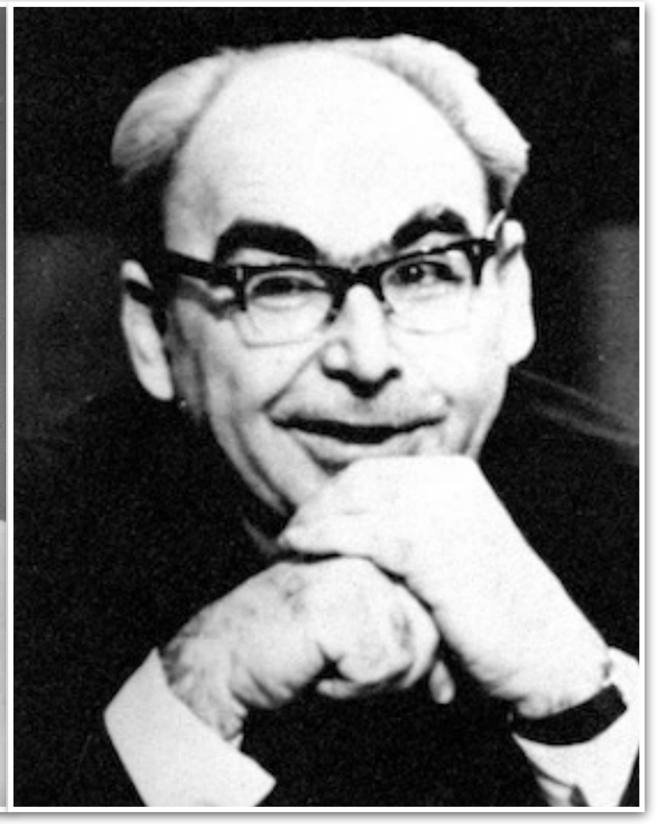
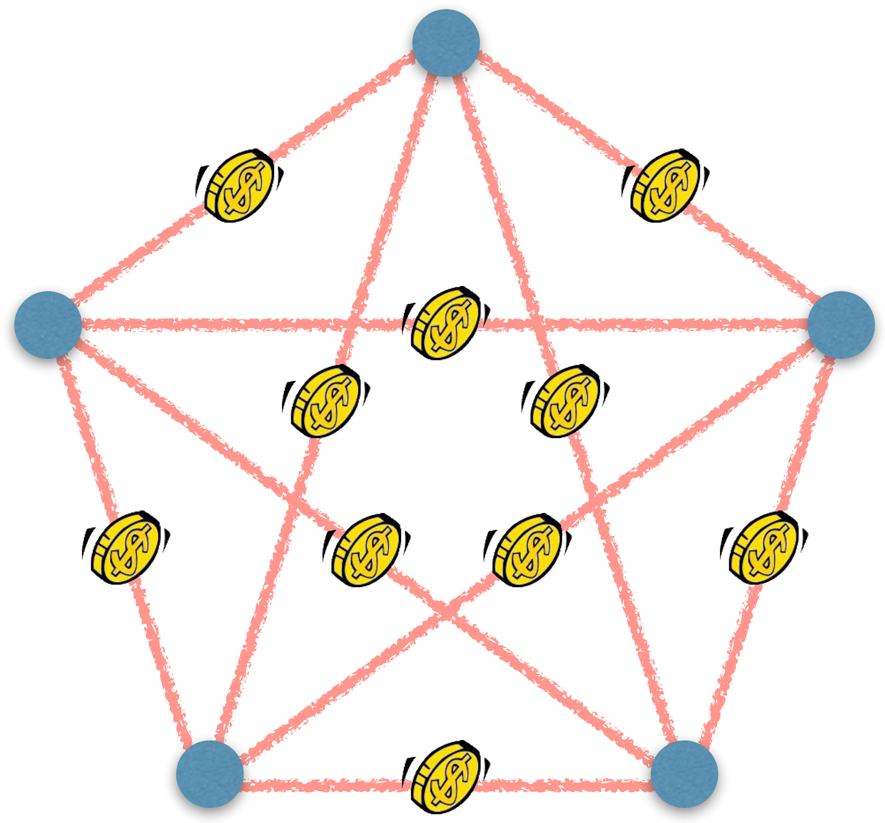
Balls into Bins

(Random mapping)



- Throw n balls into m bins **uniformly at random** (***u.a.r.***).
- Uniform random $f : [n] \to [m]$.
- The numbers of balls received in each bins $(X_1, \ldots, X_m)$ follow the **multinomial distribution** with parameters m, n and $(1/m, \ldots, 1/m)$.
  - **Birthday** problem: the property of being injective (1-1)
  - **Coupon collector** problem: the property of being surjective (onto)
  - **Occupancy** (load balancing) problem: the *maximum load* $\max_i X_i$

Random Graph
(Erdős–Rényi random graph model)

- $G \sim G(n, p)$: There are n vertices. For each pair u, v of vertices, an *i.i.d.* Bernoulli trial with parameter p is conducted, and an edge $\{u, v\}$ is added if the trial succeeds.

- $G(n, 1/2)$ gives the **uniformly distributed** random graph on n vertices.

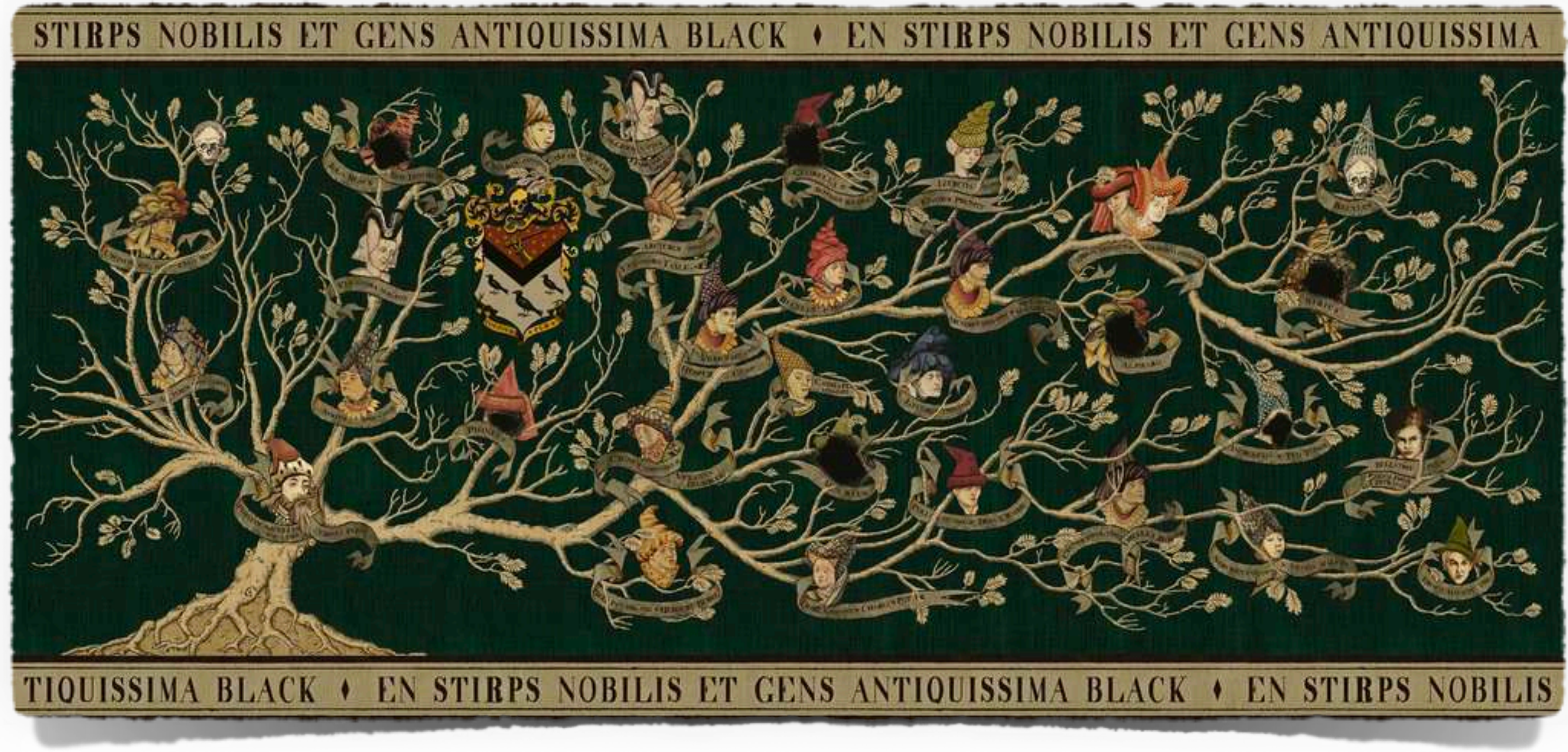
- The number of edges in $G \sim G(n, p)$ follows the binomial distribution Bin $\left(\binom{n}{2}, p\right)$.

  (Therefore, $G(n, p)$ is sometimes also called the ***binomial random graph***)

- Random variables defined by $G \sim G(n, p)$: *chromatic number* $\chi(G)$, *independence number* $\alpha(G)$, *clique number* $\omega(G)$, *diameter diam*(G), *connectivity, max-degree* $\Delta(G)$, *number of triangles, number of hamiltonian cycles,* …

Random Tree

(Galton–Watson branching process)



- A sequence of random variables $X_0, X_1, X_2, \ldots$ recursively defined by

$$X_0 = 1 \text{ and } X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n)}$$

  where $\{\xi_j^{(n)} \mid n, j \geq 0\}$ are *i.i.d.* non-negative integer-valued random variables (e.g. Poisson random variables)

- Random family tree: the jth family member in the nth generation has $\xi_j^{(n)}$ offsprings

- X_n: number of family members in the nth generation

Expectation

E [] = 233

Expectation (数学期望)

- The <u>**expectation**</u> (or <u>**mean**</u>) of a discrete random variable X is defined to be

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

  where p_X denotes the *pmf* of X and the sum is taken over all x that $p_X(x) > 0$

- $\mathbb{E}[X]$ may be ∞ (we assume *absolute convergence* for $\mathbb{E}[X] < \infty$)
  - **Example I**: $p_X(2^k) = 2^{-k}$ for $k = 1,2,\ldots$ (the St. Petersburg paradox)
  - **Example II**: $X \in \mathbb{Z}\backslash\{0\}$ and $p_X(k) = \frac{1}{ak^2}$ where $a = \sum_{k\neq 0} k^{-2} = \frac{\pi^2}{3}$

Perspectives of Expectation

- Computation of expectation:
  - straightforward computation (by definition)
  - **linearity of expectation** (by linearity)
  - **law of total expectation** (by case)
- Upper/lower bounds of expectation:
  - **Jensen's inequality** (by convexity)
  - monotonicity (by coupling)
- Implications of expectation:
  - **averaging principle** (the probabilistic method)
  - **tail inequalities** (the moment method)

Expectation of Indicator

p $\quad$ $1 - p$

- For **Bernoulli random variable** $X \in \{0,1\}$ with parameter p

$$\mathbb{E}[X] = 0 \cdot (1 - p) + 1 \cdot p = p$$

- For the **indicator** random variable $X = I(A)$ of event A, where $X = 1$ if A occurs and $X = 0$ if otherwise (i.e. $\forall \omega \in \Omega$, $X(\omega) = 1$ if $\omega \in A$ and $X(\omega) = 0$ if $\omega \notin A$)

$$\mathbb{E}[X] = 0 \cdot \Pr(A^c) + 1 \cdot \Pr(A) = \Pr(A)$$

Poisson Distribution (泊松分布)

- Expectation of **Poisson random variable** $X \sim \text{Pois}(\lambda)$

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k\geq 0} k\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \\
&= \sum_{k\geq 1} \frac{\mathrm{e}^{-\lambda}\lambda^k}{(k-1)!} \\
&= \sum_{k\geq 0} \frac{\mathrm{e}^{-\lambda}\lambda^{k+1}}{k!} = \lambda\sum_{k\geq 0}\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!} \\
&= \lambda
\end{aligned}$$

Change of Variables
(Law Of The Unconscious Statistician, *LOTUS*)

- For $f : \mathbb{R} \to \mathbb{R}$, for discrete X and $\boldsymbol{X} = (X_1, \ldots, X_n)$:
  - $\mathbb{E}[f(X)] = \sum_x f(x)p_X(x)$
  - $\mathbb{E}[f(X_1, \ldots, X_n)] = \sum_{(x_1,\ldots,x_n)} f(x_1, \ldots, x_n)p_{\boldsymbol{X}}(x_1, \ldots, x_n)$

Proof: Let $Y = f(X_1, \ldots, X_n)$. Then

$$\mathbb{E}[f(X_1, \ldots, X_n)] = \sum_y y \Pr(Y = y) = \sum_y y \sum_{(x_1,\ldots,x_n)\in f^{-1}(y)} \Pr((X_1, \ldots, X_1) = (x_1, \ldots, x_n))$$

$$= \sum_{(x_1,\ldots,x_n)} f(x_1, \ldots, x_n) \Pr((X_1, \ldots, X_1) = (x_1, \ldots, x_n))$$

$$= \sum_{(x_1,\ldots,x_n)} f(x_1, \ldots, x_n)p_{\boldsymbol{X}}(x_1, \ldots, x_n)$$

Linearity of Expectation

- For $a, b \in \mathbb{R}$ and random variables X and Y:
  - $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Proof: $\mathbb{E}[aX + b] = \sum_x (ax + b)p_X(x) = a\sum_x xp_X(x) + b\sum_x p_X(x) = a\mathbb{E}[X] + b$

$$\begin{aligned}
\mathbb{E}[X + Y] &= \sum_{x,y} (x + y) \Pr((X, Y) = (x, y)) \\
&= \sum_x x \sum_y \Pr((X, Y) = (x, y)) + \sum_y y \sum_x \Pr((X, Y) = (x, y)) \\
&= \sum_x x \Pr(X = x) + \sum_y y \Pr(Y = y) = \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}$$

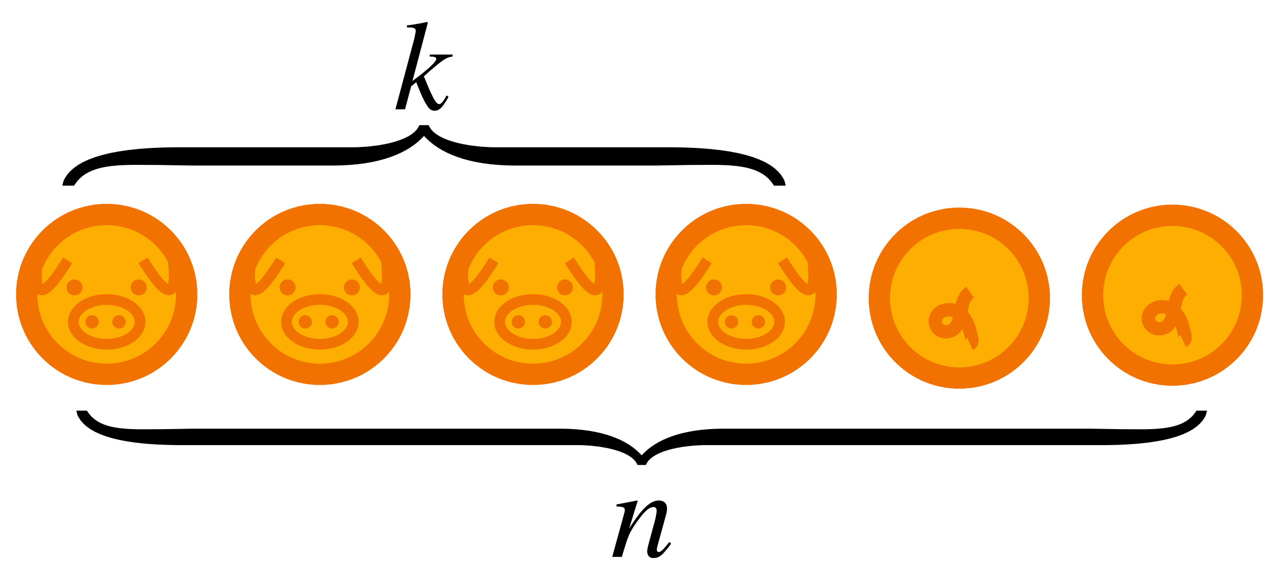
Linearity of Expectation

- For $a, b \in \mathbb{R}$ and random variables X and Y:
  - $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
  - $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
- For **linear** (**affine**) function f on random variables $X_1, \ldots, X_n$

$$\mathbb{E}[f(X_1, \ldots, X_n)] = f(\mathbb{E}[X_1], \ldots, \mathbb{E}[X_n])$$

- It holds for *arbitrarily dependent* $X_1, \ldots, X_n$

Binomial Distribution (二项分布)



- For binomial random variable $X \sim \text{Bin}(n, p)$

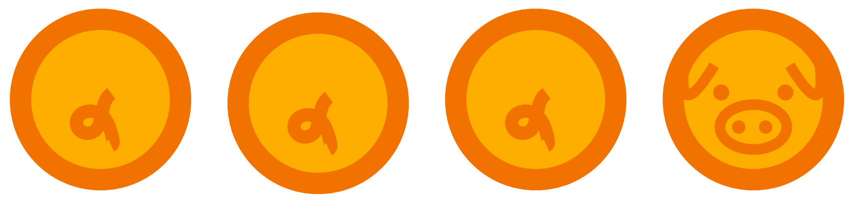
$$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

- **Observation**: $X \sim \text{Bin}(n, p)$ can be expressed as $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with parameter p

- Linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \ = np$$

Geometric Distribution (几何分布)



I_k : 1 1 1 1 0 …

- For geometric random variable $X \sim \text{Geo}(p)$

$$\mathbb{E}[X] = \sum_{k \geq 1} k(1-p)^{k-1} p$$

- **Observation**: $X \sim \text{Geo}(p)$ can be calculated by $X = \sum_{k \geq 1} I_k$, where $I_k \in \{0,1\}$ *indicates* whether <u>all of the first $(k-1)$ trials fail</u>

- Linearity of expectation:

$$\mathbb{E}[X] = \sum_{k \geq 1} \mathbb{E}[I_k] = \sum_{k \geq 1} (1-p)^{k-1} = \frac{1}{p}$$

Negative Binomial Distribution (负二项分布)

- For **negative binomial random variable** X with parameters r, p

$$\mathbb{E}[X] = \sum_{k \geq 1} k \binom{k + r - 1}{k} (1 - p)^k p^r$$

- **Observation**: X can be expressed as $X = (X_1 - 1) + \cdots + (X_r - 1)$, where $X_1, \ldots, X_r$ are i.i.d. geometric random variables with parameter p

- **Linearity of expectation:**

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_r] - r = r(1 - p)/p$$

Hypergeometric Distribution (超几何分布)

- For hypergeometric random variable X with parameters N, M, n

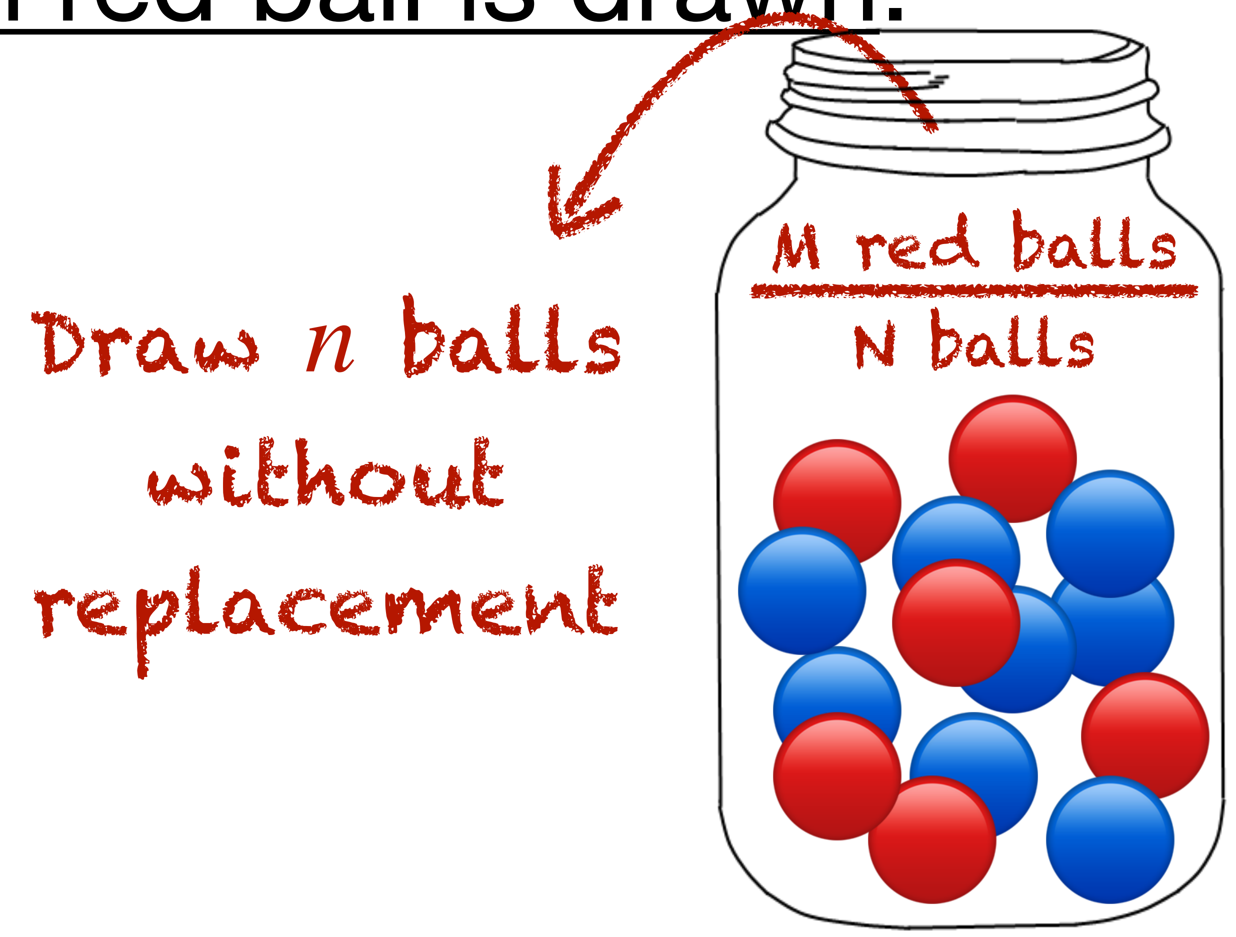
$$\mathbb{E}[X] = \sum_{k=0}^{n} k \binom{M}{k} \binom{N-M}{n-k} \Big/ \binom{N}{n}$$

- **Observation:** each red ball (success) is drawn with probability $\binom{N-1}{n-1} / \binom{N}{n} = \frac{n}{N}$.

  Then $X = X_1 + \cdots + X_M$, where $X_i \in \{0,1\}$ *indicates* whether the ith red ball is drawn.

- **Linearity of expectation:**

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_M] = \frac{nM}{N}$$



Pattern Matching

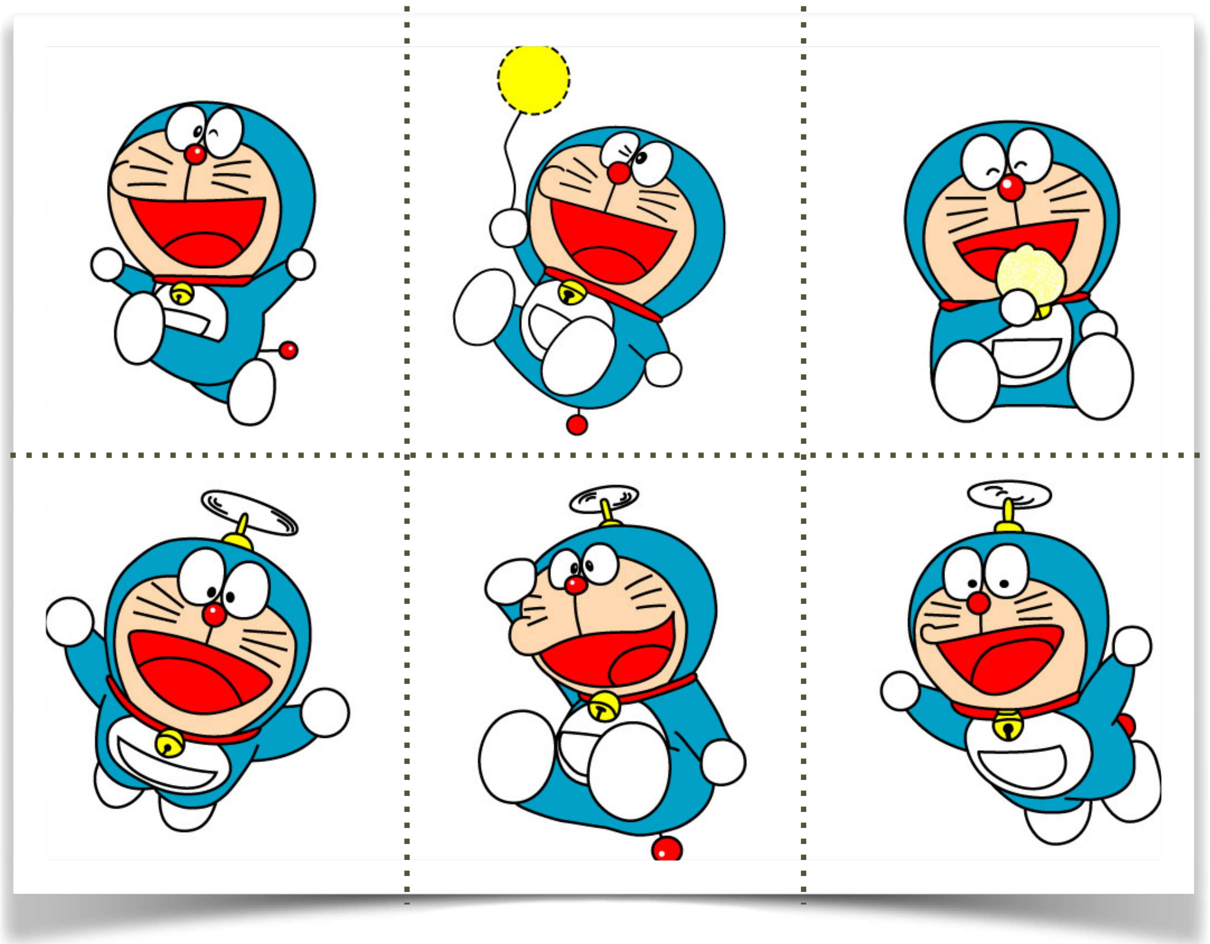
- $s = (s_1, \ldots, s_n) \in Q^n$: uniform random string of n letters from alphabet Q with $|Q| = q$
- For **pattern** $\pi \in Q^k$, let X be the number of appearances of π in s as substring
- Let $I_i \in \{0,1\}$ *indicate* that $\pi = (s_i, s_{i+1}, \ldots, s_{i+k-1})$. Then $X = \sum_{i=1}^{n-k+1} I_i$
- **Linearity of expectation:**

$$\mathbb{E}[X] = \sum_{i=1}^{n-k+1} \mathbb{E}[I_i] = (n-k+1)q^{-k}$$

- Expected time (position) for the first appearance? It may depend on the pattern π.

Optional Stopping Theorem (OST)

Coupon Collector

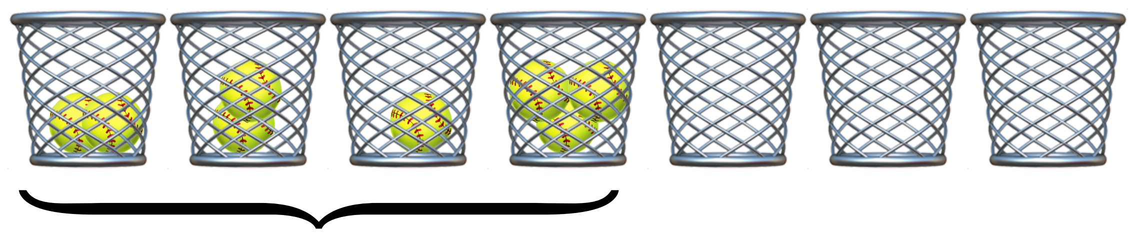


- Each cookie box comes with a uniform random coupon.
  - Number of cookie boxes opened to collect all n types of coupons
- **Balls-into-bins model:** throw balls one-by-one ***u.a.r.*** to occupy all n bins
  - X : total number of balls thrown to make all n bins nonempty
  - X_i : number of balls thrown while there are exactly $(i-1)$ nonempty bins
- X_i is **geometric** with parameter $p_i = 1 - \dfrac{i-1}{n}$ and $X = \sum_{i=1}^{n} X_i$

- **Linearity of expectation:**

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n\sum_{i=1}^{n}\frac{1}{i} = nH(n) \approx n\ln n$$

(Harmonic number)



$i-1$

Double Counting

- For **nonnegative** random variable X that takes values in $\{0,1,2,\ldots\}$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} \Pr[X > k]$$

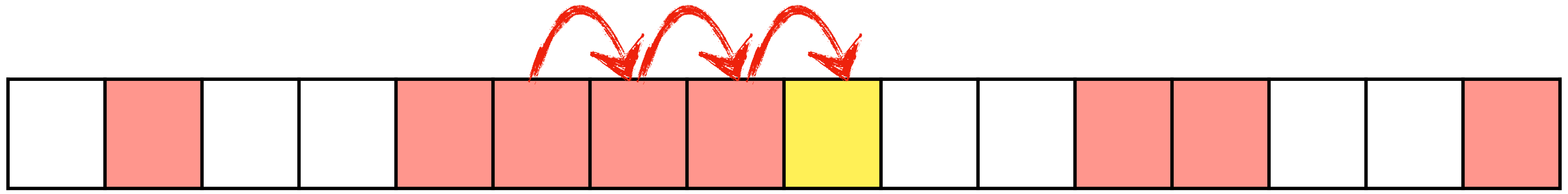
- **Proof I** (Double Counting):

$$\mathbb{E}[X] = \sum_{x \geq 0} x \Pr[X = x] = \sum_{x \geq 0} \sum_{k=0}^{x-1} \Pr[X = x] = \sum_{k \geq 0} \sum_{x > k} \Pr[X = x] = \sum_{k \geq 0} \Pr[X > k]$$

- **Proof II (Linearity of Expectation)**: Let $I_k \in \{0,1\}$ indicate whether $X > k$.

  Then $X = \sum_{k \geq 0} I_k$. By linearity, $\mathbb{E}[X] = \sum_{k \geq 0} \mathbb{E}[I_k] = \sum_{k \geq 0} \Pr[X > k]$

Open Addressing with Uniform Hashing



- **Hash table:** n *keys* from a universe U are mapped to m *slots* by hash function

$$h : U \to [m]$$

- **Open addressing** (开放寻址): hash collision is resolved by a probing strategy — when searching for a key $x \in U$, the ith probed slot is given by $h(x, i)$
  - **Linear probing:** $h(x,i) = h(x) + i \pmod{m}$
  - **Quadratic probing:** $h(x,i) = h(x) + c_1 i + c_2 i^2 \pmod{m}$
  - **Double hashing:** $h(x,i) = h_1(x) + i \cdot h_2(x) \pmod{m}$
  - **Uniform hashing:** $h(x,i) = \pi(i)$ where π is a uniform random permutation of $[m]$

Open Addressing with Uniform Hashing

- In a hash table with **load factor** $\alpha = n/m$, assuming uniform hashing, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$.

- **Proof**: Let X be the number of probes in an unsuccessful search.

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} \Pr(X > k) = 1 + \sum_{k=1}^{\infty} \Pr(X > k)$$

$$= 1 + \sum_{k=1}^{\infty} \Pr\left(\cap_{i=1}^{k} A_i\right)$$ (where A_i is the event that the ith probed slot is occupied)

$$= 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \Pr\left(A_i \mid \cap_{j<i} A_j\right)$$ (by chain rule)

$$= 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{n-i+1}{m-i+1} \leq 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{n}{m} = 1 + \sum_{k=1}^{\infty} \alpha^k = \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

Principle of Inclusion-Exclusion

- Let $I(A) \in \{0,1\}$ be the **indicator random variable** of event A. It's easy to verify:
  - ★ $I(A^c) = 1 - I(A)$
  - ✤ $I(A \cap B) = I(A) \cdot I(B)$
- For events $A_1, A_2, \ldots, A_n$:

$$I\left(\bigcup_{i=1}^{n} A_i\right) \overset{(★)}{=} 1 - I\left[\left(\bigcup_{i=1}^{n} A_i\right)^c\right] \overset{\text{(De Morgan's law)}}{=} 1 - I\left(\bigcap_{i=1}^{n} A_i^c\right) \overset{(✤)}{=} 1 - \prod_{i=1}^{n} I(A_i^c) \overset{(★)}{=} 1 - \prod_{i=1}^{n} (1 - I(A_i))$$

$$\overset{\text{(binomial theorem)}}{=} 1 - \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \prod_{i \in S} I(A_i) \overset{(✤)}{=} \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} I\left(\bigcap_{i \in S} A_i\right)$$

Principle of Inclusion-Exclusion

- Let $I(A) \in \{0,1\}$ be the **indicator random variable** of event A.
- For events $A_1, A_2, \ldots, A_n$:

$$I\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} I\left(\bigcap_{i \in S} A_i\right)$$

- By **linearity of expectation:**

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \Pr\left(\bigcap_{i \in S} A_i\right)$$

Boole-Bonferroni Inequality

- For events $A_1, A_2, \ldots, A_n$:

$$I\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n}(1 - I(A_i)) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{S \in \binom{\{1,\ldots,n\}}{k}} I\left(\bigcap_{i \in S} A_i\right)$$

- **Observation:** $X_k \triangleq \binom{\sum_{i=1}^{n} I(A_i)}{k} = \sum_{S \in \binom{\{1,\ldots,n\}}{k}} \prod_{i \in S} I\left(A_i\right) = \sum_{S \in \binom{\{1,\ldots,n\}}{k}} I\left(\bigcap_{i \in S} A_i\right)$

  and X_k as a binomial coefficient is *unimodal* in k

- For unimodal sequence X_k: $\sum_{k \le 2t}(-1)^{k-1}X_k \le \sum_{k=1}^{n}(-1)^{k-1}X_k \le \sum_{k \le 2t+1}(-1)^{k-1}X_k$

- Take expectation. By linearity of expectation $\Longrightarrow$ Bonferroni inequality

Limitation of Linearity

- Infinite sum: $X_1, X_2, \ldots$

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i\right] = \sum_{i=1}^{\infty} \mathbb{E}[X_i] \text{ if the } \textit{absolute convergence} \sum_{i=1}^{\infty} \mathbb{E}[\,|X_i|\,] < \infty \text{ holds}$$

This is possible: $\mathbb{E}\left[\sum_{i=1}^{\infty} X_i\right] < \infty$ and $\sum_{i=1}^{\infty} \mathbb{E}[X_i] < \infty$ but $\mathbb{E}\left[\sum_{i=1}^{\infty} X_i\right] \neq \sum_{i=1}^{\infty} \mathbb{E}[X_i]$

Counterexample: the martingale betting strategy in a fair gambling game

- A random number of random variables: $X_1, X_2, \ldots, X_N$ for random N

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i\right] = \mathbb{E}[N]\mathbb{E}[X_1] \textbf{ ?}$$

Conditional Expectation (条件期望)

- The <u>conditional expectation</u> of a discrete random variable X given that event A occurs, is defined by

$$\mathbb{E}[X \mid A] = \sum_x x \Pr(X = x \mid A)$$

where the sum is taken over all x that $\Pr(X = x \mid A) > 0$

- To be well-defined, assume:
  - $\Pr(A) > 0$
  - the sum $\sum_x x \Pr(X = x \mid A)$ converges absolutely

Conditional Distribution (条件分布)

- The <u>probability mass function</u> $p_{X|A} : \mathbb{Z} \to [0,1]$ of a discrete random variable X given that event A occurs, is given by

$$p_{X|A}(x) = \Pr(X = x \mid A)$$

- $(X \mid A)$ can now be seen as a well-defined discrete random variable, whose distribution is described by the *pmf* $p_{X|A}$
- $\mathbb{E}[X \mid A] = \sum_x x \Pr(X = x \mid A)$ is just the expectation of $(X \mid A)$
- $\mathbb{E}[X \mid A]$ satisfies the properties of expectation, e.g. **linearity of expectation**

Law of Total Expectation

- Let X be a discrete random variable with finite $\mathbb{E}[X]$. Let events $B_1, B_2, \ldots, B_n$ be a partition of Ω such that $\Pr(B_i) > 0$ for all i.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid B_i] \Pr(B_i)$$

- The **law of total probability** is now a special case with $X = I(A)$

Proof: $\mathbb{E}[X] = \sum_{x} x \Pr(X = x) = \sum_{x} x \sum_{i=1}^{n} \Pr(X = x \mid B_i) \Pr(B_i)$ (law of total prob.)

$$= \sum_{i=1}^{n} \Pr(B_i) \sum_{x} x \Pr(X = x \mid B_i) = \sum_{i=1}^{n} \mathbb{E}[X \mid B_i] \Pr(B_i)$$

Analysis of *QuickSort*

QSort(A): an array A of n distinct entries

If $n > 1$ then do:

choose a **pivot** $x = A[1]$;

partition A into L with all entries $< x$,

and R with all entries $> x$;

QSort(L) and QSort(R);

- A *comparison-based* sorting algorithm
  - worst-case complexity: $O(n^2)$
  - average-case complexity: ? $t(n) = O(n \ln n)$ verified by induction

- Let $t(n) = \mathbb{E}[X_n]$, where X_n is the number of comparisons used in QSort(A) on a uniform random permutation A of n distinct numbers.

- **Law of total expectation:** Let B_i be the event that $A[1]$ is the ith smallest in A.

$$t(n) = \mathbb{E}[X_n] = \sum_{i=1}^{n} \mathbb{E}[X_n \mid B_i] \Pr(B_i) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[n - 1 + X_{i-1} + X_{n-i}] = n - 1 + \frac{2}{n} \sum_{i=0}^{n-1} t(i)$$

$$t(0) = t(1) = 0$$

Analysis of *QuickSort*

QSort(A): an array A of n distinct entries

If $n > 1$ then do:

  choose a **pivot** $x = A[1]$;

  partition A into L with all entries $< x$,

    and R with all entries $> x$;

  QSort(L) and QSort(R);

- Uniform random input:
  - A is a uniform random permutation of $a_1 < \cdots < a_n$
- Let $X_{ij} \in \{0,1\}$ *indicate* whether a_i and a_j are compared within QSort(A).
  - **Observation I**: each pair of a_i, a_j are compared at most once.

    $\Longrightarrow$ total number of comparisons is $X = \sum_{i<j} X_{ij}$
  - **Observation II**: if a_i, a_j are still in the same array, then so are all a_k for $i < k < j$.

    a_i, a_j are compared iff one of them is chosen as pivot when they are in the same array.

    $$\Longrightarrow \mathbb{E}[X_{ij}] = \Pr(a_i, a_j \text{ are compared}) = \Pr(\{a_i, a_j\} \mid \{a_i, a_{i+1}, \ldots, a_j\}) = \frac{2}{j-i+1}$$
- **Linearity of expectation:**

$$\mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} \frac{2}{j-i+1} = \sum_{i=1}^{n} \sum_{k=2}^{n-i+1} \frac{2}{k} \le 2 \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{k} = 2nH(n) = 2n \ln n + O(n)$$

Conditional Expectation (条件期望)

Y \ X	x_1	x_2	x_3	x_4	$p_Y(y)$ ↓
y_1	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{8}{32}$
y_2	$\frac{3}{32}$	$\frac{6}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{15}{32}$
y_3	$\frac{9}{32}$	0	0	0	$\frac{9}{32}$
$p_X(x)$ →	$\frac{16}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{32}{32}$

- For random variables X, Y, the conditional expectation:

$$\mathbb{E}[X \mid Y]$$

is a random variable $f(Y)$ whose value is $f(y) = \mathbb{E}[X \mid Y = y]$ when $Y = y$

- Naturally generalized to $\mathbb{E}[X \mid Y, Z]$ for random variables X, Y, Z
- **Examples**:
  - $\mathbb{E}[X \mid Y]$: average height of the country of a random person on earth
  - $\mathbb{E}[X \mid Y, Z]$: average height of the gender of the country of a random person

Y \ X	x_1	x_2	x_3	x_4	$p_Y(y)$ ↓
y_1	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{8}{32}$
y_2	$\frac{3}{32}$	$\frac{6}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{15}{32}$
y_3	$\frac{9}{32}$	0	0	0	$\frac{9}{32}$
$p_X(x)$ →	$\frac{16}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{32}{32}$

Conditional Expectation (条件期望)

- For random variables X, Y, the conditional expectation:

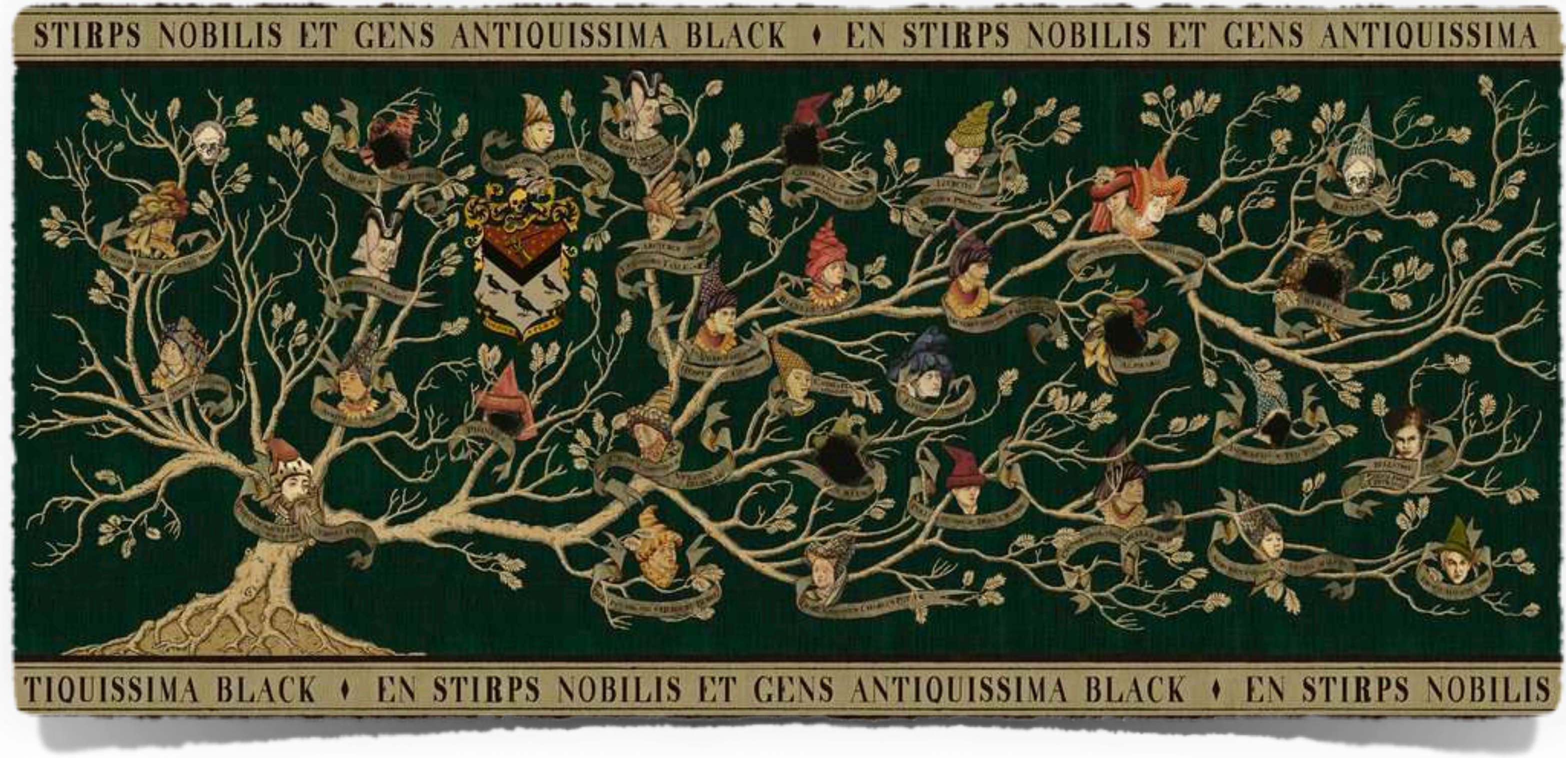
$$\mathbb{E}[X \mid Y]$$

is a random variable $f(Y)$ whose value is $f(y) = \mathbb{E}[X \mid Y = y]$ when $Y = y$

- **Law of Total Expectation:** $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$
- **Proof:** $\mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_y \mathbb{E}[X \mid Y = y] \Pr(Y = y)$ (by definition)

$$= \mathbb{E}[X]$$ (law of total expectation)

Random Family Tree



- $X_0, X_1, X_2, \ldots$ is defined by $X_0 = 1$ and $X_{n+1} = \sum_{j=1}^{X_n} \xi_j^{(n)}$
  where $\xi_j^{(n)} \in \mathbb{Z}_{\geq 0}$ are *i.i.d.* random variables with mean value $\mu = \mathbb{E}[\xi_j^{(n)}]$

- $X_0 = 1$ and $\mathbb{E}[X_1] = \mathbb{E}[\xi_1^{(0)}] = \mu$

- $\mathbb{E}[X_n \mid X_{n-1} = k] = \mathbb{E}\left[\sum_{j=1}^{k} \xi_j^{(n-1)} \middle| X_{n-1} = k\right] = k\mu \implies \mathbb{E}[X_n \mid X_{n-1}] = X_{n-1}\mu$

- $\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n \mid X_{n-1}]] = \mathbb{E}[X_{n-1}\mu] = \mathbb{E}[X_{n-1}] \cdot \mu = \mu^n$

$$\implies \mathbb{E}\left[\sum_{n \geq 0} X_n\right] = \sum_{n \geq 0} \mathbb{E}[X_n] = \sum_{n \geq 0} \mu^n = \begin{cases} \frac{1}{1-\mu} & \text{if } 0 < \mu < 1 \\ \infty & \text{if } \mu \geq 1 \end{cases}$$

Jensen's Inequality

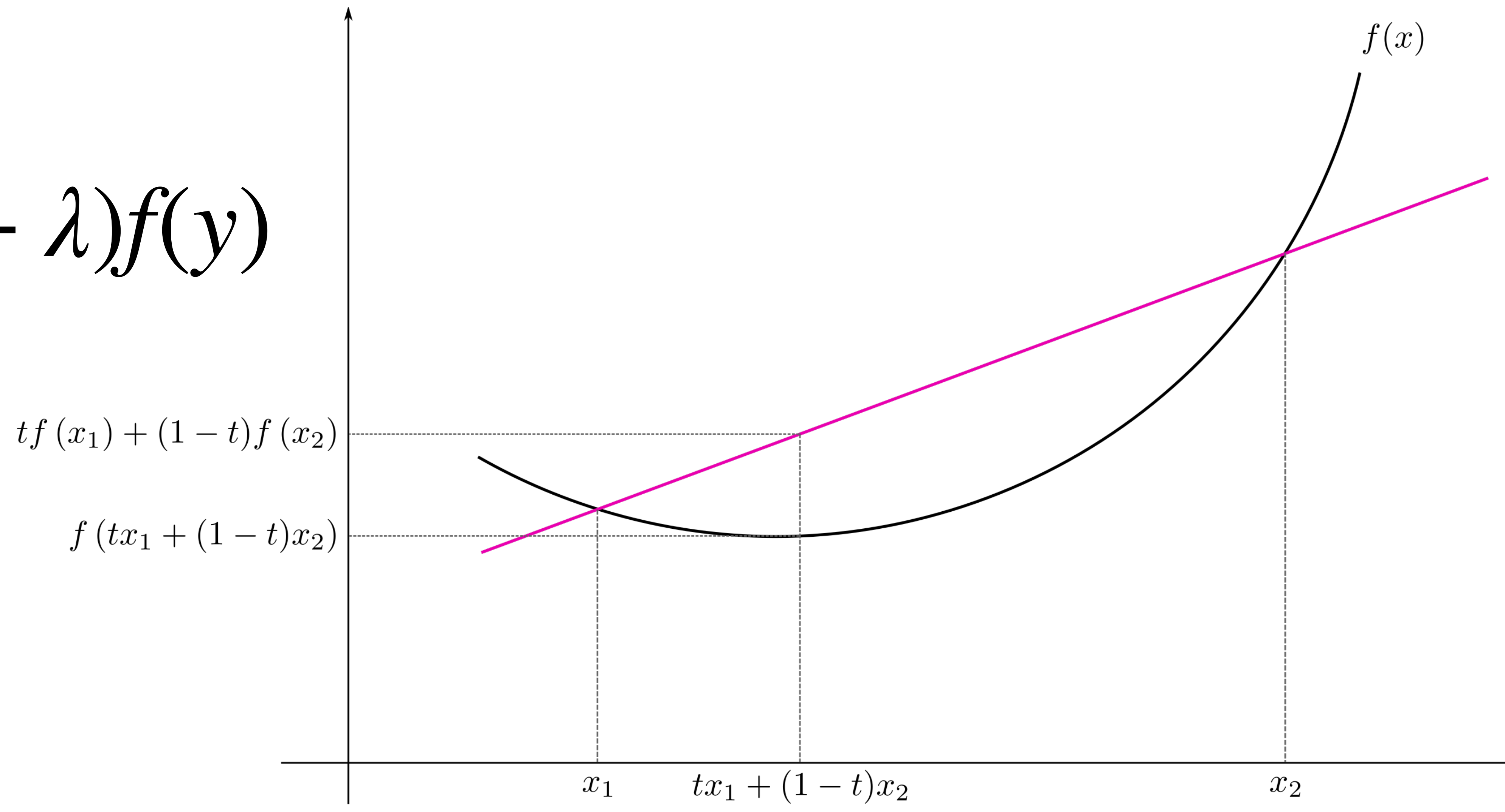
- For general (non-linear) function $f(X)$ of random variable X

  we don't have $\mathbb{E}[f(X)] = f(\mathbb{E}[X])$

- But if the convexity of f is known, then the **Jensen's inequality** applies:
  - f is **convex** $\iff f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$

    $\implies \mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$
  - f is **concave** $\iff f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$

    $\implies \mathbb{E}[f(X)] \leq f(\mathbb{E}[X])$



Monotonicity of Expectation

- For random variables X and Y, for $c \in \mathbb{R}$:
  (Y stochastically dominates X)
  - If $X \leq Y$ **a.s.** (*almost surely*, i.e. $\Pr(X \leq Y) = 1$), then $\mathbb{E}[X] \leq \mathbb{E}[Y]$
  - If $X \leq c$ ($X \geq c$) **a.s.**, then $\mathbb{E}[X] \leq c$ ($\mathbb{E}[X] \geq c$)
  - $\mathbb{E}[\,|X|\,] \geq |\mathbb{E}[X]| \geq 0$

Proof: $\mathbb{E}[X] = \sum_x x \Pr(X = x) = \sum_x x \sum_y \Pr((X, Y) = (x, y))$

$$= \sum_x x \sum_{y \geq x} \Pr((X, Y) = (x, y)) = \sum_y \sum_{x \leq y} x \Pr((X, Y) = (x, y))$$

$$\leq \sum_y \sum_{x \leq y} y \Pr((X, Y) = (x, y)) \leq \sum_y y \Pr(Y = y) = \mathbb{E}[Y]$$

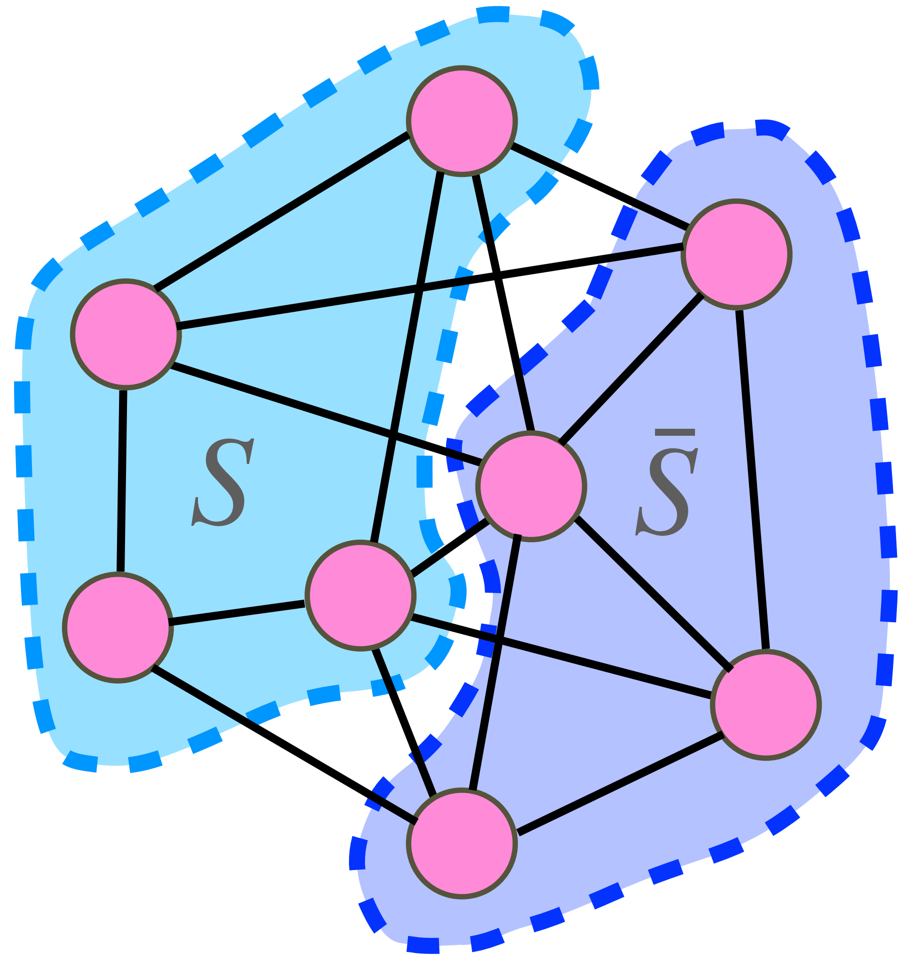
Averaging Principle

- $\Pr(X \geq \mathbb{E}[X]) > 0 \quad \Longleftarrow \quad$ if $\Pr(X < c) = 1$ then $\mathbb{E}[X] < c$
- $\Pr(X \leq \mathbb{E}[X]) > 0 \quad \Longleftarrow \quad$ if $\Pr(X > c) = 1$ then $\mathbb{E}[X] > c$
- By the Probabilistic Method:

$$\exists \omega \in \Omega \text{ such that } X(\omega) \geq \mathbb{E}[X]$$

$$\exists \omega \in \Omega \text{ such that } X(\omega) \leq \mathbb{E}[X]$$

Maximum Cut



- For an undirected graph $G(V, E)$:
  - Find an $S \subseteq V$ with largest **cut** $\delta S \triangleq \{\{u, v\} \in E \mid u \in S \wedge v \notin S\}$
- **NP-hard** problem (very unlikely to have efficient algorithms)

The average cut generated by pairwise independent bits is $\geq |E|/2$.

Proposition: There always exists a large enough cut of size $|\delta S| \geq |E|/2$.

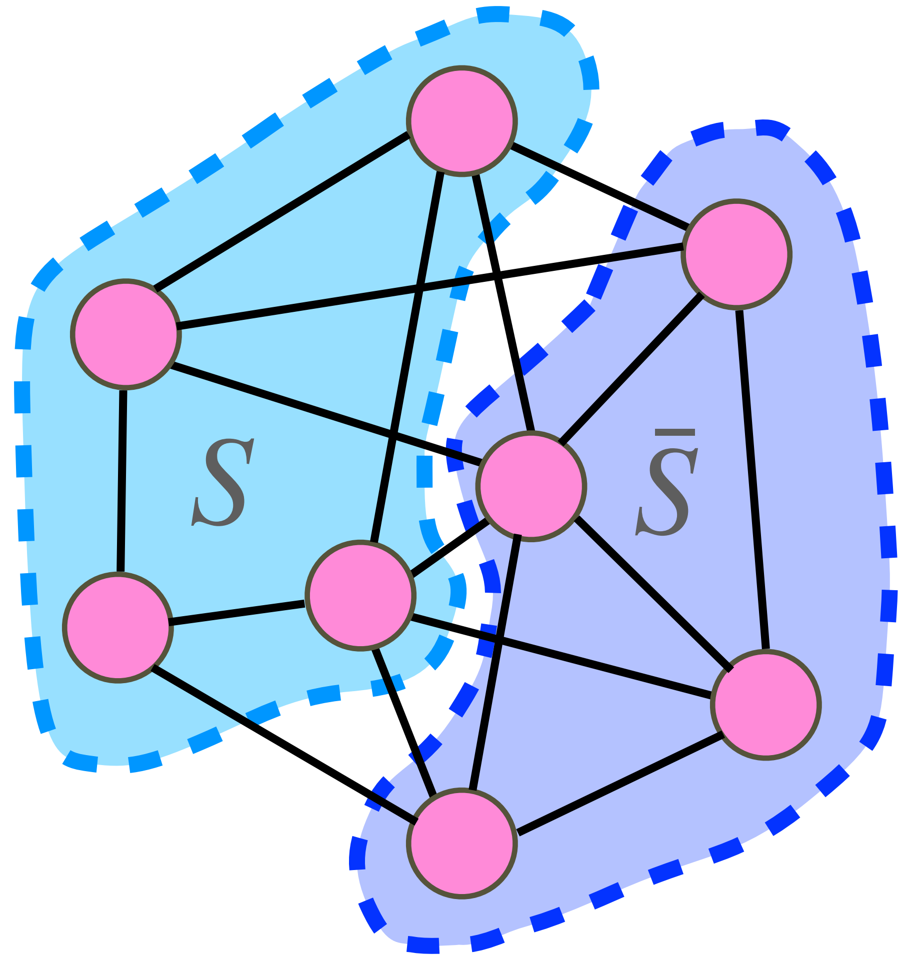
Proof: Let $Y_v \in \{0,1\}$, for $v \in V$, be ~~mutually~~ pairwise independent uniform random bits.

Each $v \in V$ joins S iff $Y_v = 1$. Then it holds that $|\delta S| = \sum_{\{u,v\}\in E} I(Y_u \neq Y_v)$.

By **linearity of expectation**: $\mathbb{E}[|\delta S|] = \sum_{\{u,v\}\in E} \Pr(Y_u \neq Y_v) = |E|/2$.

Due to **the probabilistic method**: There exists such $S \subseteq V$ with $|\delta S| \geq |E|/2$.

Maximum Cut



- For an undirected graph $G(V, E)$:
  - Find an $S \subseteq V$ with largest **cut** $\delta S \triangleq \{\{u, v\} \in E \mid u \in S \wedge v \notin S\}$
- **NP-hard** problem (very unlikely to have efficient algorithms)

Parity Search:

for all $\boldsymbol{b} \in \{0,1\}^{\lceil \log_2(n+1) \rceil}$:

  initialize $S_{\boldsymbol{b}} = \emptyset$;

  for $i = 1,2,\ldots,n$:

    if $\bigoplus_{j:\lfloor i/2^j \rfloor \bmod 2 = 1} b_j = 1$ then v_i joins $S_{\boldsymbol{b}}$;

return the $S_{\boldsymbol{b}}$ with the largest cut $\delta S_{\boldsymbol{b}}$;

Guarantees to return an $S \subseteq V$ with $|\delta S| \geq |E|/2$.