Randomized Algorithms

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Martingales

Definition:

A sequence of random variables $X_0, X_1, \ldots$ is a **martingale** if for all $i > 0$,

$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

$\forall x_0, x_1, \ldots, x_{i-1}$,

$$\mathbf{E}[X_i \mid X_0 = x_0, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] = x_{i-1}$$

Azuma's Inequality:

Let $X_0, X_1, \ldots$ be a martingale such that, for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c_k,$$

Then

$$\Pr[|X_n - X_0| \geq t] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

- For a sequence, if in each step:
  - averagely no change to the current value (martingale),
  - no big jump,
- the final does not deviate far from the initial.

Azuma's Inequality:

Let $X_0, X_1, \ldots$ be a martingale such that, for all $k \geq 1$,

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Then

$$\Pr\left[|X_n - X_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

1. Represent total difference as sum of step-wise differences.

   Let $Y_i = X_i - X_{i-1}$. $\quad X_n - X_0 = \sum_{i=1}^{n} Y_i$

2. Apply Markov's inequality to the moment generating function.

   $$\Pr[\sum_{i=1}^{n} Y_i \geq t] = \Pr[e^{\lambda \sum_{i=1}^{n} Y_i} \geq e^{\lambda t}] \leq \frac{\mathbf{E}[e^{\lambda \sum_{i=1}^{n} Y_i}]}{e^{\lambda t}}$$

3. Bound the moment generating function.

   by martingale property & convexity of MGF

Generalization

Definition:

$Y_0, Y_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$ if, for all $i \geq 0$,

- Y_i is a function of $X_0, X_1, \ldots, X_i$;
- $\mathbf{E}[Y_{i+1} \mid X_0, \ldots, X_i] = Y_i$.

Azuma's Inequality (general version):

Let $Y_0, Y_1, \ldots$ be a martingale with respect to $X_0, X_1, \ldots$ such that, for all $k \geq 1$,

$$|Y_k - Y_{k-1}| \leq c_k,$$

Then

$$\Pr\left[|Y_n - Y_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

Doob Sequence

Definition (Doob sequence):

The Doob sequence of a function f with respect to a sequence $X_1, \ldots, X_n$ is

$$Y_i = \mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i]$$

$$Y_0 = \mathbf{E}[f(X_1, \ldots, X_n)] \dashrightarrow Y_n = f(X_1, \ldots, X_n)$$

Doob sequence:

$$Y_i = \mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i]$$

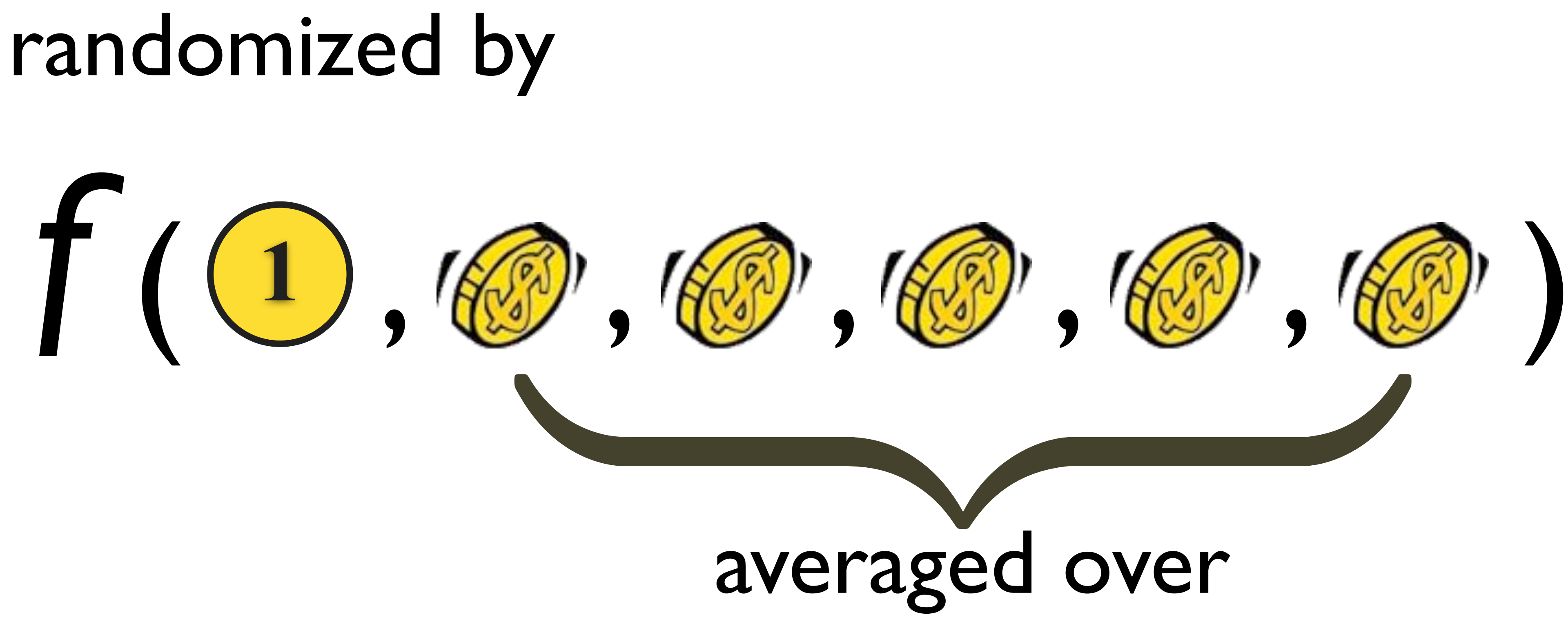
Doob sequence is a martingale:

$$\mathbf{E}[Y_i \mid X_1, \ldots, X_{i-1}] = Y_{i-1}$$

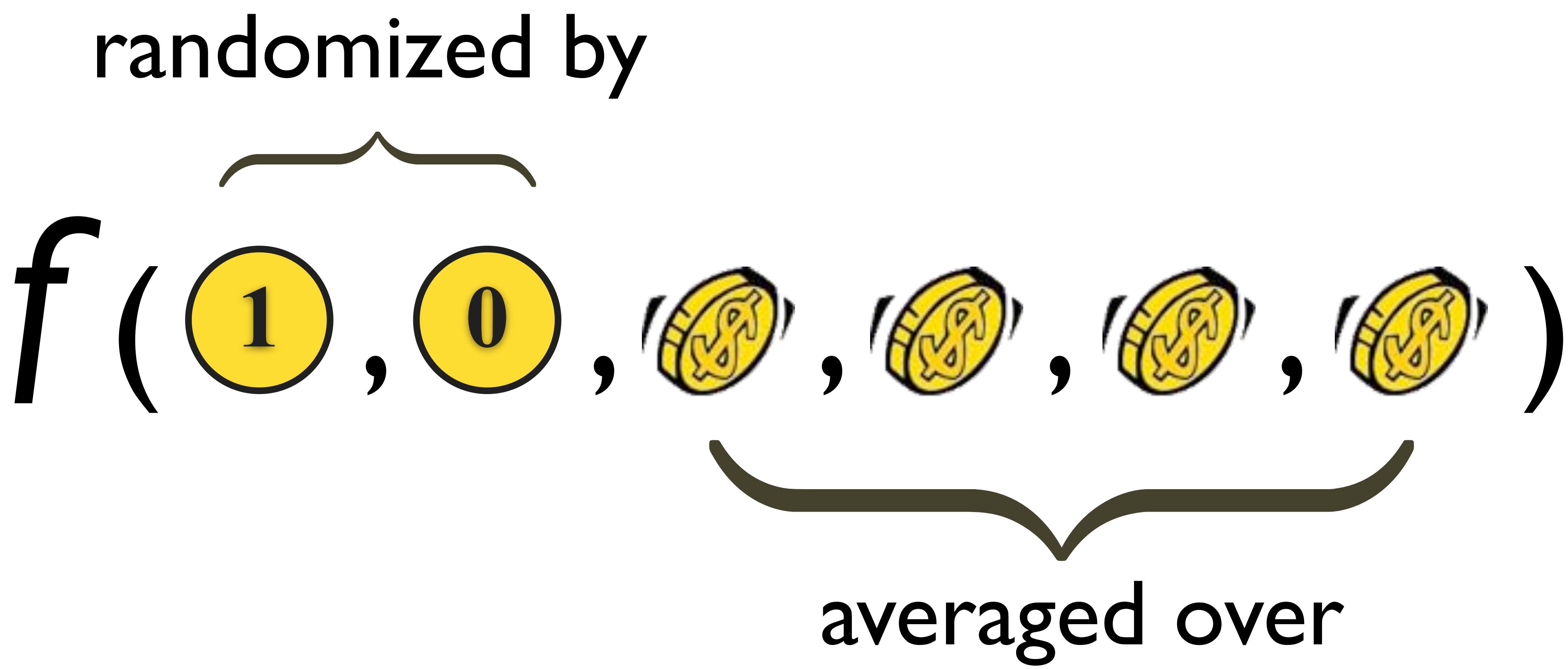
Proof:

$$\begin{aligned}
&\mathbf{E}[Y_i \mid X_1, \ldots, X_{i-1}] \\
&= \mathbf{E}[\mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i] \mid X_1, \ldots, X_{i-1}] \\
&= \mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_{i-1}] \\
&= Y_{i-1}
\end{aligned}$$

Doob Sequence



Doob Sequence



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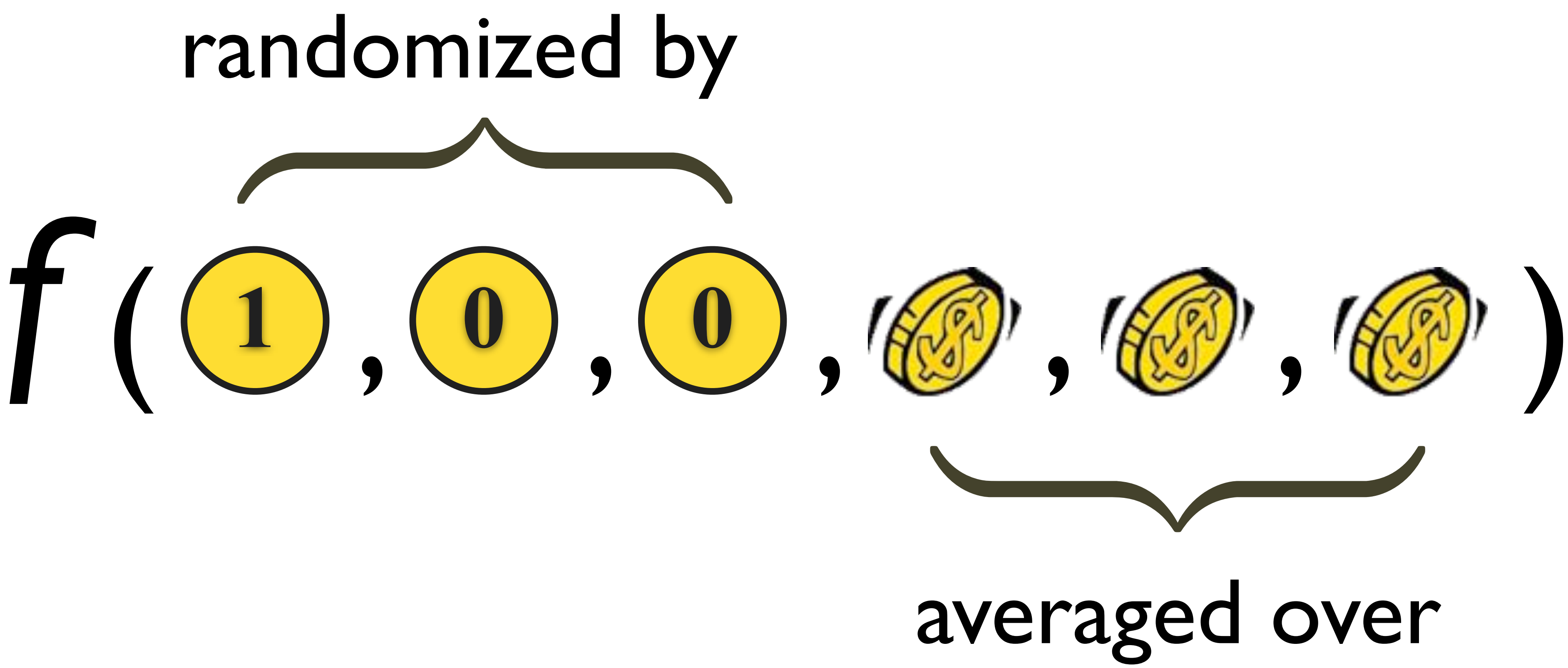

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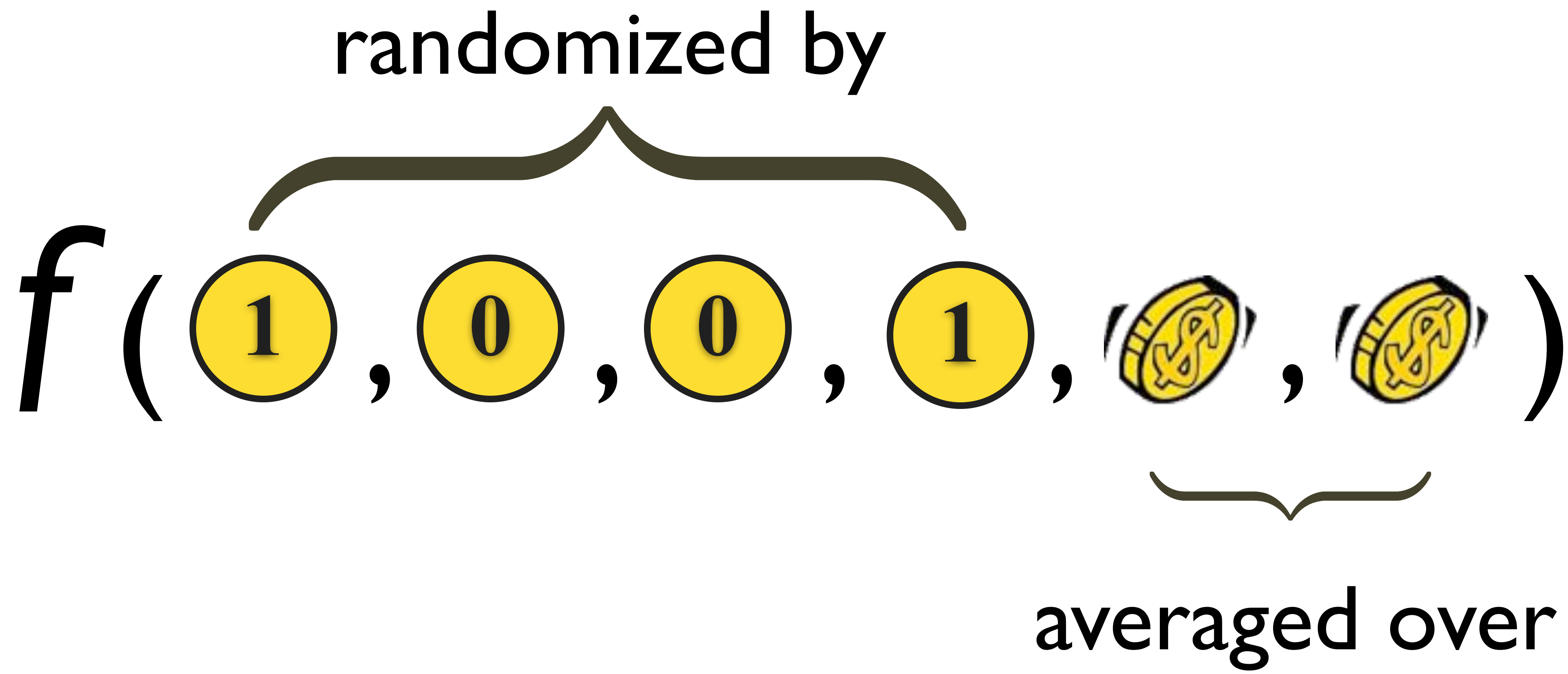
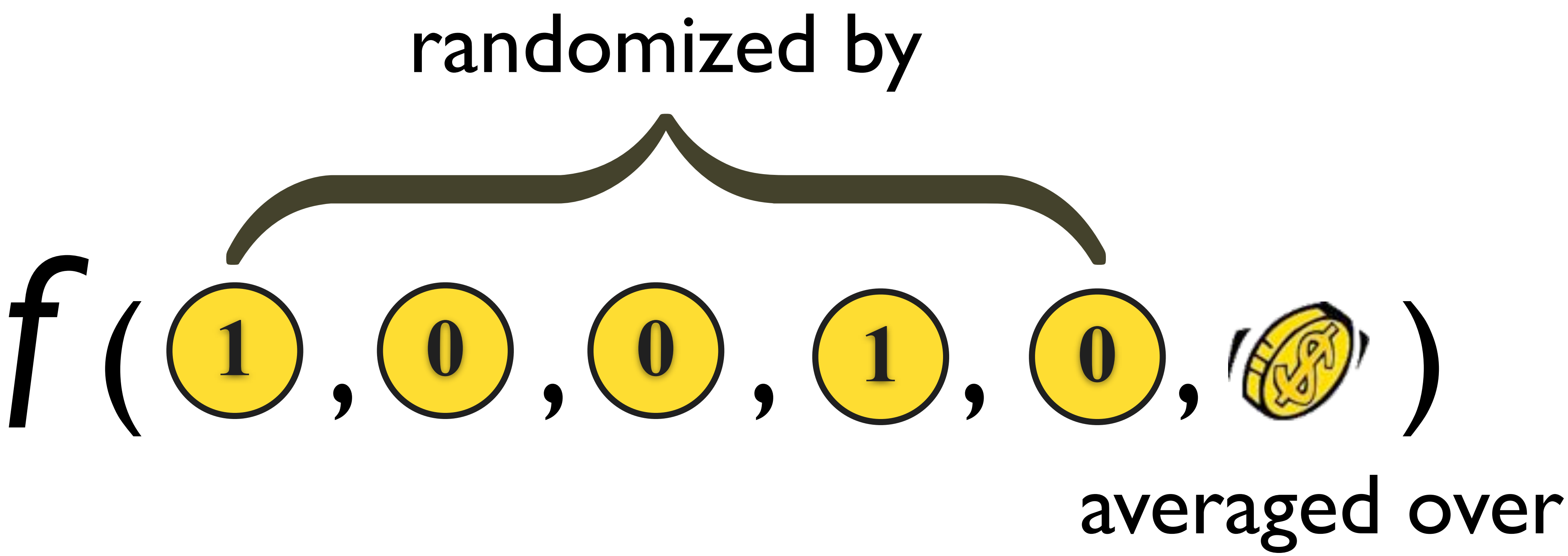
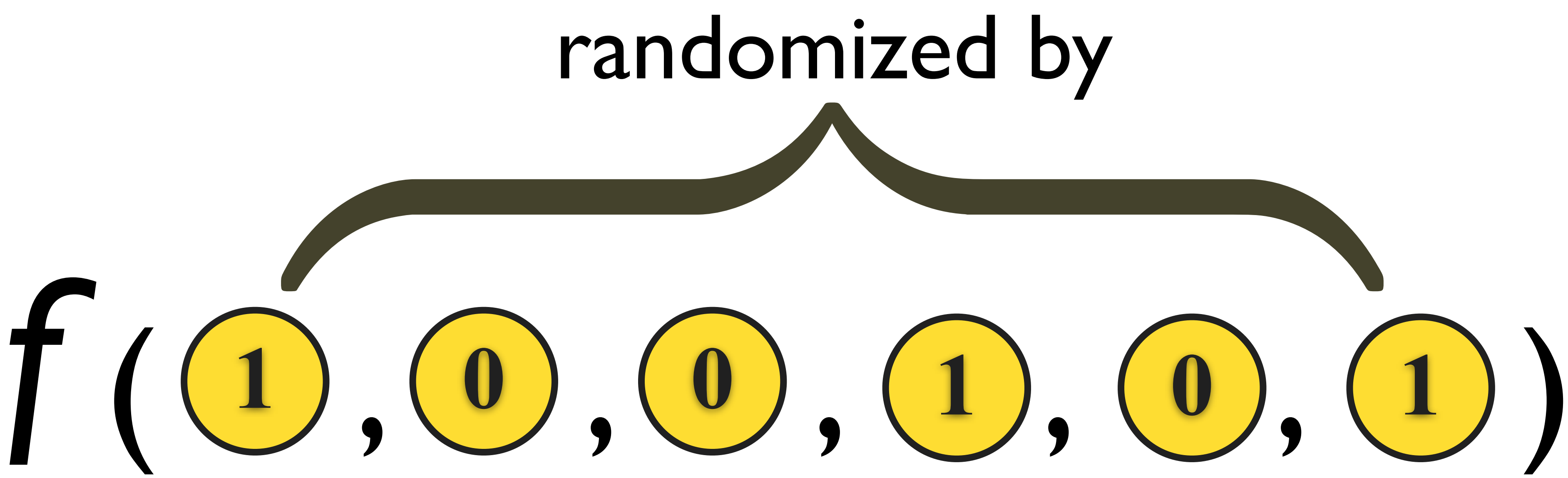


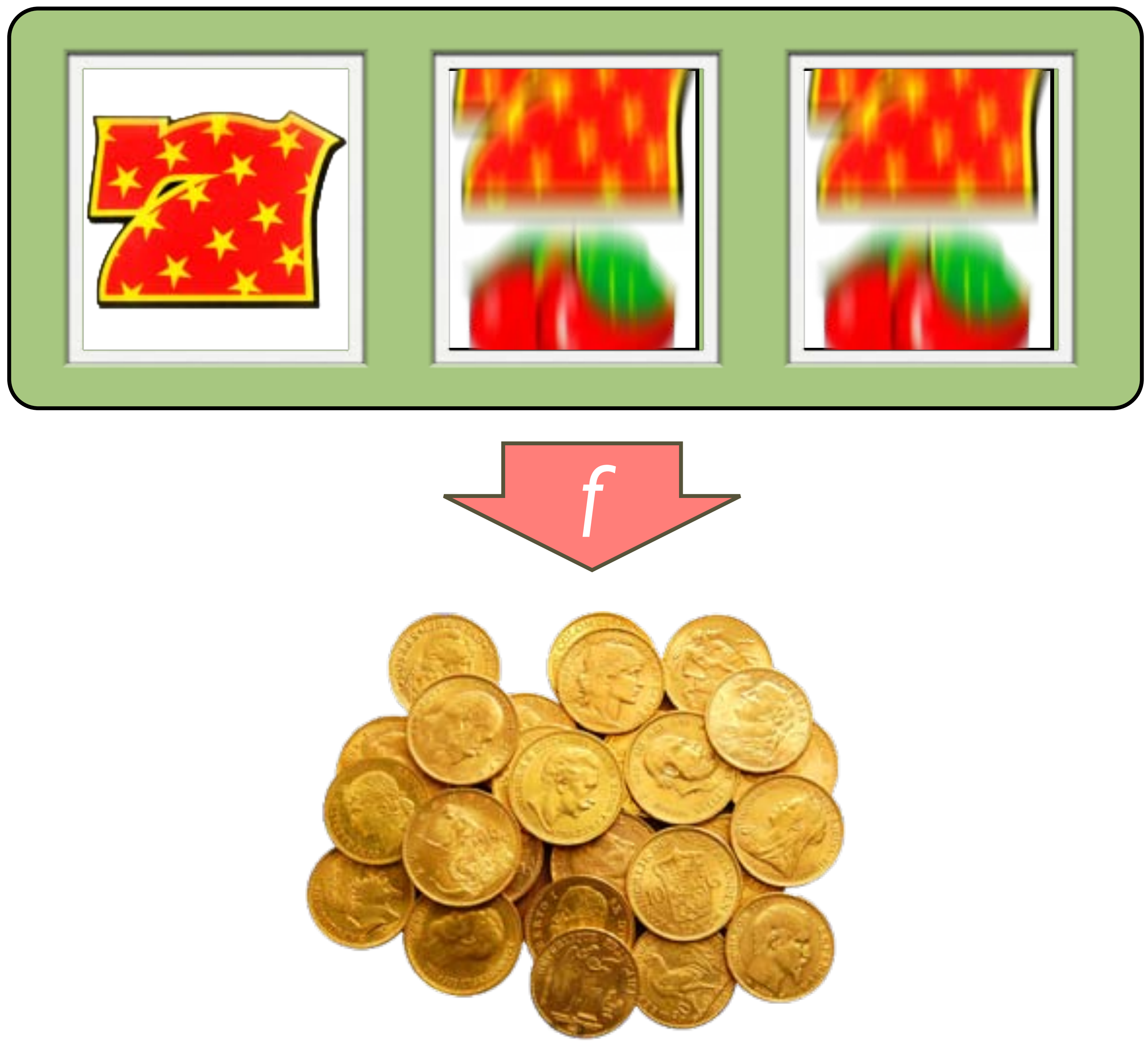
Doob Sequence

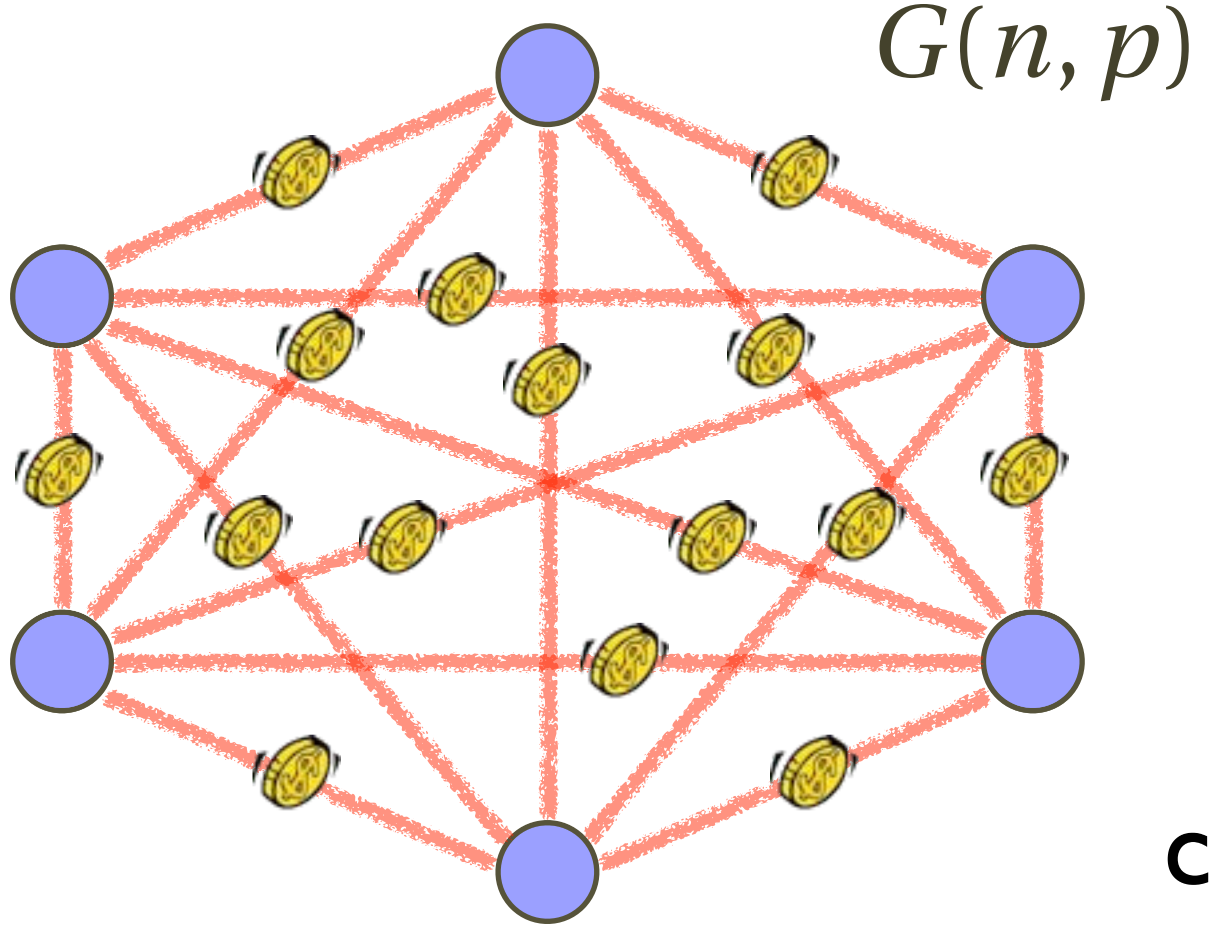


Doob Sequence



Doob Martingale





Graph parameter:

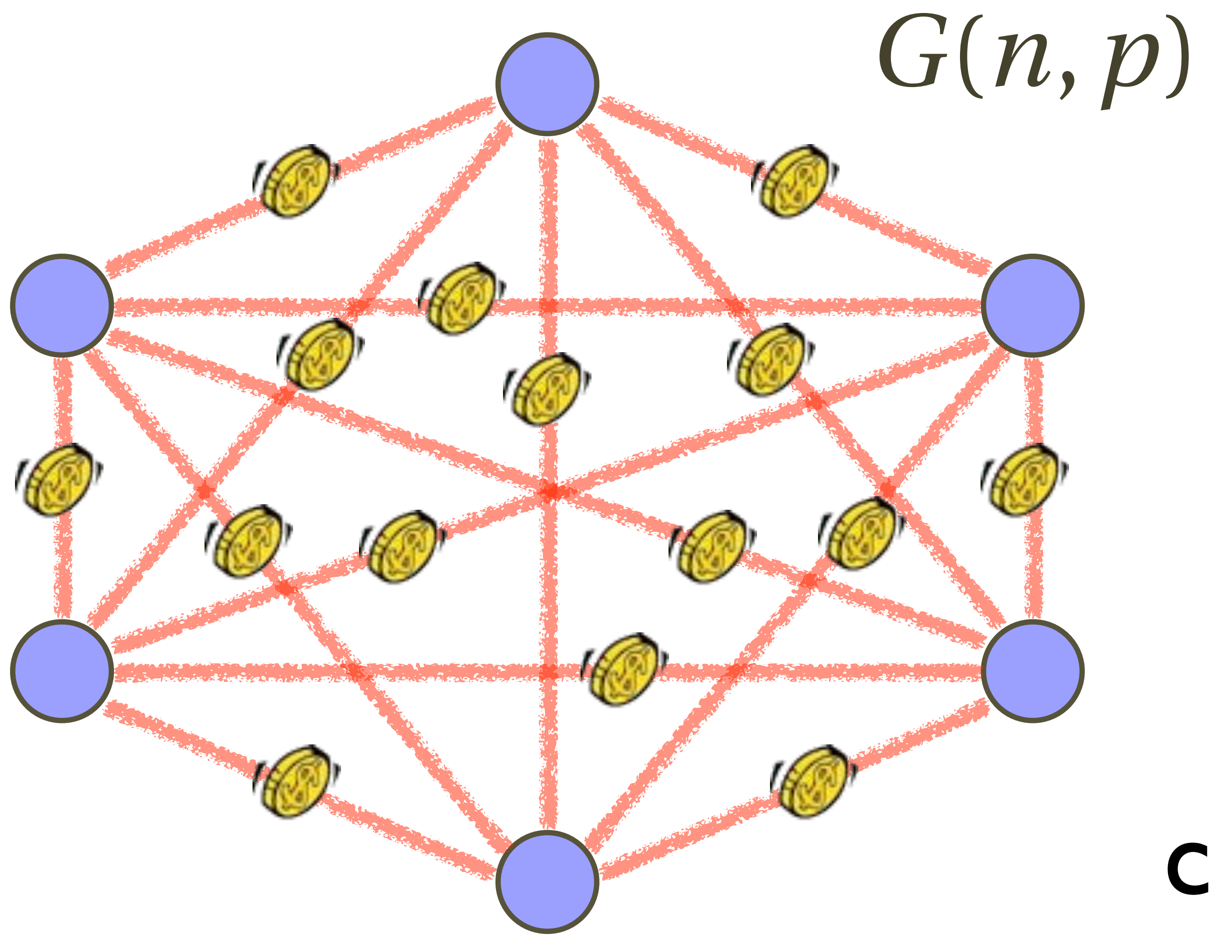
$$f(G)$$

example: chromatic #, components, diameter ...

numbering all vertex-pairs: $1, 2, 3, \ldots, \binom{n}{2}$

$$I_j = \begin{cases} 1 & \text{edge } j \in G \\ 0 & \text{edge } j \notin G \end{cases} \qquad Y_i = \mathbf{E}[f(G) \mid I_1, \ldots, I_i]$$

$$Y_0 = \mathbf{E}[f(G)] \dashrightarrow Y_{\binom{n}{2}} = f(G)$$



Graph parameter:

$$f(G)$$

example: chromatic #, components, diameter ...

numbering all vertices: $1, 2, 3, \ldots, n$

X_i : subgraph of G induced by the first i vertices

$$Y_i = \mathbf{E}[f(G) \mid X_1, \ldots, X_i]$$

$$Y_0 = \mathbf{E}[f(G)] \dashrightarrow Y_n = f(G)$$

Martingales induced by a random graph

- Edge exposure martingale:

$$I_j \text{ indicates the } j\text{th edge}$$

$$Y_i = \mathbf{E}[f(G) \mid I_1, \ldots, I_i]$$

- Vertex exposure martingale:

$$X_i = G([i])$$

$$Y_i = \mathbf{E}[f(G) \mid X_1, \ldots, X_i]$$

martingale $X_0, X_1, X_2, \ldots$

$\mathrm{E}[X_i \mid X_0, X_1, \ldots, X_{i-1}] = X_{i-1}$

generalization

martingale $Y_0, Y_1, Y_2, \ldots$

w.r.t. $X_0, X_1, X_2, \ldots$

$Y_i = f(X_0, X_1, \ldots, X_i)$

$\mathrm{E}[Y_i \mid X_0, X_1, \ldots, X_{i-1}] = Y_{i-1}$

special case

Doob martingale

$Y_i = \mathrm{E}[f(X_0, X_1, \ldots, X_n) \mid X_0, X_1, \ldots, X_{i-1}]$

special cases
in random graphs

edge-exposure martingale

vertex-exposure martingale

Hoeffding's Inequality

Hoeffding's Inequality:

Let $X = \sum_{i=1}^{n} X_i$, where $X_1, \ldots, X_n$ are independent random variables with

$$a_i \le X_i \le b_i.$$

Then

$$\Pr[|X - \mathbf{E}[X]| \ge t] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

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Then

$$\Pr[|X - \mathbf{E}[X]| \geq t] \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Proof: X is a function (sum) of $X_1, \ldots, X_n$.

Doob martingale: $Y_i = \mathbf{E}[X \mid X_1, \ldots, X_i]$

$$|Y_i - Y_{i-1}| = |X_i - \mathbf{E}[X_i]| \leq b_i - a_i$$

Azuma → Done!

The Power of Doob + Azuma

- For a function of (dependent) random variables: $f(X_1,\ldots,X_n)$
- Doob martingale:

$$Y_i = \mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_i]$$

$$Y_0 = \mathbf{E}[f(X_1,\ldots,X_n)] \qquad Y_n = f(X_1,\ldots,X_n)$$

- If the differences $|Y_i - Y_{i-1}|$ are bounded,
- Azuma $\Longrightarrow$ $|Y_n - Y_0|$ is small

$f(X_1,\ldots,X_n)$ is tightly concentrated to its mean

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and let f be a function of $X_0, X_1, \ldots, X_n$ satisfying that, for all $1 \le i \le n$,

$$|\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]| \le c_i,$$

Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

The method of bounded differences:

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$$\left|\underbrace{\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i]}_{Y_i} - \underbrace{\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]}_{Y_{i-1}}\right| \le c_i,$$

Then

(Azuma)

$$\Pr\left[\left|\underbrace{f(\boldsymbol{X})}_{Y_n} - \underbrace{\mathbf{E}[f(\boldsymbol{X})]}_{Y_0}\right| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

Doob martingale: $Y_i = \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i]$

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ and let f be a function of $X_0, X_1, \ldots, X_n$ satisfying that, for all $1 \le i \le n$,

$$\left|\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]\right| \le c_i,$$

hard to check!

Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

$$|\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]| \le c_i,$$

Lipschitz Condition:

$f(x_1, \ldots, x_n)$ satisfies the **Lipschitz condition** with constants c_i, $1 \le i \le n$, if

$$\Big| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \Big| \le c_i.$$

Average-case:

$$|\mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_i] - \mathbf{E}[f(\boldsymbol{X}) \mid X_1, \ldots, X_{i-1}]| \le c_i,$$

Worst-case:

Lipschitz Condition:

$f(x_1, \ldots, x_n)$ satisfies the **Lipschitz condition** with constants c_i, $1 \le i \le n$, if

$$\Big| f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \Big| \le c_i.$$

The method of bounded differences:

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n independent random variables and let f be a function satisfying the Lipschitz condition with constants c_i, $1 \leq i \leq n$. Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$

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Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be n independent random variables and let f be a function satisfying the Lipschitz condition with constants c_i, $1 \le i \le n$. Then

$$\Pr\left[|f(\boldsymbol{X}) - \mathbf{E}[f(\boldsymbol{X})]| \ge t\right] \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n c_i^2}\right).$$

Proof:

Lipschitz condition

+

independence

⇒ bounded averaged differences

Applications of the Method

- A function *f* of independent random variable:

$$X_1, X_2, \ldots, X_n$$

- Lipschitz condition of f:
  - changes of any variable makes little change to the value of f.
- $\Rightarrow \; f(X_1, X_2, \ldots, X_n)$ is tightly concentrated to its mean.

Occupancy Problem

- m-balls-into-n-bins:
- number of empty bins?

$$X_i = \begin{cases} 1 & \text{bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$$

\# empty bins: $X = \sum_{i=1}^{n} X_i$

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] \quad = n\left(1 - \frac{1}{n}\right)^m$$

Xi are dependent

deviation: $\Pr[|X - \mathbf{E}[X]| \geq t] \leq ?$

Occupancy Problem

- m-balls-into-n-bins:
- number of empty bins?

empty bins: X

deviation:

$$\Pr[|X - \mathbf{E}[X]| \geq t] \leq ?$$

Y_j : the bin of ball *j* (Independent!)

$$X = f(Y_1, \ldots, Y_m) \quad = |[n] - \{Y_1, \ldots, Y_m\}|$$

Lipschitz:

changing any Y_j can change X for at most 1

$$\Rightarrow \quad \Pr[|X - \mathbf{E}[X]| \geq t\sqrt{m}] \leq 2e^{-t^2/2}$$

Pattern Matching

- a random string of length n,
- a pattern of length k,
- # of matched substrings?

alphabet Σ

$$|\Sigma| = m$$

a fixed pattern: $\pi \in \Sigma^k$

uniform & independent: $X_1, \ldots, X_n \in \Sigma$

$$Y: \#\text{substrings } \pi \text{ in } (X_1, \ldots, X_n)$$

$$\mathbf{E}[Y] = (n - k + 1)\left(\frac{1}{m}\right)^k$$

Deviation?

Pattern Matching

- a random string of length n,
- a pattern of length k,
- # of matched substrings?

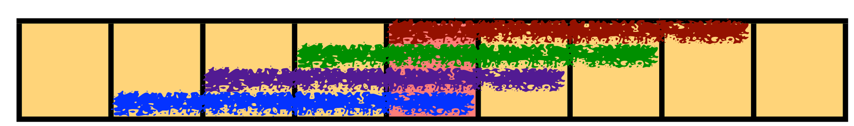
alphabet Σ

$|\Sigma| = m$

a fixed **pattern**: $\pi \in \Sigma^k$

uniform & independent: $X_1, \ldots, X_n \in \Sigma$

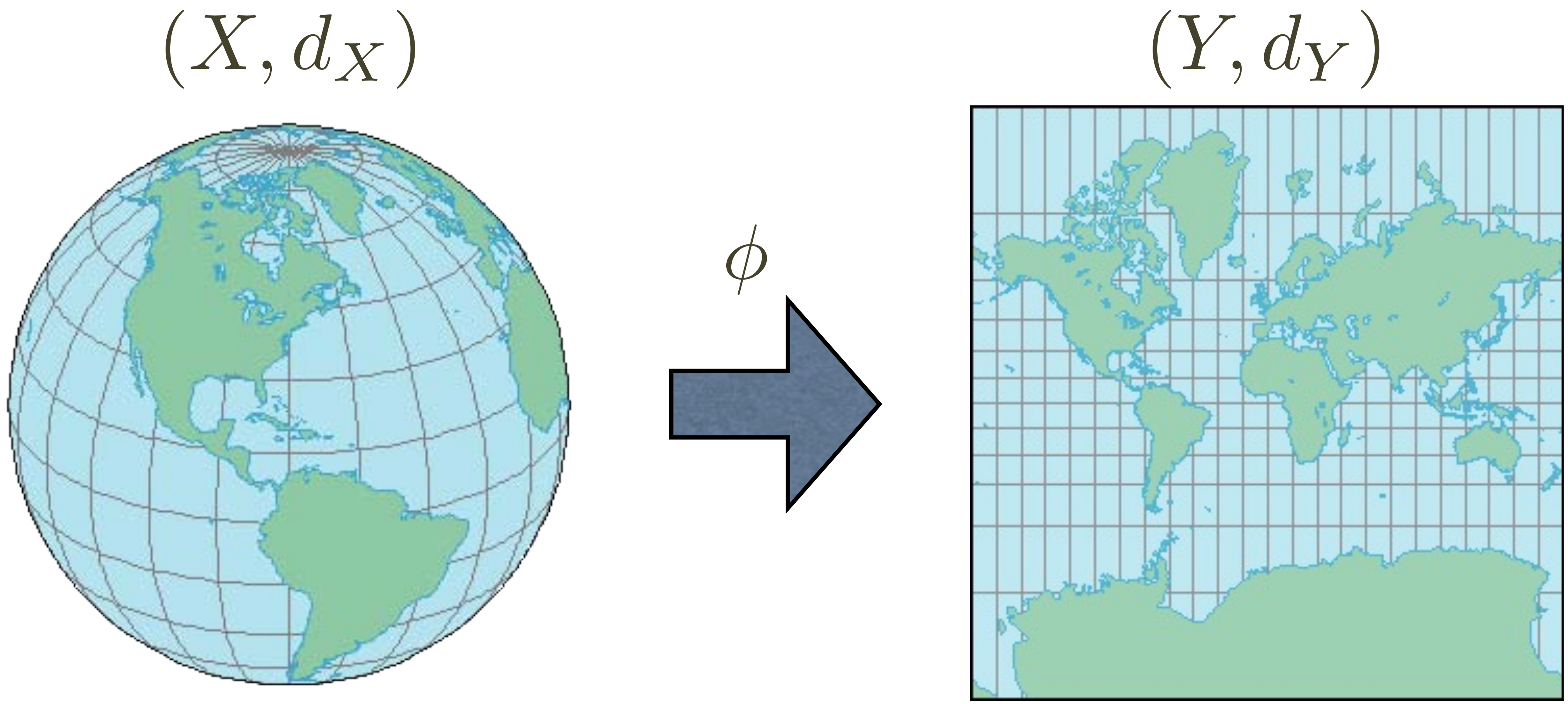
$$Y = f(X_1, \ldots, X_n)$$



changing any X_i changes f for at most k

$$\Pr\left[|Y - \mathbf{E}[Y]| \geq tk\sqrt{n}\right] \leq 2e^{-t^2/2}$$

Metric Embedding

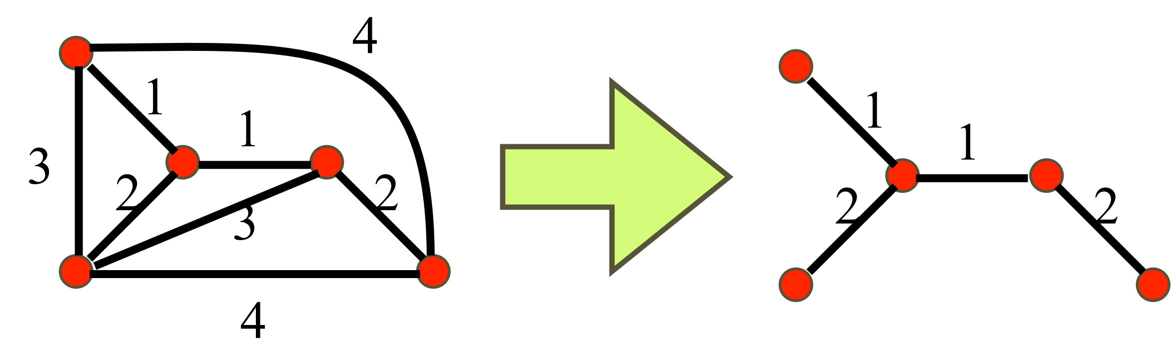


low-distortion: For a small $\alpha \geq 1$

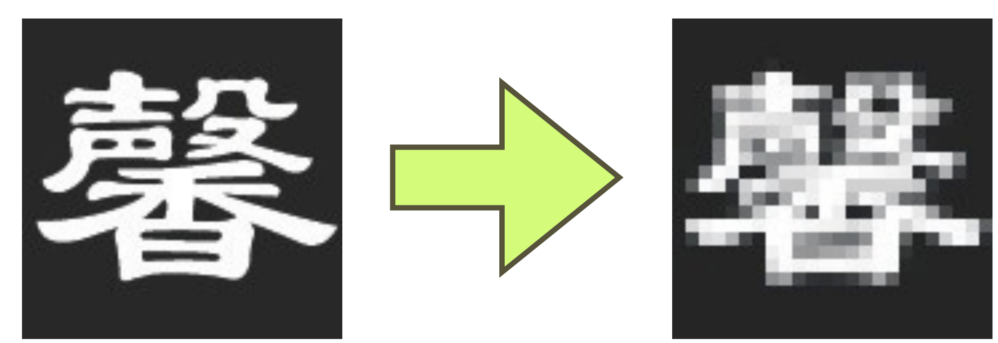
$$\forall x_1, x_2 \in X, \quad \frac{1}{\alpha} d_X(x_1, x_2) \leq d_Y(\phi(x_1), \phi(x_2)) \leq \alpha d_X(x_1, x_2)$$

Why Embedding?

- Some problems are easier to solve in simple metrics:
  - TSP in trees.
- “Curse of dimensionality”
  - proximity search;
  - learning;
  - due to volume explosion.



graph metric tree metric



high dimension low dimension

Dimension Reduction

In Euclidian space, it is always possible to embed a set of n points in arbitrary dimension to $O(\log n)$ dimension with constant distortion.

Johnson-Lindenstrauss Theorem:

For any $0 < \epsilon < 1$, for any set V of n points in $\mathbf{R}^d$, there is a map $\phi : \mathbf{R}^d \to \mathbf{R}^k$ with $k = O(\ln n)$, such that $\forall u, v \in V$,

$$(1-\epsilon)\|u-v\|^2 \leq \|\phi(u) - \phi(v)\|^2 \leq (1+\epsilon)\|u-v\|^2$$

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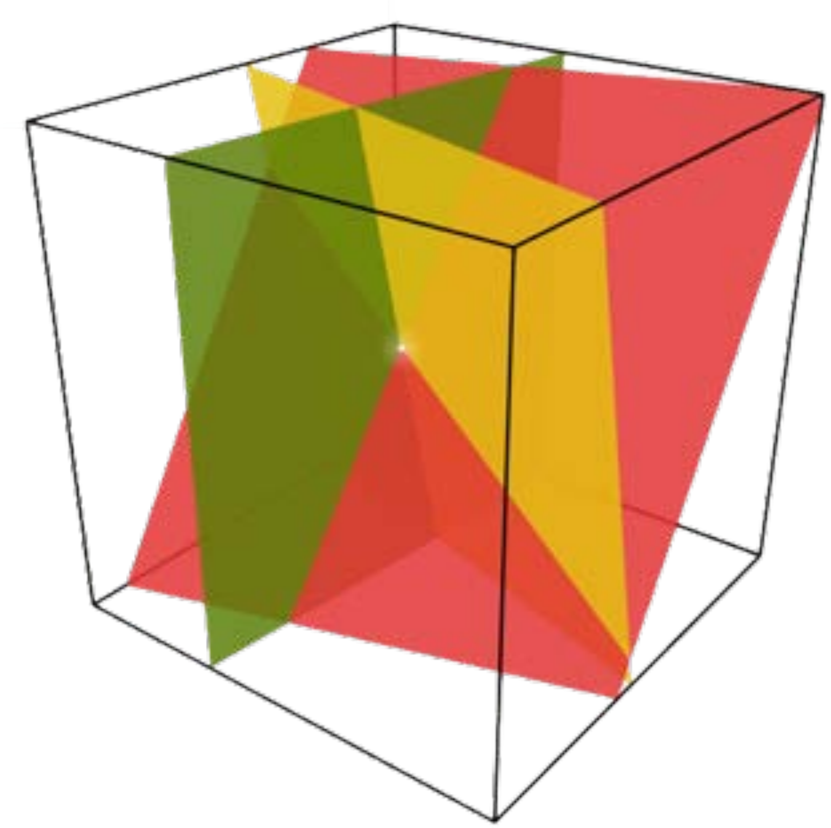
$$(1-\epsilon)\|u-v\|^2 \leq \|\phi(u)-\phi(v)\|^2 \leq (1+\epsilon)\|u-v\|^2$$

- $\phi(v) = Av$.
- A is a random projection matrix.

Random Projection

Random $k \times d$ matrix A:

- Projection onto a uniform random subspace. (Johnson-Lindenstrauss) (Dasgupta-Gupta)
- i.i.d. Gaussian entries. (Indyk-Motiwani)
- i.i.d. $-1/+1$ entries. (Achlioptas)



rows: $A_{1\cdot}, A_{2\cdot}, \ldots, A_{k\cdot}$

random orthogonal unit vectors $\in \mathbb{R}^d$