Randomized Algorithms

南京大学

尹一通

Random Walk

Stochastic Process

$\{X_t \mid t \in T\}$ $X_t \in \Omega$

time t state space Ω state $x \in \Omega$

- discrete time:

  T is countable $T = \{0, 1, 2, \ldots\}$

- discrete space:

  Ω is finite or countably infinite

$$X_0, X_1, X_2, \ldots$$

Markov Property

- dependency structure of $X_0, X_1, X_2, \ldots$
- Markov property: (memoryless)

X_{t+1} depends only on X_t

$$\forall t = 0, 1, 2, \ldots, \forall x_0, x_1, \ldots, x_{t-1}, x, y \in \Omega$$

$$\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x]$$
$$= \Pr[X_{t+1} = y \mid X_t = x]$$

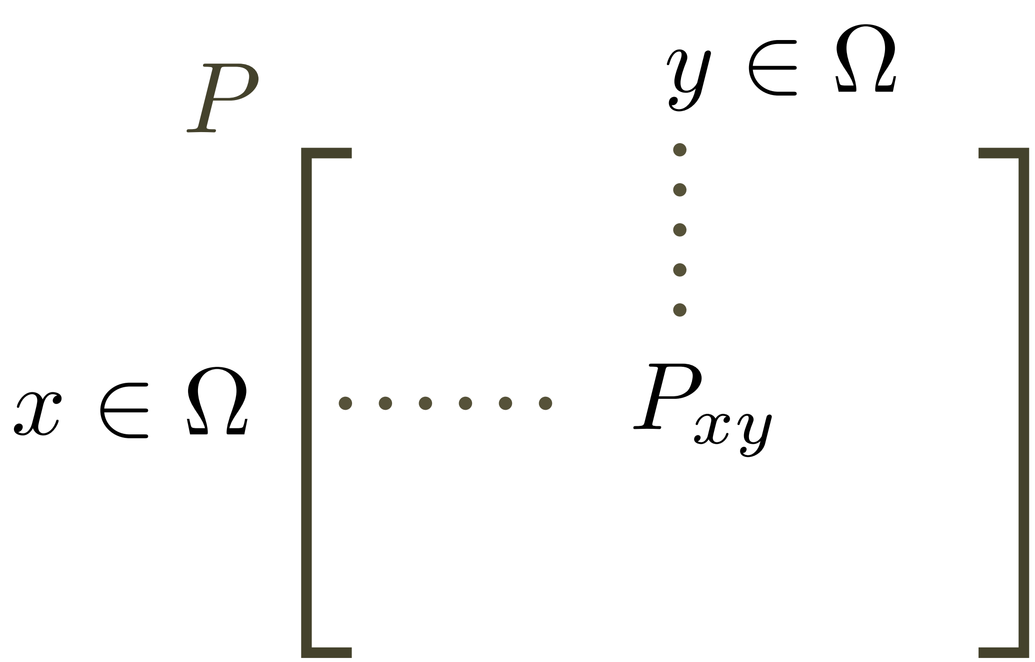
Markov chain: discrete time discrete space stochastic process with Markov property.

Transition Matrix

Markov chain: $X_0, X_1, X_2, \ldots$

$$\Pr[X_{t+1} = y \mid X_0 = x_0, \ldots, X_{t-1} = x_{t-1}, X_t = x]$$
$$= \Pr[X_{t+1} = y \mid X_t = x] = P_{xy}^{(t)} = P_{xy}$$

(time-homogenous)



stochastic matrix $P\mathbf{1} = \mathbf{1}$

chain: $X_0, \quad X_1, \quad X_2, \quad \ldots$

distribution: $\pi^{(0)} \quad \pi^{(1)} \quad \pi^{(2)} \quad \in [0,1]^{\Omega} \quad \sum_{x\in\Omega} \pi_x = 1$

$$\pi_x^{(t)} = \Pr[X_t = x]$$

$$\pi^{(t+1)} = \pi^{(t)} P$$

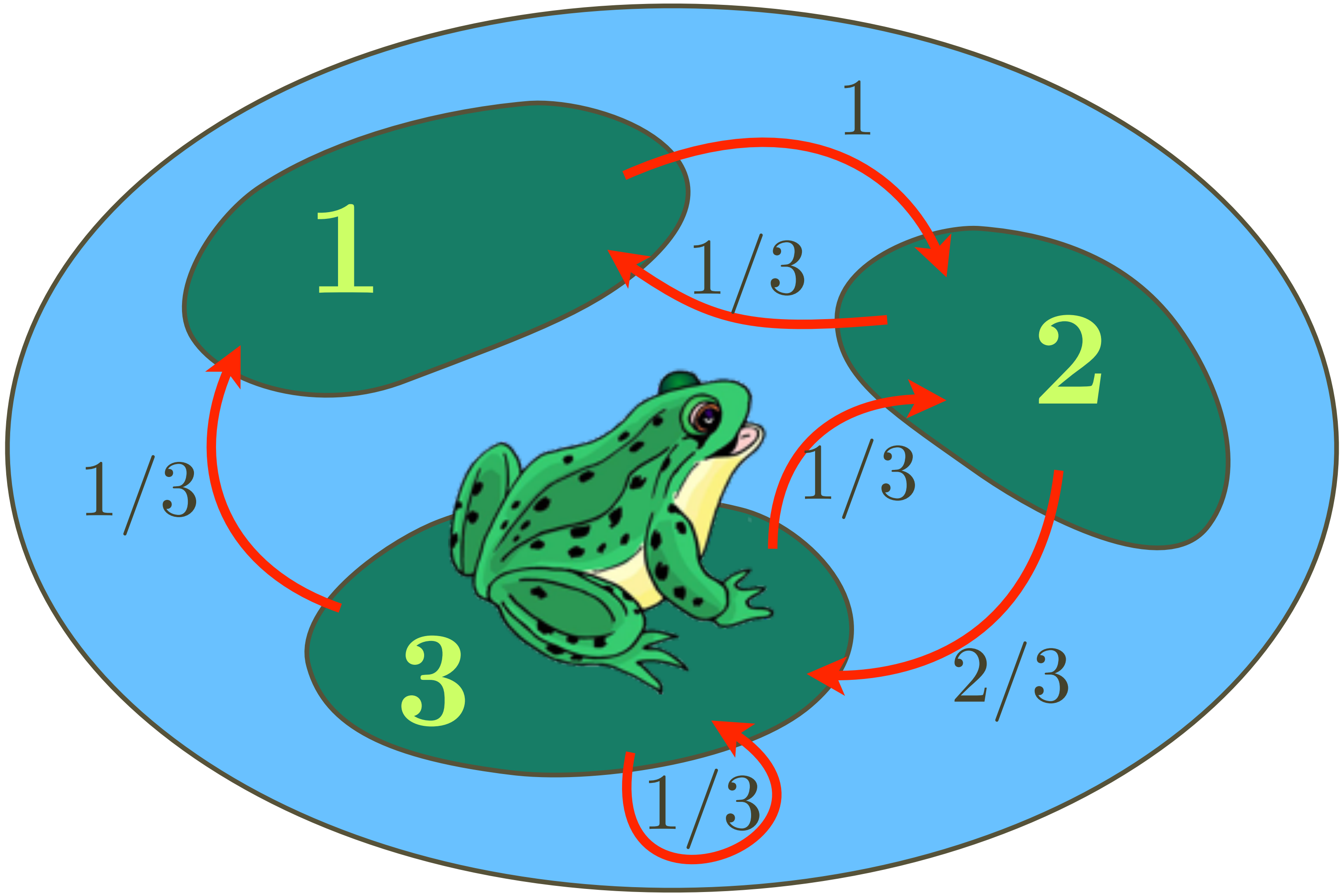
$$\begin{aligned}
\pi_y^{(t+1)} &= \Pr[X_{t+1} = y] \\
&= \sum_{x\in\Omega} \Pr[X_t = x] \Pr[X_{t+1} = y \mid X_t = x] \\
&= \sum_{x\in\Omega} \pi_x^{(t)} P_{xy} \\
&= (\pi^{(t)} P)_y
\end{aligned}$$

$$\pi^{(0)} \xrightarrow{P} \pi^{(1)} \xrightarrow{P} \cdots\cdots\pi^{(t)} \xrightarrow{P} \pi^{(t+1)} \xrightarrow{P} \cdots$$

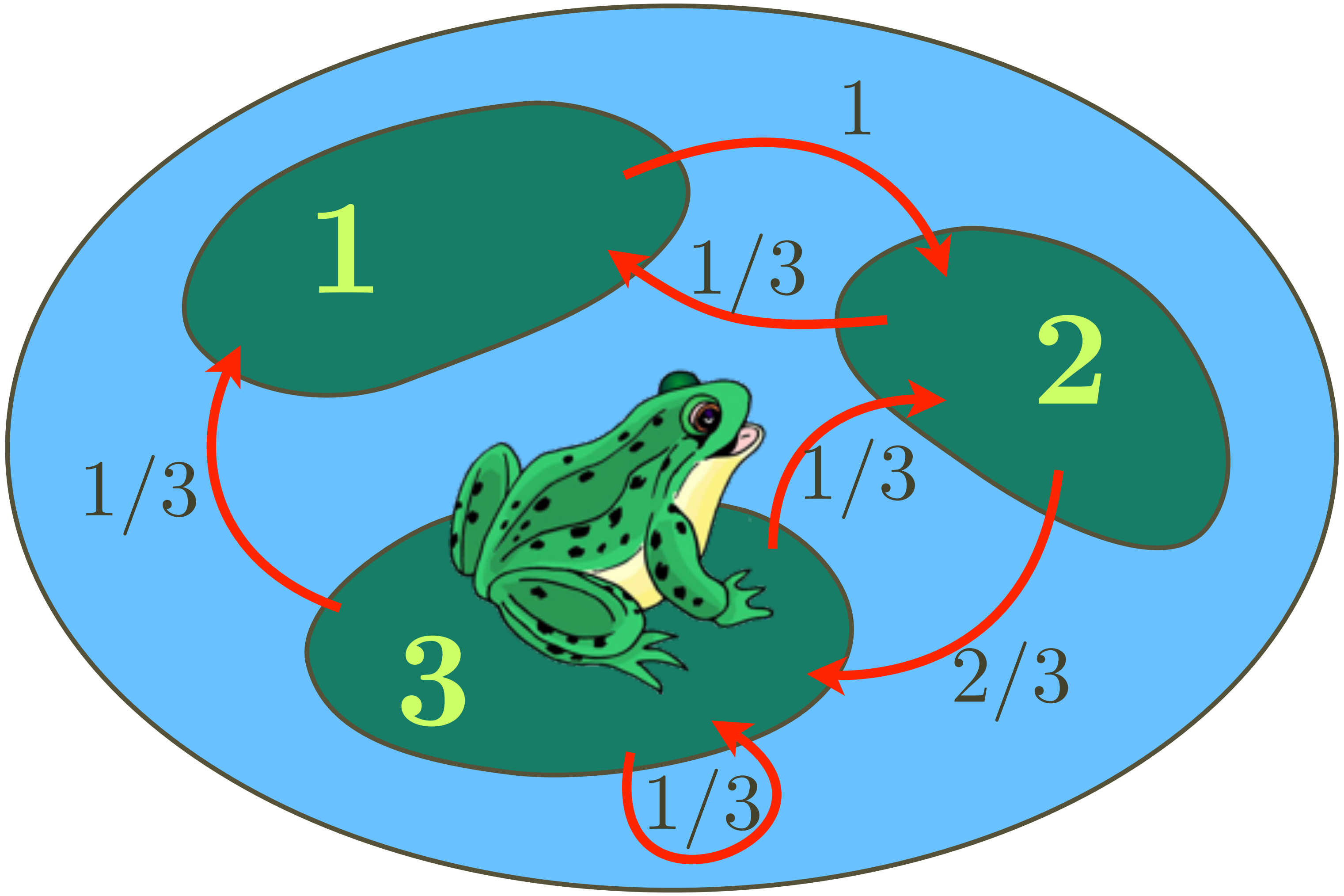
- initial distribution: $\pi^{(0)}$
- transition matrix: P

$$\pi^{(t)} = \pi^{(0)} P^t$$

Markov chain: $\mathfrak{M} = (\Omega, P)$



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$



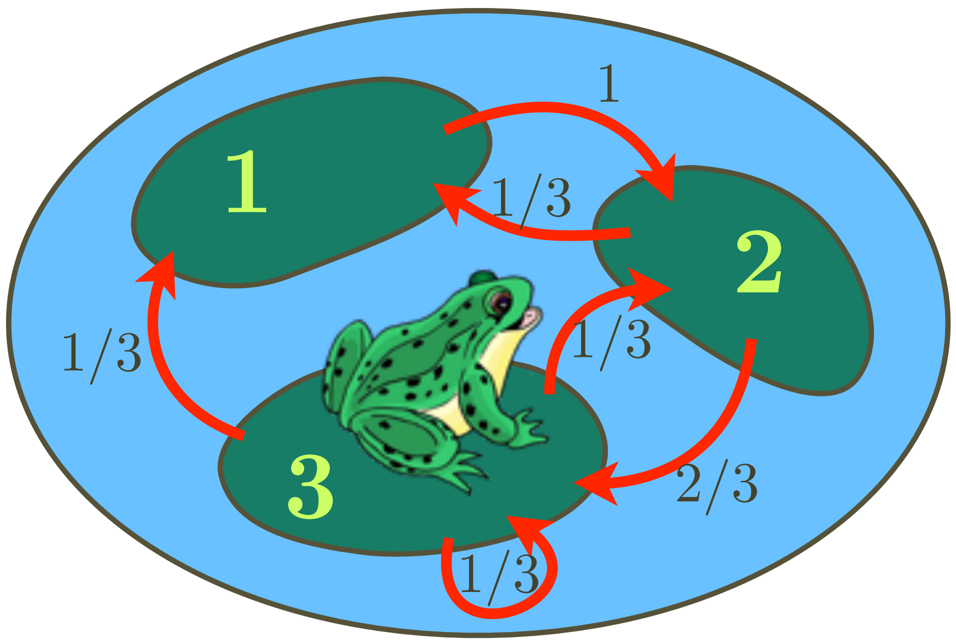
$$\pi^{(0)} = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}) \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\pi^{(1)} = \tfrac{1}{4}(0, 1, 0) + \tfrac{1}{2}(\tfrac{1}{3}, 0, \tfrac{2}{3}) + \tfrac{1}{4}(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})$$

Random Walks

- fair ± 1 random walk: flipping a fair coin, the state is the difference between heads and tails;
- random walk on a graph;
- card shuffling: random walk in a state space of permutations;
- random walk on *q*-coloring of a graph;

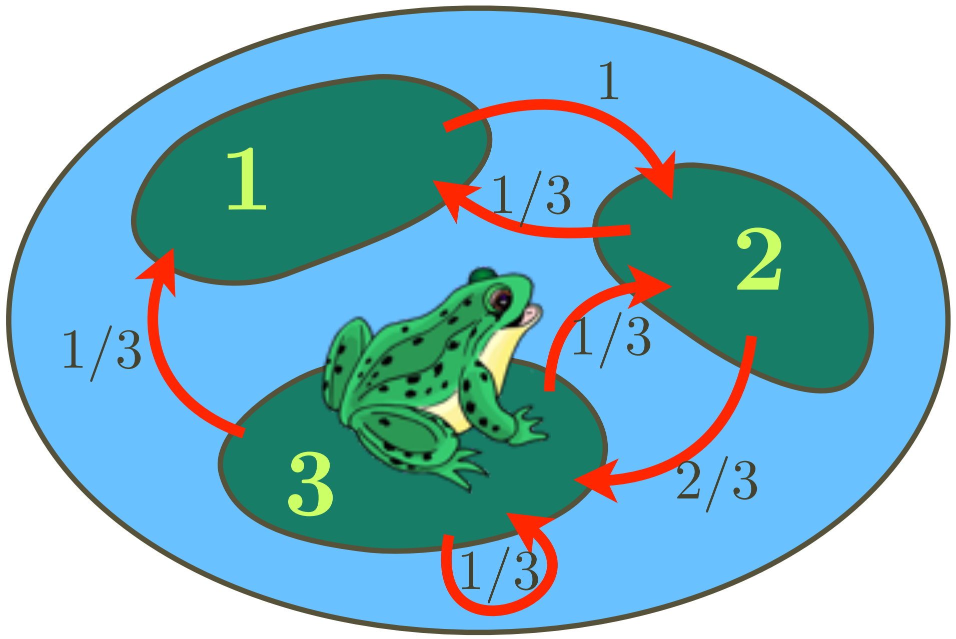
Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^2 \approx \begin{bmatrix} 0.3333 & 0 & 0.6667 \\ 0.3333 & 0.5556 & 0.2222 \\ 0.2778 & 0.6111 & 0.3333 \end{bmatrix}$$

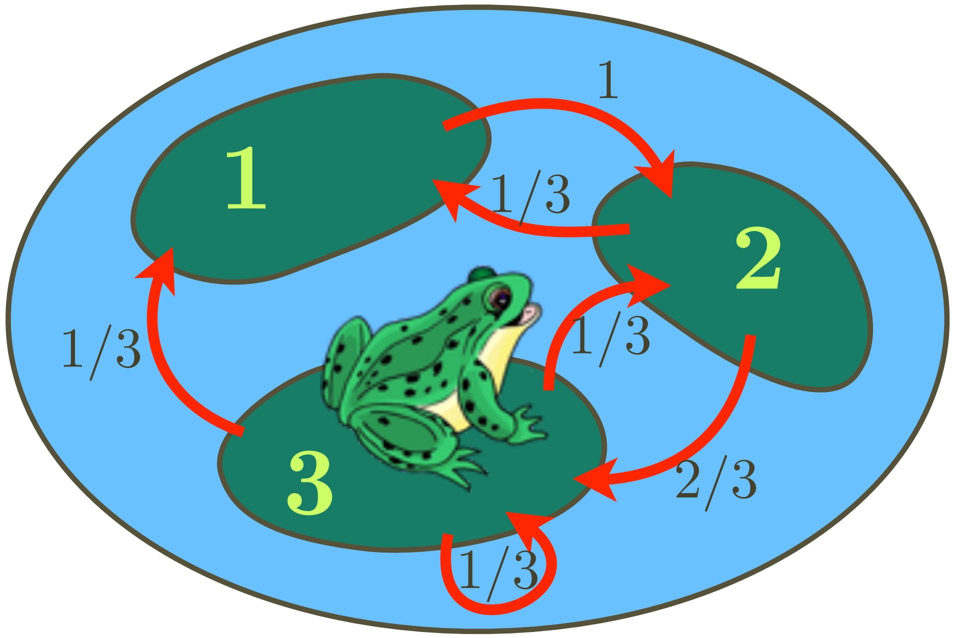
Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^5 \approx \begin{bmatrix} 0.2469 & 0.4074 & 0.3457 \\ 0.2510 & 0.3621 & 0.3868 \\ 0.2510 & 0.3663 & 0.3827 \end{bmatrix}$$

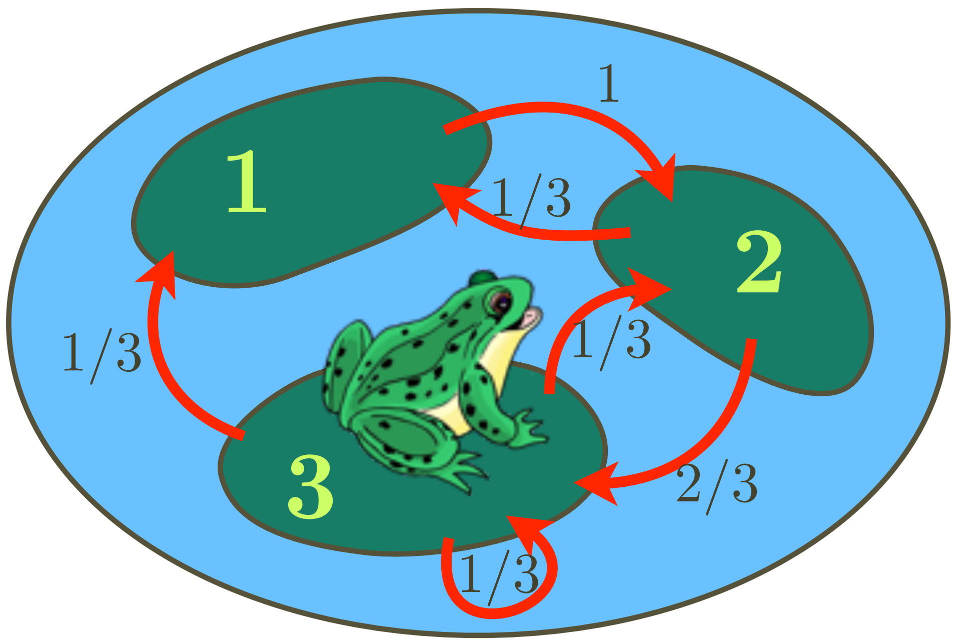
Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^{10} \approx \begin{bmatrix} 0.2500 & 0.3747 & 0.3752 \\ 0.2500 & 0.3751 & 0.3749 \\ 0.2500 & 0.3751 & 0.3749 \end{bmatrix}$$

Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix}$$

$$\forall \text{ distribution } \pi, \qquad \pi P^{20} \approx (\tfrac{1}{4}, \tfrac{3}{8}, \tfrac{3}{8})$$

Stationary Distribution

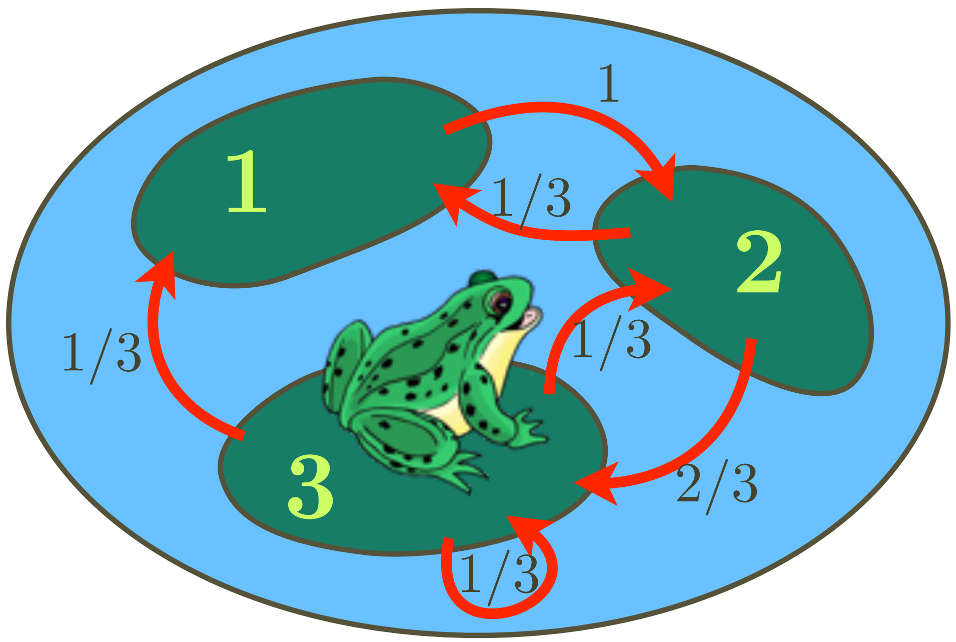
Markov chain $\mathfrak{M} = (\Omega, P)$

- stationary distribution π:

$$\pi P = \pi \qquad \text{(fixed point)}$$

- Perron-Frobenius Theorem:
  - stochastic matrix P: $P\mathbf{1} = \mathbf{1}$
  - 1 is also a left eigenvalue of P (eigenvalue of P^T)
  - the left eigenvector $\pi P = \pi$ is nonnegative
- stationary distribution always exists

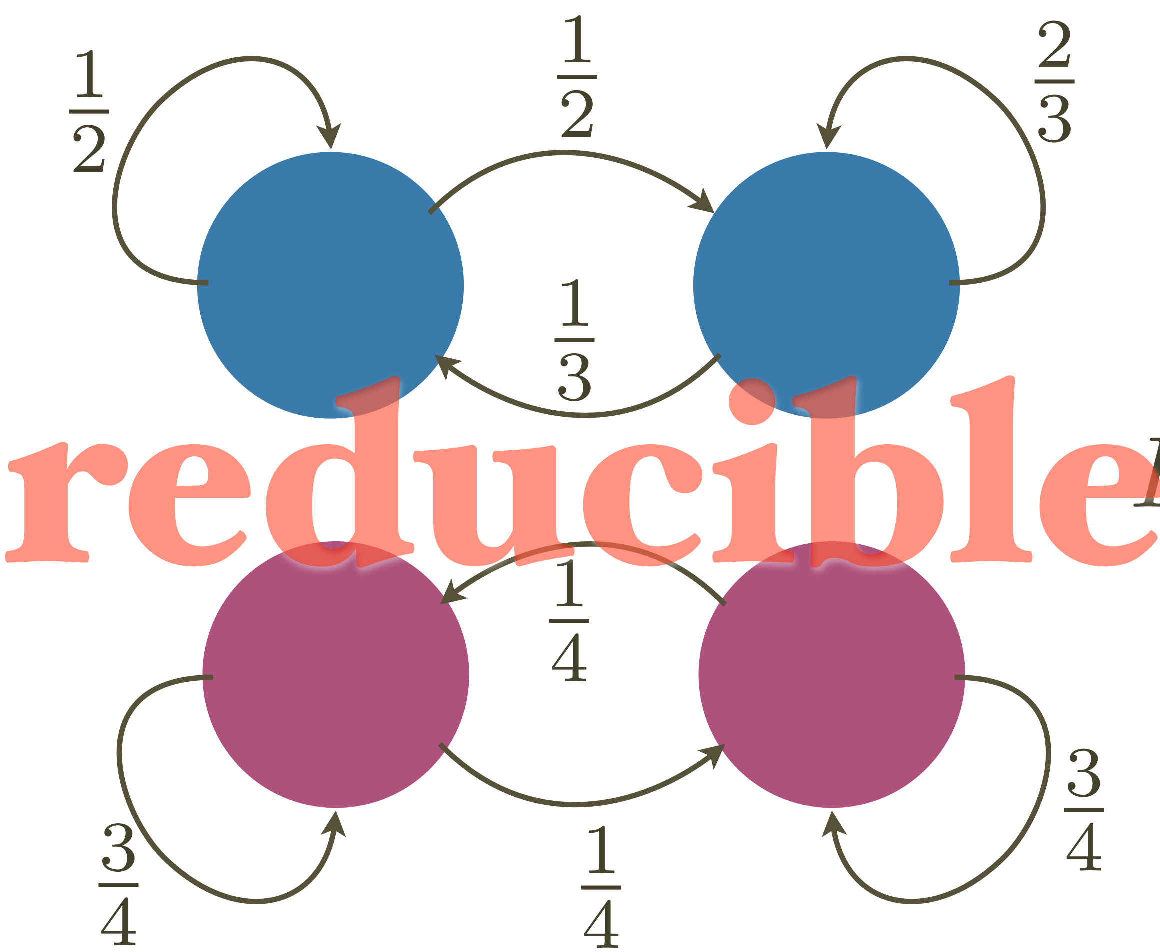
Convergence



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

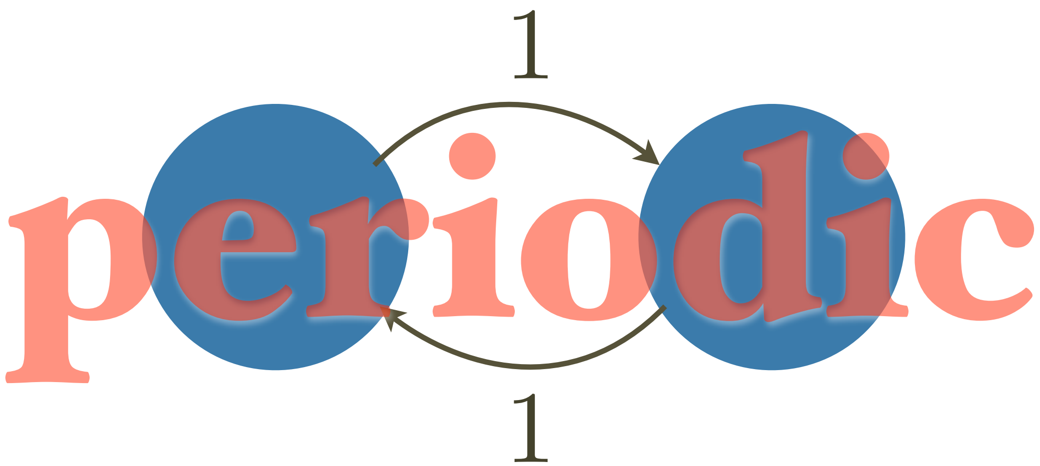
$$P^{20} \approx \begin{bmatrix} 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \\ 0.2500 & 0.3750 & 0.3750 \end{bmatrix}$$

ergodic: convergent to stationary distribution



$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}$$

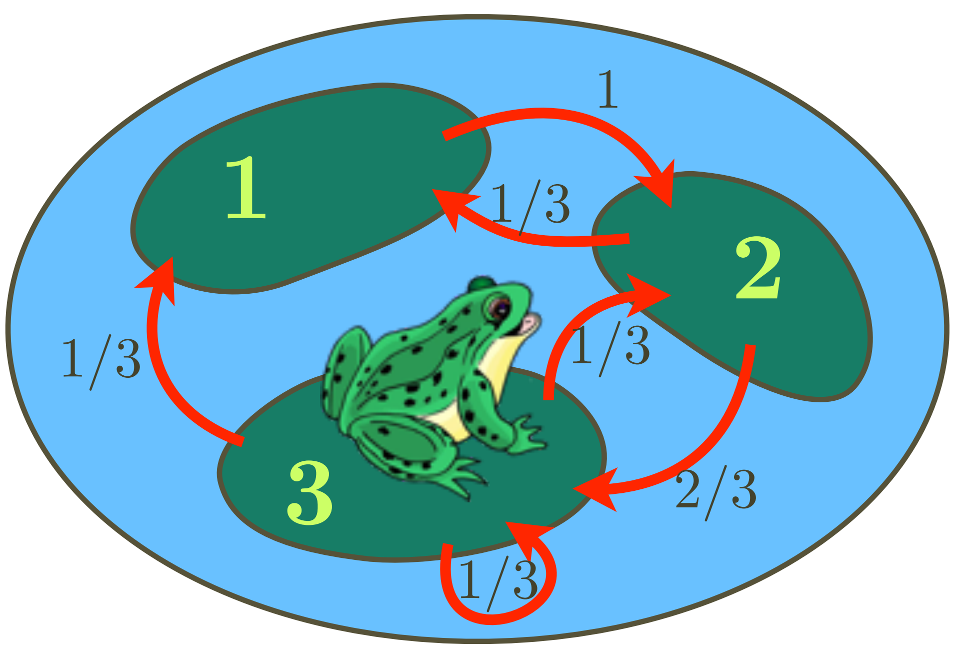
$$P^{20} \approx \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

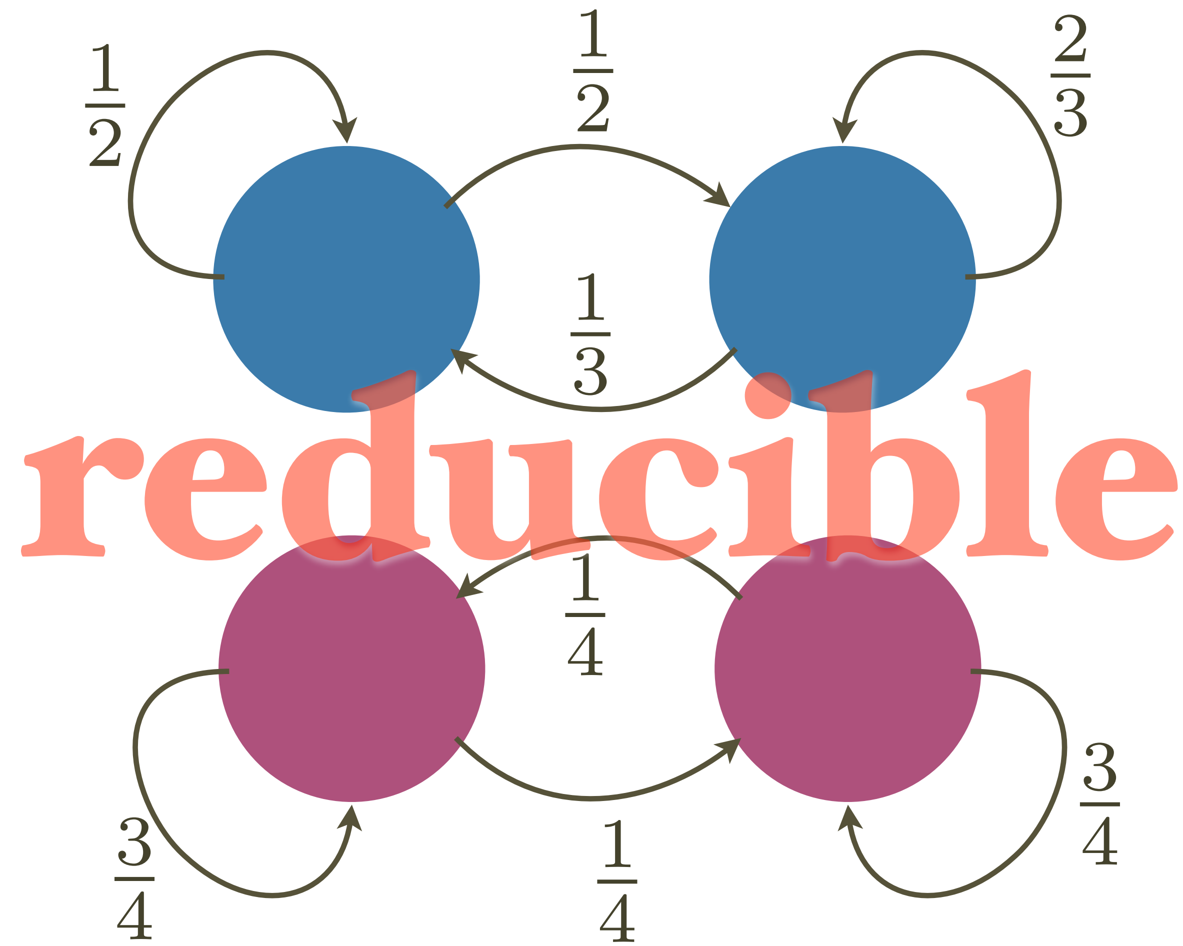


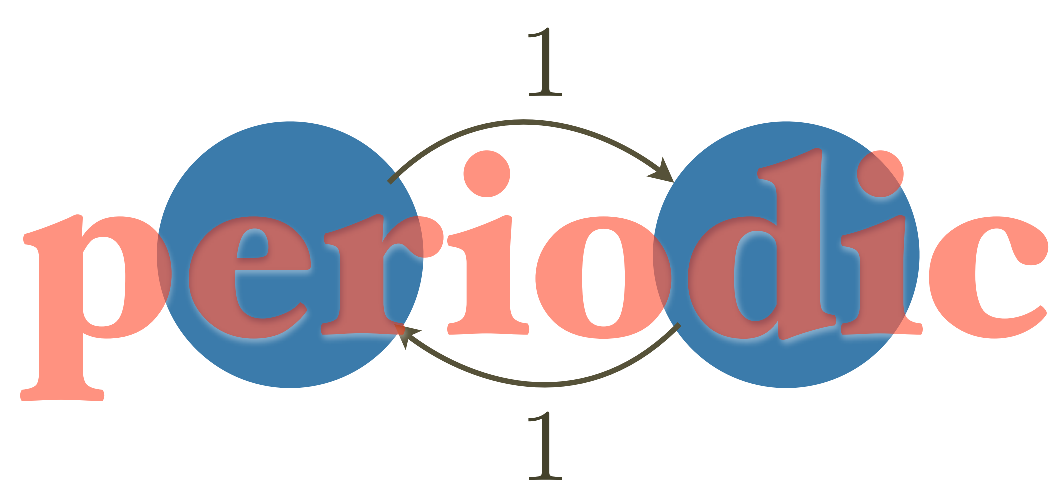
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{2k} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P^{2k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$


1
2
3
1
1/3
1/3
1/3
2/3
1/3


1/2
1/2
2/3
1/3
reducible
1/4
3/4
3/4
1/4


1
periodic
1

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

(ergodic)

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

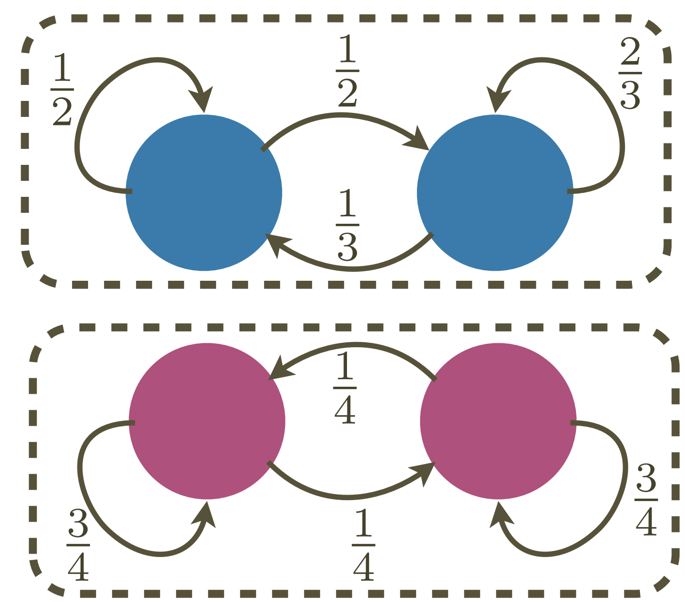
Irreducibility

- y is accessible from x:

$$\exists t, \;\; P^t(x, y) > 0$$

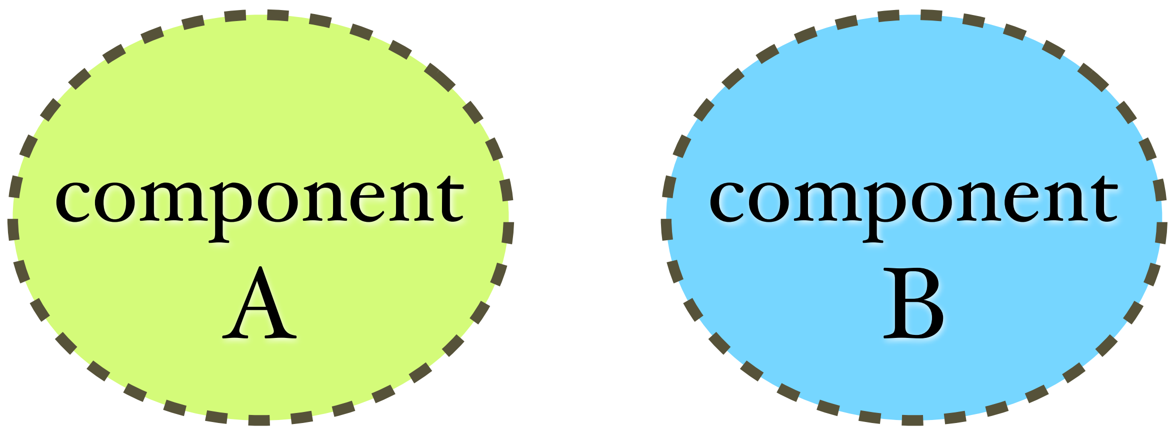
- x communicates with y:
    - x is accessible from y
    - y is accessible from x

- MC is irreducible: all pairs of states communicate



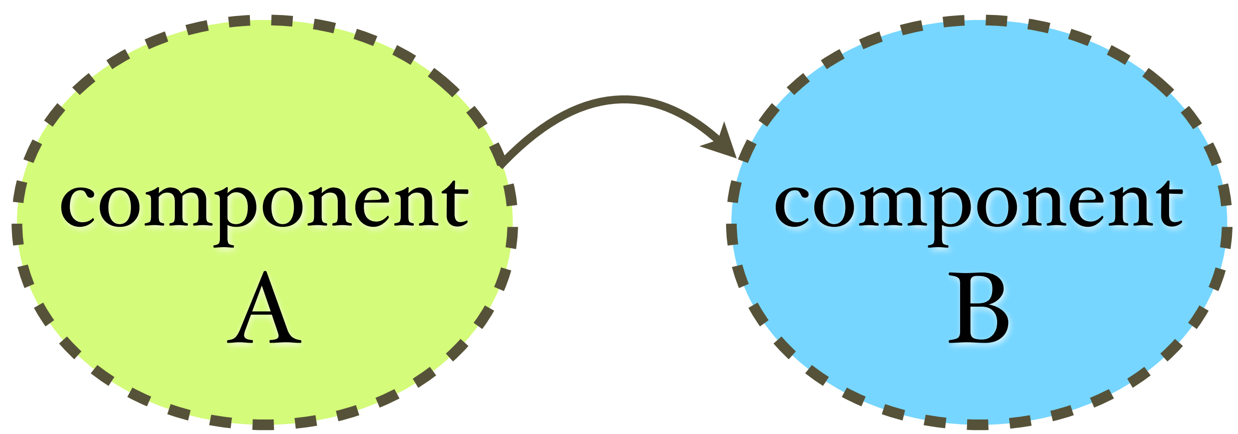
communicating classes

Reducible Chains



$$P = \begin{bmatrix} P_A & 0 \\ 0 & P_B \end{bmatrix}$$

stationary distributions: $\pi = \lambda \pi_A + (1-\lambda)\pi_B$



stationary distribution: $\pi = (\mathbf{0}, \pi_B)$

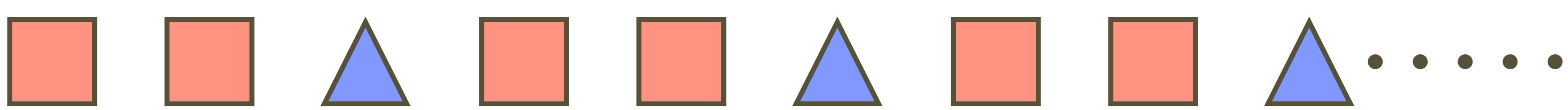
Aperiodicity

- period of state *x*:

$$d_x = \gcd\{t \mid P^t(x,x) > 0\}$$

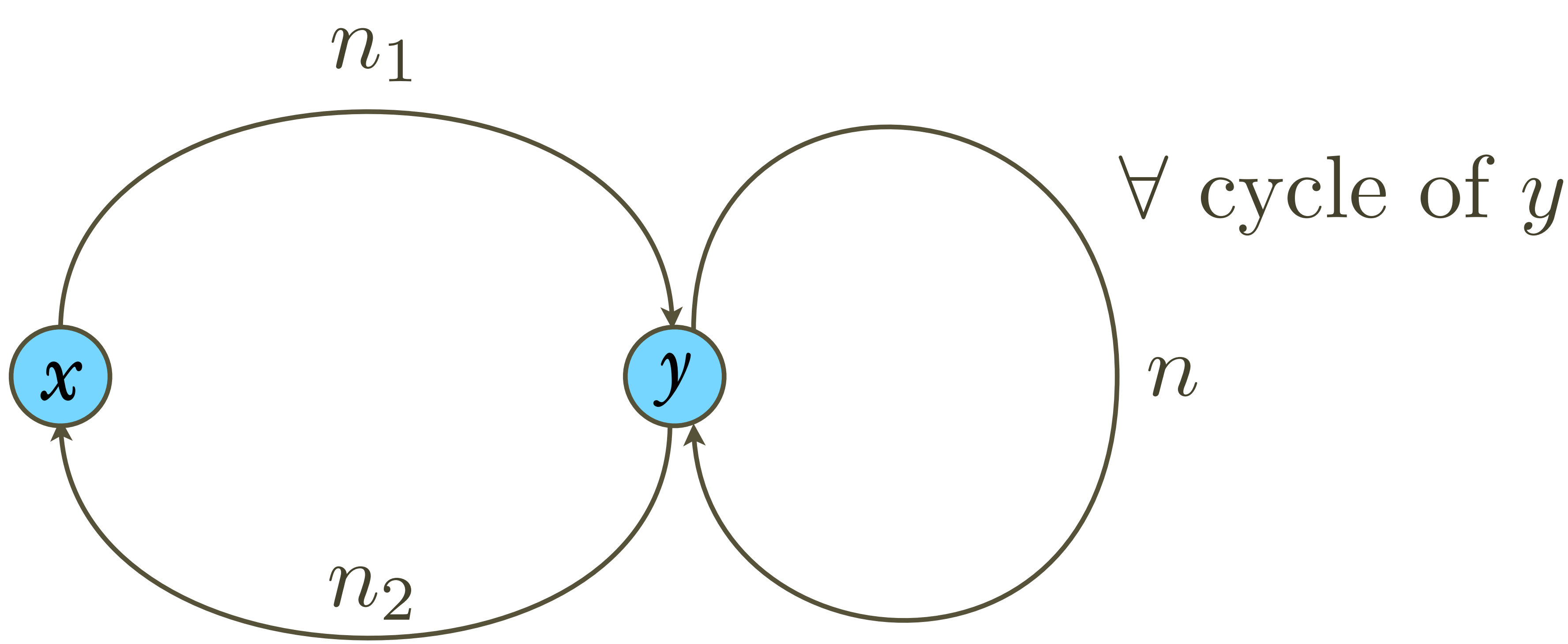
- aperiodic chain: all states have period 1
- period: the gcd of lengths of cycles

$$d_x = 3$$

x 

Lemma (period is a class property)

x and y communicate ⇨ $d_x = d_y$



$$\left.\begin{array}{l} d_x \mid (n_1 + n_2) \\ d_x \mid (n_1 + n_2 + n) \end{array}\right\} \Rightarrow d_x \mid n \quad \Rightarrow d_x \leq d_y$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t\to\infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence

irreducibility ⇒ uniqueness

} Perron-Frobenius

ergodicity ⇒ convergence

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t\to\infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence (Perron-Frobenius)

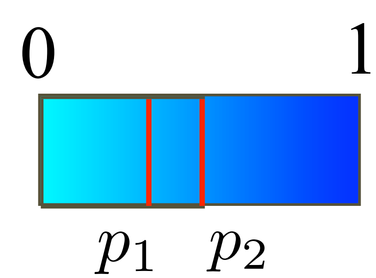
irreducibility, ergodicity ⇒ uniqueness, convergence (coupling)

Theorem

If $0 \leq p_1 \leq p_2 \leq 1$, then

$\Pr[G(n, p_1) \text{ is connected }] \leq \Pr[G(n, p_2) \text{ is connected }]$.

x_{uv} uniform over [0,1].



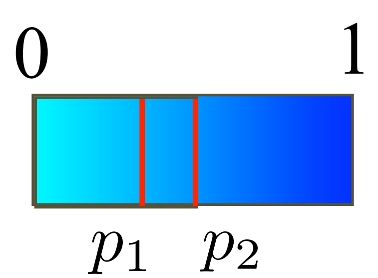
$$uv \in G(n, p_1) \Leftrightarrow x_{uv} \leq p_1 \quad \Rightarrow \quad x_{uv} \leq p_2 \Leftrightarrow uv \in G(n, p_2)$$

$$G(n, p_1) \text{ is connected} \quad \Rightarrow \quad G(n, p_2) \text{ is connected}$$

Coupling

"Compare two unrelated variables by forcing them to be related in some way."

x_{uv} uniform over $[0,1]$.



$$uv \in G(n, p_1) \Leftrightarrow x_{uv} \leq p_1 \quad \Rightarrow \quad x_{uv} \leq p_2 \Leftrightarrow uv \in G(n, p_2)$$

$$G(n, p_1) \text{ is connected} \quad \Rightarrow \quad G(n, p_2) \text{ is connected}$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t\to\infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence (Perron-Frobenius)

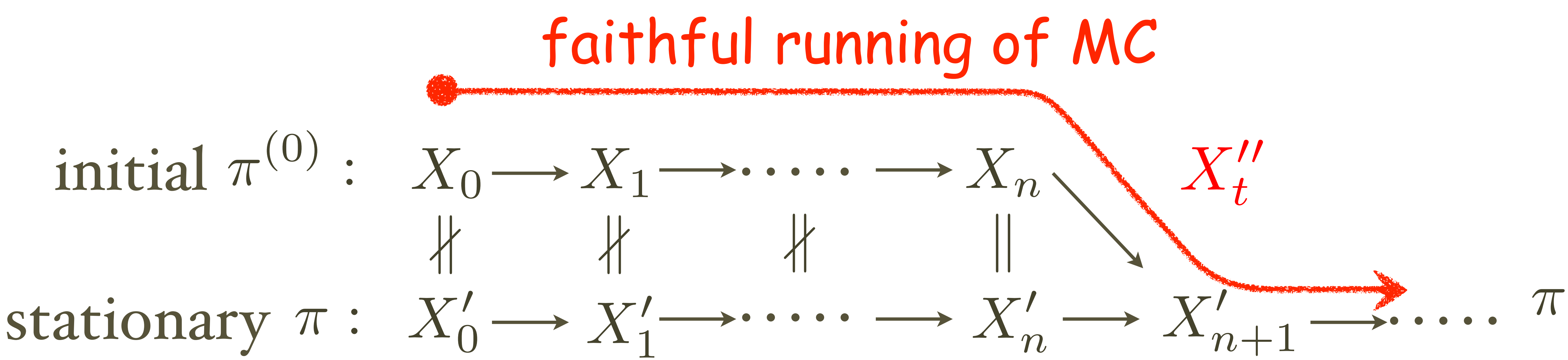
irreducibility, ergodicity ⇒ uniqueness, convergence (coupling)

Coupling of Markov Chains

Markov chain $\mathfrak{M} = (\Omega, P)$

initial $\pi^{(0)}$: $X_0, X_1, X_2, \ldots$

stationary π : $X'_0, X'_1, X'_2, \ldots$



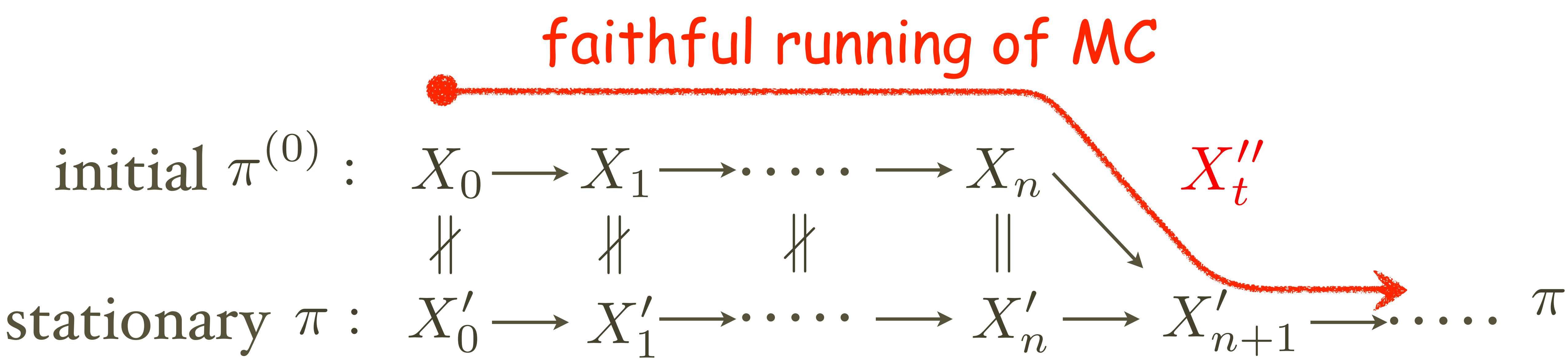
Goal: $\lim_{t\to\infty} \Pr[X''_t = X'_t] = 1$

Coupling of Markov Chains

Markov chain $\mathfrak{M} = (\Omega, P)$

initial $\pi^{(0)}$: $X_0, X_1, X_2, \ldots$

stationary π : $X'_0, X'_1, X'_2, \ldots$



Goal: conditioning on $X_0 = x, X'_0 = y$

$$\exists n, \Pr[X_n = X'_n = y] > 0$$

Markov chain $\mathfrak{M} = (\Omega, P)$

$$
\begin{array}{lcccccccc}
\text{initial } \pi^{(0)} : & X_0 & \longrightarrow & X_1 & \longrightarrow \cdots\cdots & \longrightarrow & X_n & \searrow & \\
 & \nparallel & & \nparallel & \nparallel & & \parallel & & \\
\text{stationary } \pi : & X'_0 & \longrightarrow & X'_1 & \longrightarrow \cdots\cdots & \longrightarrow & X'_n & \longrightarrow & X'_{n+1} \longrightarrow \cdots\cdots \quad \pi
\end{array}
$$

conditioning on $X_0 = x, X'_0 = y$

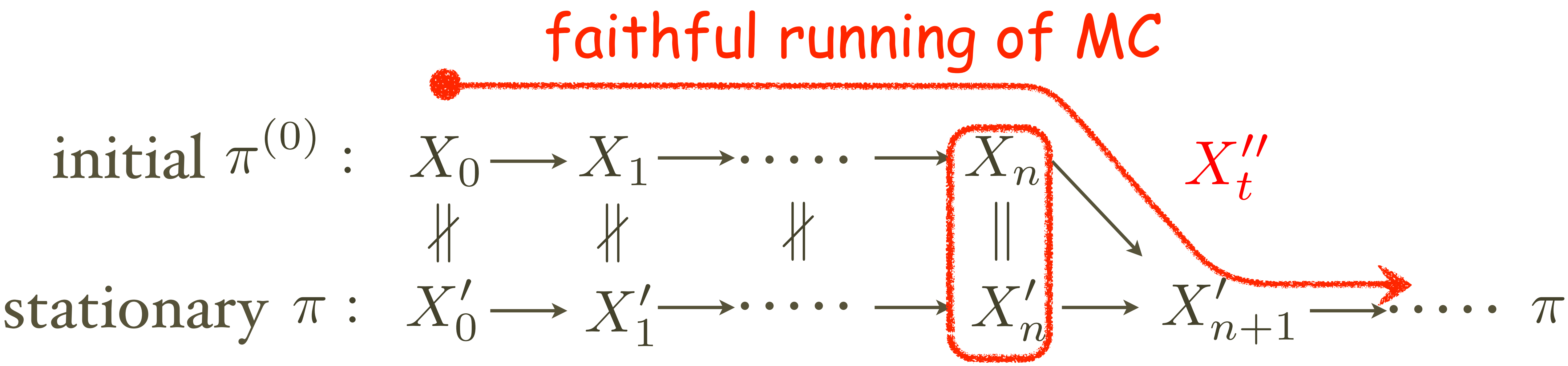
irreducible: $\exists n_1, \quad P^{n_1}(x, y) > 0$

aperiodic: $\exists n_2, \quad \begin{cases} P^{n_2}(y, y) > 0 \\ P^{n_1+n_2}(y, y) > 0 \end{cases}$

$$n = n_1 + n_2$$

$$
\begin{aligned}
\Pr[X_n = X'_n = y] &\geq \Pr[X_n = y] \Pr[X'_n = y] \\
&\geq P^{n_1}(x, y) P^{n_2}(y, y) P^{n}(y, y) \;\; > 0
\end{aligned}
$$

Markov chain $\mathfrak{M} = (\Omega, P)$



conditioning on $X_0 = x, X'_0 = y$

$$\exists n, \quad \Pr[X_n = X'_n] \geq \epsilon > 0$$

$$\lim_{t \to \infty} \Pr[X''_t = X'_t] = 1$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t\to\infty} \pi^{(0)} P^t = \pi$$

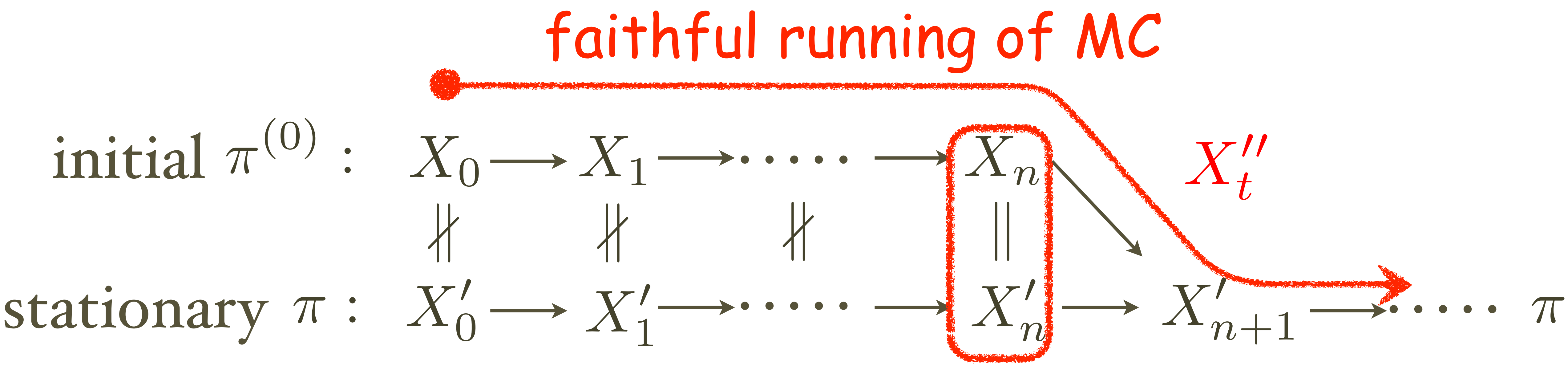
where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence (Perron-Frobenius)

irreducibility, ergodicity ⇒ uniqueness, convergence (coupling)

Markov chain $\mathfrak{M} = (\Omega, P)$



conditioning on $X_0 = x, X'_0 = y$

$$\exists n, \quad \Pr[X_n = X'_n] \geq \epsilon > 0$$

$$\lim_{t \to \infty} \Pr[X''_t = X'_t] = 1$$

Fundamental Theorem of Markov Chain:

If a *finite* Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *aperiodic*, then $\forall$ initial distribution $\pi^{(0)}$

(ergodic)

$$\lim_{t\to\infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finiteness ⇒ existence (Perron-Frobenius)

irreducibility, ergodicity ⇒ uniqueness, convergence (coupling)

Fundamental Theorem of Markov Chain:

If a Markov chain $\mathfrak{M} = (\Omega, P)$ is *irreducible* and *ergodic*, then $\forall$ initial distribution $\pi^{(0)}$

$$\lim_{t \to \infty} \pi^{(0)} P^t = \pi$$

where π is a *unique* stationary distribution satisfying

$$\pi P = \pi$$

finite: ergodic = aperiodic

infinite: ergodic = aperiodic + *non-null persistent*