Randomized Algorithms

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$\mathfrak{M} = (\Omega, P)$ stationary distribution: π

$p_x^{(t)}$: distribution at time t when initial state is x

$$\Delta_x(t) = \|p_x^{(t)} - \pi\|_{TV} \qquad \Delta(t) = \max_{x \in \Omega} \Delta_x(t)$$

$$\tau_x(\epsilon) = \min\{t \mid \Delta_x(t) \le \epsilon\} \qquad \tau(\epsilon) = \max_{x \in \Omega} \tau_x(\epsilon)$$

Theorem

P is symmetric, with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$

Let $\lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$

$$\tau(\epsilon) \le \frac{\frac{1}{2}\ln n + \ln \frac{1}{2\epsilon}}{1 - \lambda_{\max}}$$

Graph Spectrum

d-regular undirected graph $G(V,E)$

adjacency matrix A

eigenvalues: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ ← graph spectrum

Theorem

Lazy random walk on d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ has mixing rate

$$\tau(\epsilon) \leq \frac{d(\ln n + \ln \frac{1}{2\epsilon})}{d - \lambda_2}$$

d-regular undirected graph $G(V,E)$

graph spectrum : $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

1. $\forall i,\ |\lambda_i| \leq d.$
2. $\lambda_1 = d.$
3. Connected $\Leftrightarrow \lambda_1 > \lambda_2.$

spectral gap : $d - \lambda_2 = \lambda_1 - \lambda_2$

Theorem

$$\tau(\epsilon) \leq \frac{d(\ln n + \ln \frac{1}{2\epsilon})}{d - \lambda_2}$$

Expansion

undirected $G(V, E)$

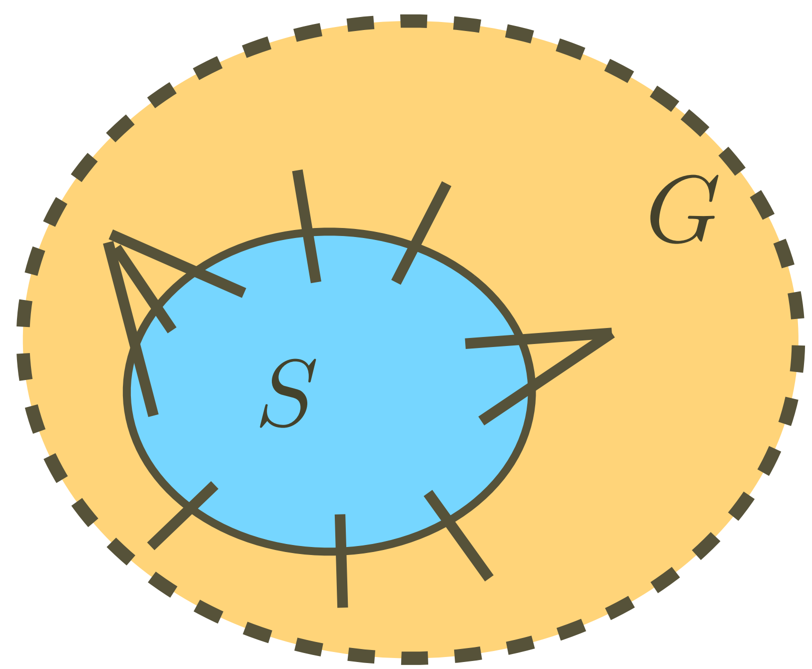
$$E(S, T) = \{uv \in E \mid u \in S, v \in T\}$$

edge boundary

$$\partial S = E(S, \bar{S})$$

expansion ratio

$$\phi(G) = \min_{\substack{S \subset V \\ |S| \leq \frac{n}{2}}} \frac{|\partial S|}{|S|}$$



Expander Graph

$$\phi(G) = \min_{\substack{S \subset V \\ |S| \le \frac{n}{2}}} \frac{|\partial S|}{|S|}$$

Expander graphs (combinatorial definition):

d-regular graphs with constant degree d

and constant expansion ratio $\phi(G)$.

- sparse;
- “expanding” (well connected);

“A Magical Graph!”

- Existence ?
  - random graph is an expander w.h.p.
- Computation ?
  - co-NP-complete

Theorem (Cheeger's inequality)

G is a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\Rightarrow \quad \frac{d-\lambda_2}{2} \leq \phi(G) \leq \sqrt{2d(d-\lambda_2)}$$

spectral gap : $d - \lambda_2 = \lambda_1 - \lambda_2$

expander graphs (algebraic definition):

d-regular graphs with constant degree d
and constant spectral gap

Eigenvalues vs Optimization

Rayleigh-Ritz Theorem

Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\lambda_1 = \max_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x} \qquad \lambda_2 = \max_{x \perp v_1} \frac{x^T A x}{x^T x}$$

Rayleigh quotient

Eigenvalues vs Optimization

Courant-Fischer Theorem

Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

$$\lambda_k = \max_{v_1, v_2, \ldots, v_{n-k} \in \mathbb{R}^n} \; \min_{\substack{x \in \mathbb{R}^n, x \neq \mathbf{0} \\ x \perp v_1, v_2, \ldots, v_{n-k}}} \frac{x^T A x}{x^T x}$$

$$= \min_{v_1, v_2, \ldots, v_{k-1} \in \mathbb{R}^n} \; \max_{\substack{x \in \mathbb{R}^n, x \neq \mathbf{0} \\ x \perp v_1, v_2, \ldots, v_{k-1}}} \frac{x^T A x}{x^T x}$$

Eigenvalues vs Optimization

Rayleigh-Ritz Theorem

Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then

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Graph Laplacian

d-regular undirected graph $G(V,E)$

adjacency matrix A

graph Laplacian $L = dI - A$

$$L(u,v) = \begin{cases} d & u = v \\ -1 & u \neq v, \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

quadratic form:

$$x^T L x = \sum_{v \in V} \sum_{uv \in E} (x_v^2 - x_u x_v) = \sum_{uv \in E} (x_u - x_v)^2$$

Variational Characterization

d-regular undirected graph $G(V,E)$

graph spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

graph Laplacian $L = dI - A$

$$d - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{x^T L x}{x^T x} = \min_{x \perp \mathbf{1}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\min_{x \perp \mathbf{1}} \frac{x^T L x}{x^T x} = \min_{x \perp \mathbf{1}} \frac{x^T (dI - A) x}{x^T x} = \min_{x \perp \mathbf{1}} \left(d - \frac{x^T A x}{x^T x} \right)$$

$$= d - \max_{x \perp \mathbf{1}} \frac{x^T A x}{x^T x} = d - \lambda_2$$ (Rayleigh-Ritz)

Variational Characterization

d-regular undirected graph $G(V,E)$

graph spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

graph Laplacian $L = dI - A$

$$d - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{x^T L x}{x^T x} = \min_{x \perp \mathbf{1}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\min_{x \perp \mathbf{1}} \frac{x^T L x}{x^T x} = d - \lambda_2$$

$$x^T L x = \sum_{uv \in E} (x_u - x_v)^2$$

d-regular undirected graph $G(V,E)$

graph spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$d - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\phi(G) = \min_{\substack{S \subset V \\ |S| \leq \frac{n}{2}}} \frac{|\partial S|}{|S|}$$

let $\chi_S(v) = \begin{cases} 1 & v \in S \\ 0 & v \notin S \end{cases}$

$$\frac{|\partial S|}{|S|} = \frac{\sum_{uv \in E} (\chi_S(u) - \chi_S(v))^2}{\sum_{v \in V} \chi_S(v)^2}$$

$$\phi(G) = \min_{\substack{x \in \{0,1\}^n \\ \|x\|_1 \leq \frac{n}{2}}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

d-regular undirected graph $G(V,E)$

graph spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$d - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{\sum_{uv \in E}(x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\phi(G) = \min_{\substack{x \in \{0,1\}^n \\ \|x\|_1 \leq \frac{n}{2}}} \frac{\sum_{uv \in E}(x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\phi(G) = \min_{\substack{S \subset V \\ |S| \leq \frac{n}{2}}} \frac{|\partial S|}{|S|} = \frac{|\partial S^*|}{|S^*|}$$

with $\begin{cases} S^* \subset V \\ |S^*| \leq \frac{n}{2} \end{cases}$

let $x \in \mathbb{R}^n$ **be** $x_v = \begin{cases} 1/|S^*| & \text{if } v \in S^* \\ -1/\left|\overline{S^*}\right| & \text{if } v \in \overline{S^*} \end{cases}$ $\Rightarrow$ $x \perp \mathbf{1}$

$$d - \lambda_2 \leq \frac{\sum_{uv \in E}(x_u - x_v)^2}{\sum_{v \in V} x_v^2} = \frac{\sum_{u \in S^*, v \in \overline{S^*}, uv \in E} \left(1/|S^*| + 1/|\overline{S^*}|\right)^2}{1/|S^*| + 1/|\overline{S^*}|}$$

$$= \left(\frac{1}{|S^*|} + \frac{1}{|\overline{S^*}|}\right) \cdot |\partial S^*| \leq \frac{2|\partial S^*|}{|S^*|} = 2\phi(G)$$

d-regular undirected graph $G(V,E)$

graph spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$d - \lambda_2 = \min_{x \perp \mathbf{1}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

$$\phi(G) = \min_{\substack{x \in \{0,1\}^n \\ \|x\|_1 \leq \frac{n}{2}}} \frac{\sum_{uv \in E} (x_u - x_v)^2}{\sum_{v \in V} x_v^2}$$

Theorem (Cheeger's inequality)

G is a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\Rightarrow \quad \frac{d - \lambda_2}{2} \leq \phi(G) \leq \sqrt{2d(d - \lambda_2)}$$

spectral gap : $d - \lambda_2 = \lambda_1 - \lambda_2$

$$\lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$$

symmetric chain

$$\tau(\epsilon) \le \frac{\frac{1}{2}\ln n + \ln \frac{1}{2\epsilon}}{1 - \lambda_{\max}}$$

lazy walk on d-regular graphs

$$\tau(\epsilon) \le \frac{d(\ln n + \ln \frac{1}{2\epsilon})}{d - \lambda_2}$$

Cheeger's inequality: $\frac{d - \lambda_2}{2} \le \phi(G) \le \sqrt{2d(d - \lambda_2)}$

$$\phi(G) = \min_{\substack{S \subset V \\ |S| \le \frac{n}{2}}} \frac{|\partial S|}{|S|}$$

lazy walk on d-regular graphs

$$\tau(\epsilon) \le \frac{2d^2(\ln n + \ln \frac{1}{2\epsilon})}{\phi^2}$$

Random walk on expander graph is rapidly mixing!

Reversibility

ergodic flow

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

time-reversible Markov chain:

$$\exists \pi, \forall, x, y \in \Omega, \quad \pi(x)P(x,y) = \pi(y)P(y,x)$$

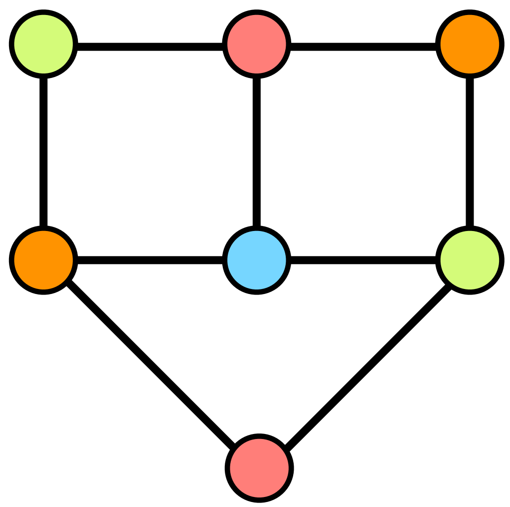
stationary distribution:

$$(\pi P)y = \sum_x \pi(x)P(x,y) = \sum_x \pi(y)P(y,x) = \pi(y)$$

time-reversible: when start from π

$$\Pr[X_0 = x_0 \wedge X_1 = x_1 \wedge \ldots \wedge X_n = x_n]$$

$$= \Pr[X_0 = x_n \wedge X_1 = x_{n-1} \wedge \ldots \wedge X_n = x_0]$$



$G(V,E)$ of max degree Δ

proper q-coloring

Markov Chain (Glauber dynamics):

at each step:

- randomly pick a vertex $v \in V$ and a color $c \in [q]$;
- change the color of v to c if it is proper;

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

$$\forall x \neq y, \quad P(x,y) = P(y,x) = \begin{cases} \frac{1}{qn} \\ 0 \end{cases}$$

π is uniform

Lazy Random Walk

undirected $G(V,E)$

$$P(u,v) = \begin{cases} \frac{1}{2} & u = v \\ \frac{1}{2d(u)} & u \sim v \\ 0 & \text{otherwise} \end{cases}$$

detailed balance equation:

$$\pi(u)P(u,v) = \pi(v)P(v,u)$$

$$\pi(u) = \frac{d(u)}{2m}$$

Metropolis Algorithm

neighborhood structure: undirected $G(\Omega, E)$

max degree Δ

Goal: uniform stationary distribution

choose $p \leq \frac{1}{2\Delta}$

$$P(x,y) = \begin{cases} p & \text{if } x \neq y \text{ and } xy \in E \\ 0 & \text{if } x \neq y \text{ and } xy \notin E \\ 1 - p \cdot d(x) & \text{if } x = y \end{cases}$$

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

Spectral Theory for Reversible Chains

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

$$\left.\begin{array}{l} \sqrt{\frac{\pi_x}{\pi_y}}P(x,y) = \sqrt{\frac{\pi_y}{\pi_x}}P(y,x) \\ \text{let } S(x,y) = \sqrt{\frac{\pi_x}{\pi_y}}P(x,y) \end{array}\right\} \Rightarrow$$

S is symmetry

$$\Pi(x,y) = \begin{cases} \sqrt{\pi_x} & x = y \\ 0 & x \neq y \end{cases} \Rightarrow \quad S = \Pi P \Pi^{-1}$$

$$qP^t = q(\Pi^{-1}S\Pi)^t = q\Pi^{-1}S^t\Pi$$

Spectral Theory for Reversible Chains

detailed balance equation:

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

$$\Pi(x,y) = \begin{cases} \sqrt{\pi_x} & x = y \\ 0 & x \neq y \end{cases} \qquad S = \Pi P \Pi^{-1}$$

$$qP^t = q(\Pi^{-1}S\Pi)^t = q\Pi^{-1}S^t\Pi$$

$$\Delta_x(t) = \|p_x^t - \pi\|_{TV} = \|(e_x - \pi)P^t\|_1$$

$$= \|(e_x - \pi)\Pi^{-1}S^t\Pi\|_1 \le \lambda_{\max}^t \sqrt{\frac{1-\pi_x}{\pi_x}}$$

where $\lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$

$\mathfrak{M} = (\Omega, P)$ stationary distribution: π

$p_x^{(t)}$: distribution at time t when initial state is x

$$\Delta_x(t) = \|p_x^{(t)} - \pi\|_{TV} \qquad \Delta(t) = \max_{x \in \Omega} \Delta_x(t)$$

$$\tau_x(\epsilon) = \min\{t \mid \Delta_x(t) \leq \epsilon\} \qquad \tau(\epsilon) = \max_{x \in \Omega} \tau_x(\epsilon)$$

Theorem

(Ω, P) is reversible, with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$

Let $\lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$

$$\tau(\epsilon) \leq \max_{x \in \Omega} \frac{\frac{1}{2} \ln \frac{1}{\pi_x} + \ln \frac{1}{2\epsilon}}{1 - \lambda_{\max}}$$

Conductance

reversible Markov chain $\mathfrak{M} = (\Omega, P)$

$$\pi(x)P(x,y) = \pi(y)P(y,x)$$

conductance:

$$\Phi = \min_{\substack{S\subset\Omega \\ 0<\pi(S)\le\frac{1}{2}}} \frac{\sum_{x\in S, y\not\in S} \pi_x P(x,y)}{\pi(S)}$$

graph expansion:

$$\phi(G) = \min_{\substack{S\subset V \\ |S|\le\frac{n}{2}}} \frac{|\partial S|}{|S|} = \min_{\substack{S\subset V \\ |S|\le\frac{n}{2}}} \frac{|\{uv\in E \mid u\in S, v\not\in S\}|}{|S|}$$

conductance:

$$\Phi = \min_{\substack{S\subset\Omega \\ 0<\pi(S)\le\frac{1}{2}}} \frac{\sum_{x\in S, y\notin S} \pi_x P(x,y)}{\pi(S)}$$

lazy walk on $G(V,E)$ $\qquad \pi_v = \frac{d(v)}{2m} \qquad \pi_u P(u,v) = \frac{1}{2m}$

$$\Phi = \min_{\substack{S\subset\Omega \\ \sum_{v\in S} d(v)\le m}} \frac{|\{uv\in E \mid u\in S, v\notin S\}|}{\sum_{v\in S} d(v)}$$

graph expansion:

$$\phi(G) = \min_{\substack{S\subset V \\ |S|\le\frac{n}{2}}} \frac{|\partial S|}{|S|} = \min_{\substack{S\subset V \\ |S|\le\frac{n}{2}}} \frac{|\{uv\in E \mid u\in S, v\notin S\}|}{|S|}$$

on *d*-regular graph $\qquad \Phi = \frac{\phi}{d}$

Theorem (Cheeger's inequality)

G is a d-regular graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\Rightarrow \quad \frac{d - \lambda_2}{2} \leq \phi(G) \leq \sqrt{2d(d - \lambda_2)}$$

Theorem (Jerrum-Sinclair 1988)

For reversible chain with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\Rightarrow \quad \frac{1 - \lambda_2}{2} \leq \Phi \leq \sqrt{2(1 - \lambda_2)}$$

Theorem (Jerrum-Sinclair 1988)

For reversible chain with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

$$\Rightarrow \quad \frac{1-\lambda_2}{2} \leq \Phi \leq \sqrt{2(1-\lambda_2)}$$

$$\tau(\epsilon) \leq \max_{x\in\Omega} \frac{\frac{1}{2}\ln\frac{1}{\pi_x} + \ln\frac{1}{2\epsilon}}{1-\lambda_{\max}} \qquad \lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$$

Theorem: For lazy reversible Markov chain

$$\tau(\epsilon) \leq \max_{x\in\Omega} \frac{\ln\frac{1}{\pi_x} + 2\ln\frac{1}{2\epsilon}}{\Phi^2}$$

Canonical Path

reversible Markov chain $\mathfrak{M} = (\Omega, P)$

canonical paths: $\Gamma = \{\gamma_{xy} \mid x, y \in \Omega, x \neq y\}$

γ_{xy} : path from x to y in transition graph

congestion: $$\rho = \max_{uv \in E} \frac{1}{\pi_u P(u,v)} \sum_{\gamma_{xy} \ni uv} \pi_x \pi_y$$

conductance: $$\Phi = \min_{\substack{S \subset \Omega \\ 0 < \pi(S) \leq \frac{1}{2}}} \frac{\sum_{x \in S, y \notin S} \pi_x P(x,y)}{\pi(S)}$$

Theorem (Jerrum-Sinclair 1988)

For any canonical paths, $\Phi \geq \frac{1}{2\rho}$

$$\Gamma = \{\gamma_{xy} \mid x, y \in \Omega, x \neq y\}$$

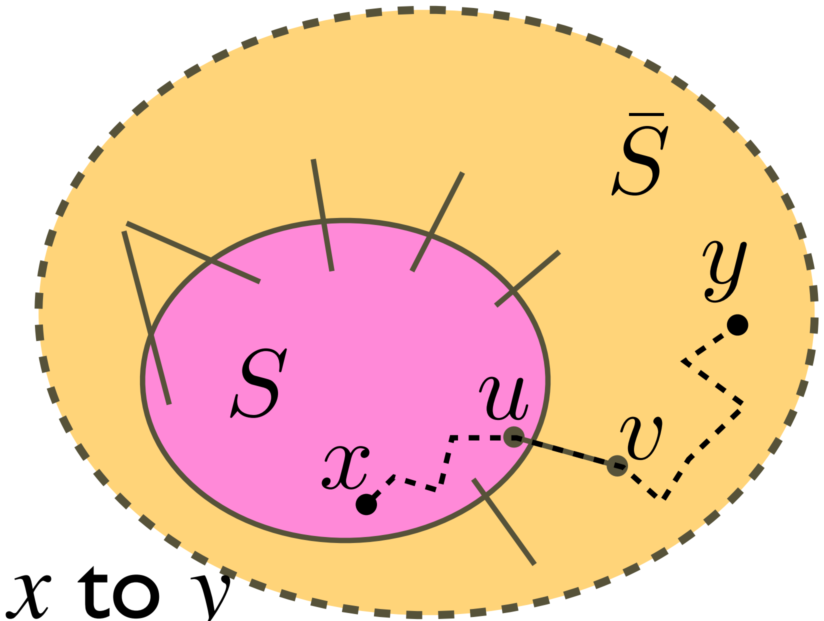
$$\rho = \max_{uv \in E} \frac{1}{\pi_u P(u,v)} \sum_{\gamma_{xy} \ni uv} \pi_x \pi_y$$

$$\Phi = \min_{\substack{S \subset \Omega \\ 0 < \pi(S) \le \frac{1}{2}}} \frac{\sum_{x \in S, y \notin S} \pi_x P(x,y)}{\pi(S)}$$

$$\Phi \ge \frac{1}{2\rho}$$

say:

$$\Phi = \frac{\sum_{x \in S, y \notin S} \pi_x P(x,y)}{\pi(S)}$$



each path routes $\pi_x \pi_y$ units of flow from x to y

total flow from S to $\bar{S}$:

$$\frac{\pi(S)}{2} \le \pi(S)\pi(\bar{S}) = \sum_{x \in S, y \notin S} \pi_x \pi_y$$

$$\le \sum_{u \in S, v \notin S} \sum_{\gamma_{xy} \ni uv} \pi_x \pi_y \le \rho \sum_{u \in S, v \notin S} \pi_u P(u,v)$$

$$\lambda_{\max} = \max\{|\lambda_2|, |\lambda_n|\}$$

reversible chain:

$$\tau_x(\epsilon) \le \frac{\frac{1}{2}\ln\frac{1}{\pi_x} + \ln\frac{1}{2\epsilon}}{1-\lambda_{\max}}$$

conductance:

$$\Phi = \min_{\substack{S\subset\Omega \\ 0<\pi(S)\le\frac{1}{2}}} \frac{\sum_{x\in S, y\notin S}\pi_x P(x,y)}{\pi(S)}$$

$$\frac{1-\lambda_2}{2} \le \Phi \le \sqrt{2(1-\lambda_2)}$$

$$\Phi \ge \frac{1}{2\rho}$$

canonical paths: $\Gamma = \{\gamma_{xy} \mid x, y \in \Omega, x \neq y\}$

congestion: $\rho = \max_{uv\in E} \frac{1}{\pi_u P(u,v)} \sum_{\gamma_{xy}\ni uv} \pi_x \pi_y$

lazy reversible chain:

$$\tau_x(\epsilon) \le \frac{\ln\frac{1}{\pi_x} + 2\ln\frac{1}{2\epsilon}}{\Phi^2}$$

lazy reversible chain:

$$\tau_x(\epsilon) \le 4\left(\ln\frac{1}{\pi_x} + 2\ln\frac{1}{2\epsilon}\right)\rho^2$$

canonical paths: $\Gamma = \{\gamma_{xy} \mid x, y \in \Omega, x \neq y\}$

congestion: $\rho = \max_{uv \in E} \frac{1}{\pi_u P(u,v)} \sum_{\gamma_{xy} \ni uv} \pi_x \pi_y$

lazy reversible chain:

$$\tau_x(\epsilon) \leq 4\left(\ln\frac{1}{\pi_x} + 2\ln\frac{1}{2\epsilon}\right)\rho^2$$

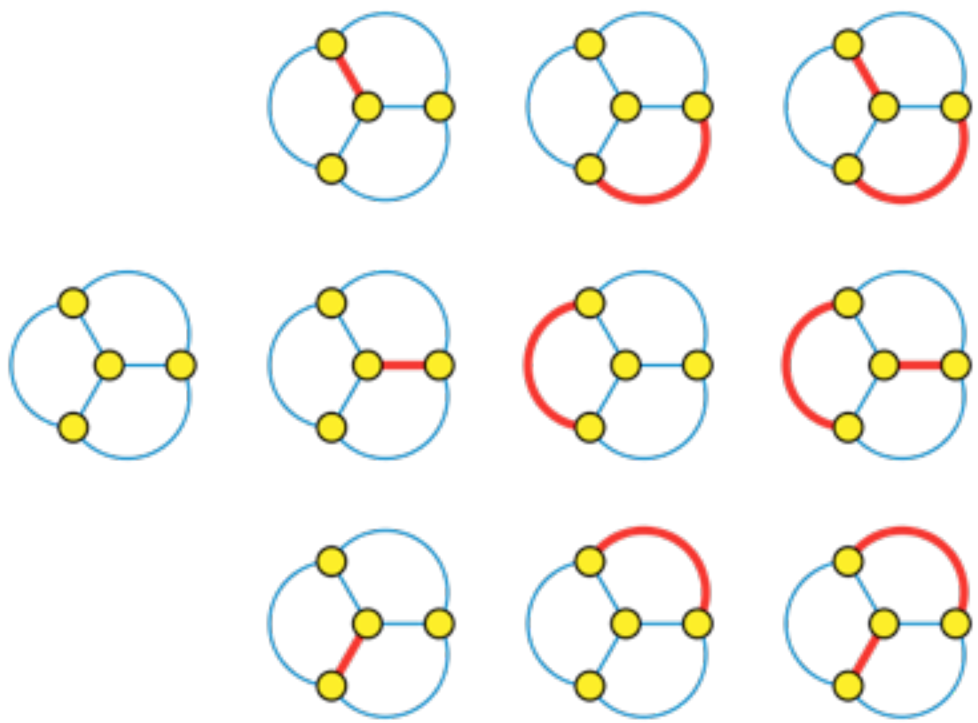
for uniform stationary distribution: $\pi(x) = \frac{1}{N}$

$$\rho = \max_{uv \in E} \frac{|\{\gamma_{xy} \ni uv\}|}{N \cdot P(u,v)}$$

usually $\frac{1}{P(x,y)} = \mathrm{polylog}(N)$

Goal: $|\{\gamma_{xy} \ni uv\}| = O(N \cdot \mathrm{polylog}(N))$

Sample Matchings



$G(V,E)$

matching: $M \subseteq E$

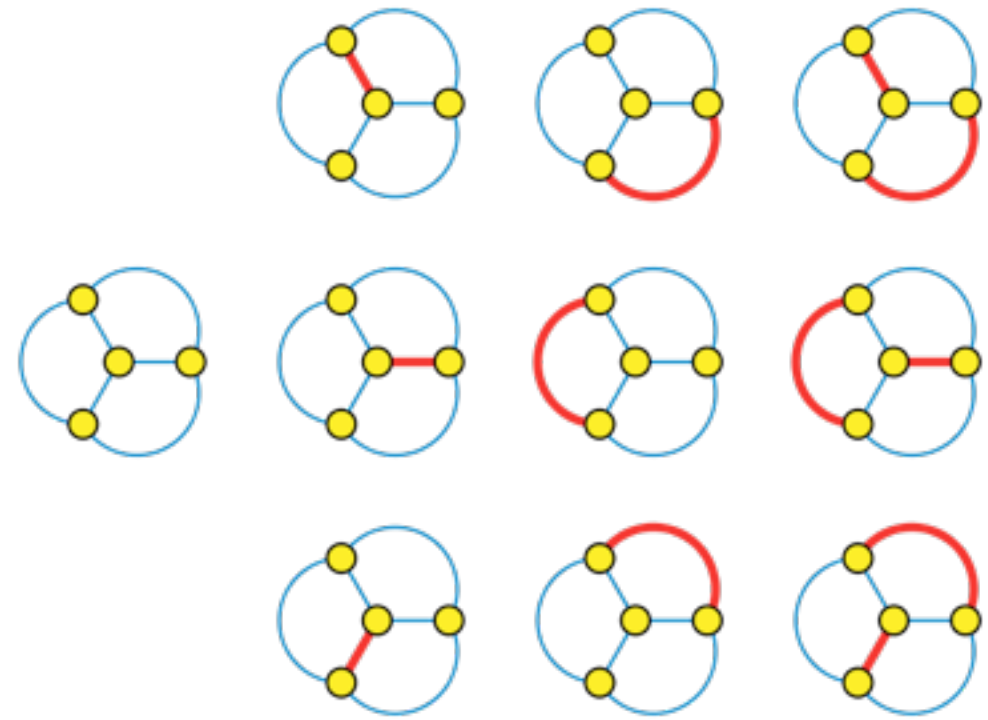
$\forall e_1, e_2 \in M$

no common vertex

input: $G(V,E)$

sampling: uniformly sample a matching in *G*

counting: count # of matchings in *G* #P-hard the Jerrum-Sinclair random walk of matchings



at each step: M

- with probability $1/2$, do nothing;
- pick a uniform random edge $e=uv\in E$;
- ($\uparrow$) if none of u, v is matched, add e to M;

  ($\downarrow$) if $e\in M$, remove e from M;

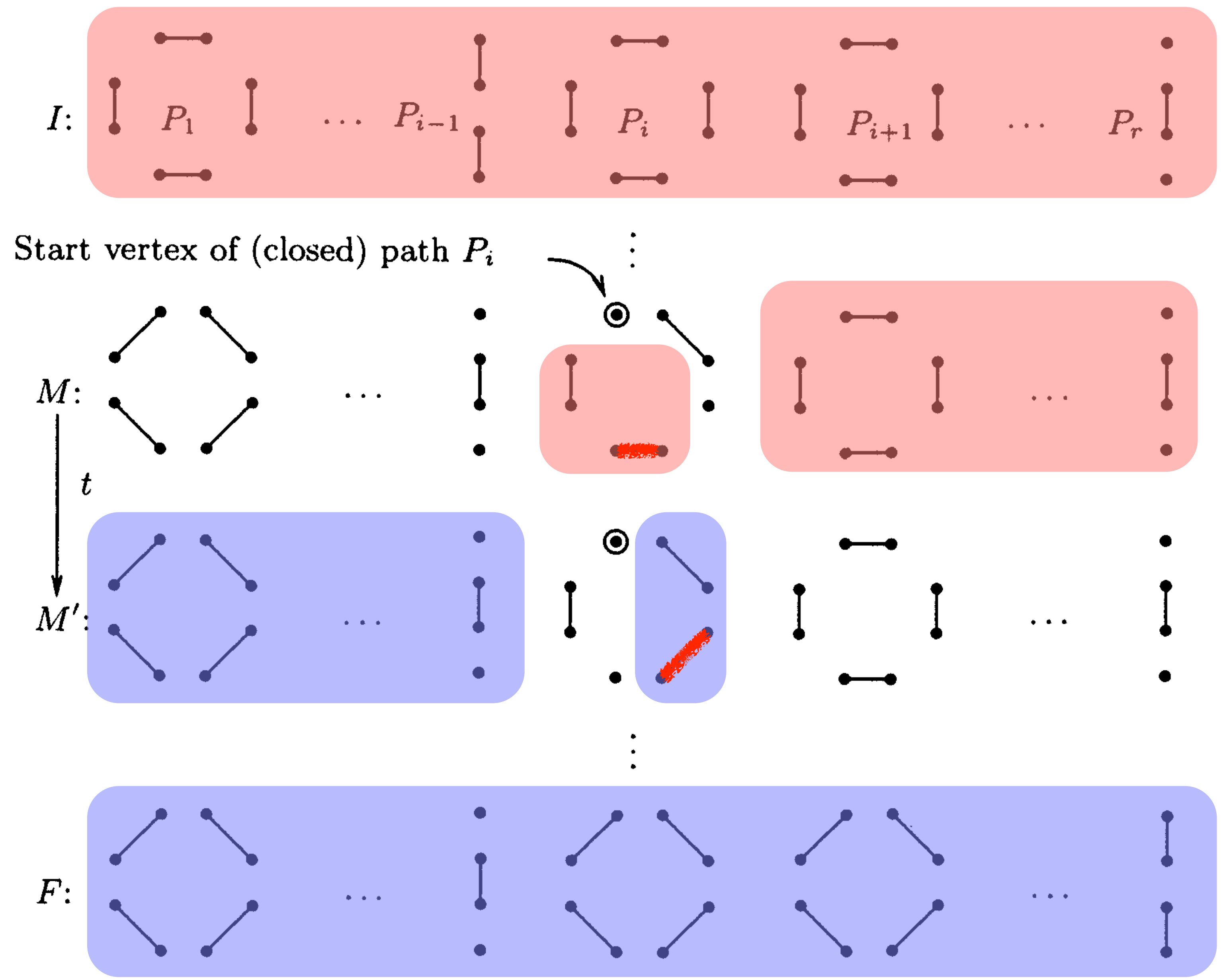
  ($\leftrightarrow$) if one of u, v is matched by some $e'\in M$,

  replace e' by e;

initial matching
I

canonical path
γ_{IF}

final matching
F



$I \oplus F$: disjoint cycles and paths (in lexicographic order)

γ_{IF} : fix each cycle and path by ↑, ↓, ↔

transition (M, M') $\quad |\{(I,F) \mid (M,M') \in \gamma_{IF}\}| = N \cdot \mathrm{poly}(n)$

at each step: M

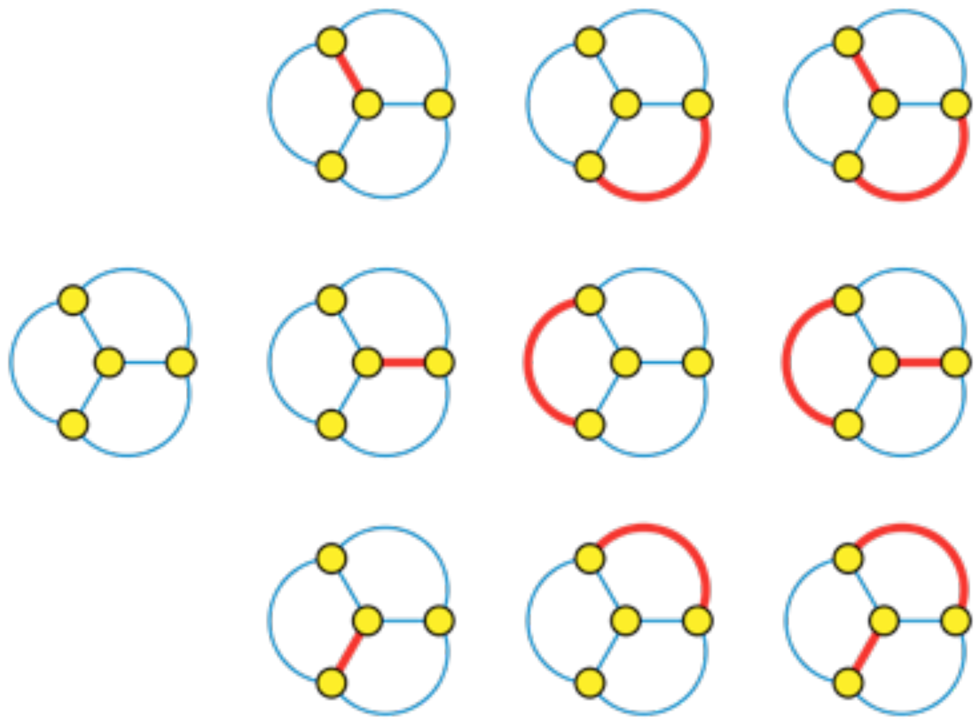
- with probability $1/2$, do nothing;
- pick a uniform random edge $e=uv\in E$;
- (↑) if none of u, v is matched, add e to M;
  (↓) if $e\in M$, remove e from M;
  (↔) if one of u, v is matched by some $e'\in M$,
  replace e' by e;

$$|\{(I,F)\mid (M,M')\in\gamma_{IF}\}| = N\cdot\mathrm{poly}(n)$$

$$\begin{aligned}\rho &= \max_{M,M'}\frac{1}{\pi_M P(M,M')}\sum_{\gamma_{IF}\ni(M,M')}\pi_I\pi_F\\ &= \max_{M,M'}\frac{|\{(I,F)\mid (M,M')\in\gamma_{IF}\}|}{2mN} = \mathrm{poly}(n)\end{aligned}$$

$$\tau(\epsilon)\le 4\left(\ln N + 2\ln\frac{1}{2\epsilon}\right)\rho^2 \qquad \tau_{\mathrm{mix}} = \mathrm{poly}(n)$$

Counting and Sampling



input: $G(V,E)$

counting:
count # of matchings in G

suppose there are N matchings in G

uniform random matching M

say: $E = \{e_1, e_2, \ldots, e_m\}$

$$\frac{1}{N} = \Pr[M = \emptyset] = \prod_{i=1}^{m} \Pr[e_i \notin M \mid e_1, \ldots, e_{i-1} \notin M]$$

$$N = \prod_{i=1}^{m} \frac{1}{\Pr[e_i \notin M \mid e_1, \ldots, e_{i-1} \notin M]}$$

uniform random matching M say: $E = \{e_1, e_2, \ldots, e_m\}$

$$N = \prod_{i=1}^{m} \frac{1}{\Pr[e_i \notin M \mid e_1, \ldots, e_{i-1} \notin M]}$$

$$\Pr[e_i \notin M \mid e_1, \ldots, e_{i-1} \notin M] = \Pr_{M \subseteq G \setminus \{e_1, \ldots, e_{i-1}\}}[e_i \notin M]$$

sample M almost uniformly ⇒ accurately estimate $\Pr[e_i \notin M]$

⇒ well approximate N

FPRAS (Fully Poly-time Randomized Approximation Scheme) :

For any input G and ε, algorithm returns $A(G)$

s.t. $\Pr[(1-\epsilon)N \leq A(G) \leq (1+\epsilon)N] \geq 1-\delta$

in time $\mathrm{poly}(n, \frac{1}{\epsilon}, \log \frac{1}{\delta})$.

The End