Randomized Algorithms

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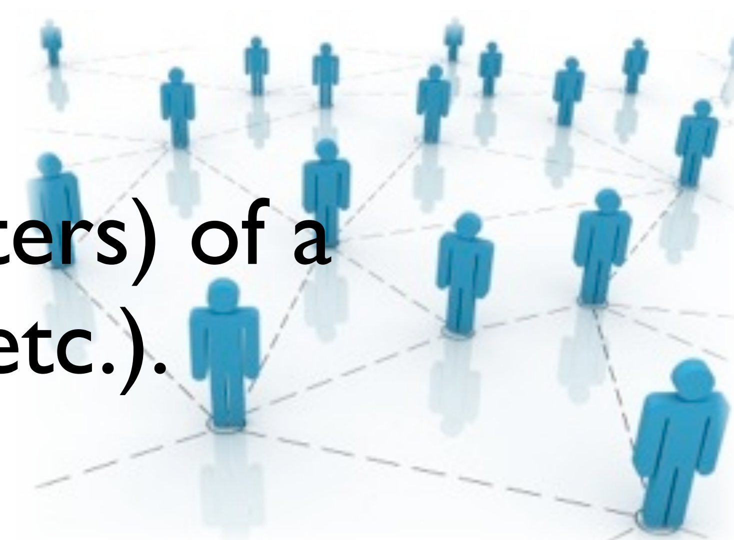
- Keep flipping a fair coin, the difference between #HEADs and #TAILs.



- Chromatic number (or other parameters) of a random graph (web, social networks, etc.).
- Number of appearances of a substring in a long random sequence (DNA pattern matching).

- A sequence of dependent random variables:

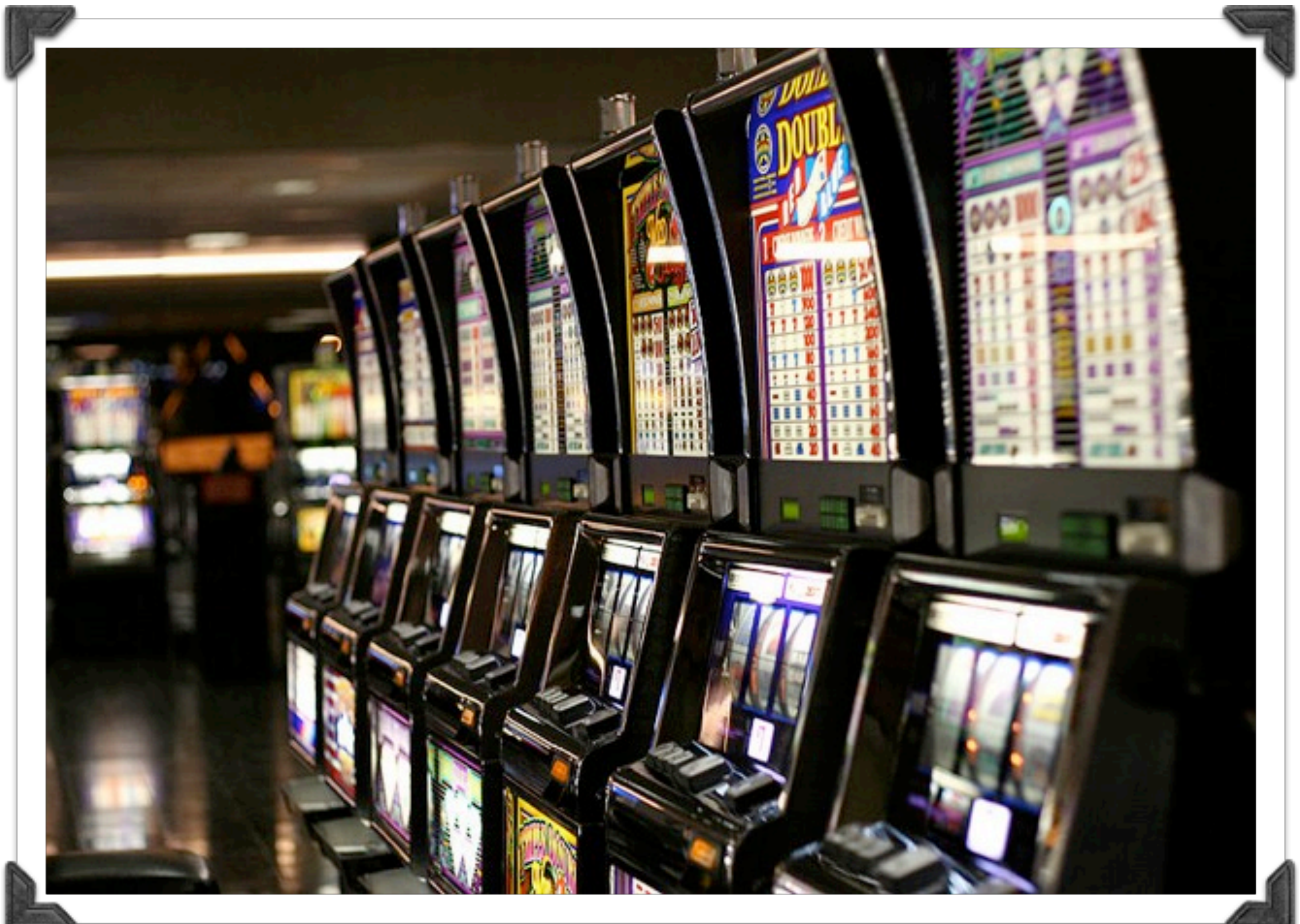
$$X_0, X_1, X_2 \ldots, X_n \qquad\qquad |X_n - X_0| < \text{ ?}$$

- A function (not just sum) of random variables:

$$f(X_1, X_2, \ldots, X_n)$$

$$|f(X_1, X_2, \ldots, X_n) - \mathbf{E}[f(X_1, X_2, \ldots, X_n)]| < \text{ ?}$$

Martingales



Martingale

Definition:

A sequence of random variables $X_0, X_1, \ldots$ is a **martingale** if for all $i > 0$,

$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

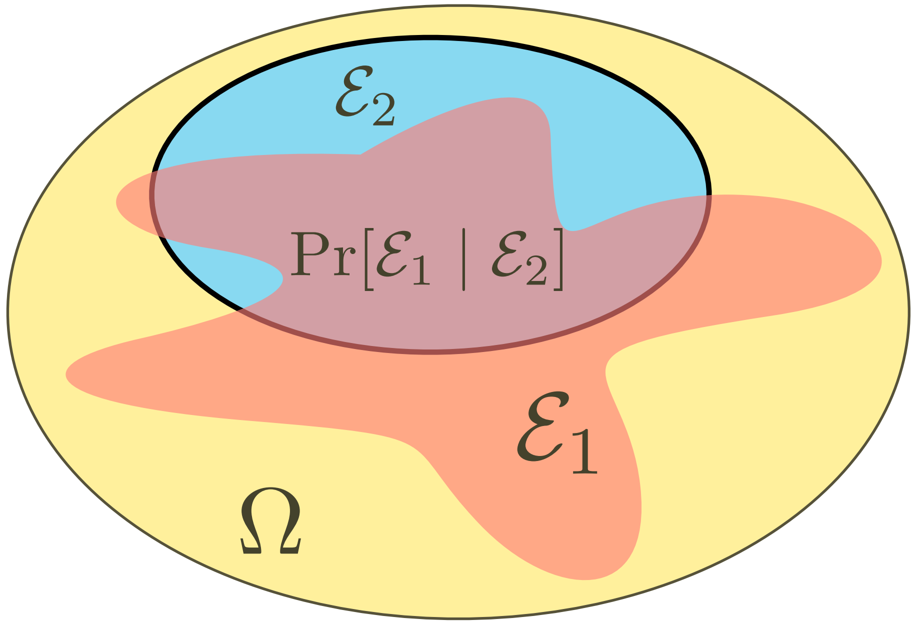
What does this mean?

Conditional Probability

Definition:

The **conditional probability** that event $\mathcal{E}_1$ occurs given that event $\mathcal{E}_2$ occurs is

$$\Pr[\mathcal{E}_1 \mid \mathcal{E}_2] = \frac{\Pr[\mathcal{E}_1 \wedge \mathcal{E}_2]}{\Pr[\mathcal{E}_2]}.$$



Conditional Expectation

Conditional expectation of Y with respect to $\mathcal{E}$:

$$\mathbf{E}[Y \mid \mathcal{E}] = \sum_y y \Pr[Y = y \mid \mathcal{E}]$$

Example: for a uniform random human being

Y: height

X: country

$$\mathbf{E}[Y \mid X = \text{"China"}]$$

$$f(x) = \mathbf{E}[Y \mid X = x]$$

Conditional Expectation

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X: country

$$\mathbf{E}[Y \mid X = \text{“China”}]$$

$$f(X) = \mathbf{E}[Y \mid X = x]$$

a random variable

uniform random person,

the average height of his/her country

$$\mathbf{E}[Y \mid X, Z]$$ gender

the average male/female height of his/her country

Fundamental facts about conditional expectation:

1. $\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Y]]$

average height over all individuals
= average height country-by-country

2. $\mathbf{E}[X \mid Z] = \mathbf{E}[\mathbf{E}[X \mid Y, Z] \mid Z]$

average height over all individuals with the same sex
= do it on a country-by-country basis

3. $\mathbf{E}[\mathbf{E}[f(X)g(X,Y) \mid X]] = \mathbf{E}[f(X) \cdot \mathbf{E}[g(X,Y) \mid X]]$

conditioning on any $X = x$, $f(x)$ is a constant

Martingales

- Origin: a betting strategy
  - "double your bet after every loss"

  a win after n losses: $2^n - \sum_{i=1}^{n} 2^{i-1} = 1$ positive!

- For a fair game, with any betting strategy

$$\text{our wealth: } X_0, X_1, X_2, \ldots$$

  - Conditioning on the past, we expect no change to current value. (the game is fair)

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$\forall x_0, x_1, \ldots, x_{i-1},$

$$\mathbf{E}[X_i \mid X_0 = x_0, X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] = x_{i-1}$$

$$\mathbf{E}[X_i - X_{i-1} \mid X_0, \ldots, X_{i-1}] = 0$$

$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

Example I:
coin flipping

flip a fair coin for many times,
the differences between #HEADs and #TAILs

independent uniform $Z_1, Z_2, \ldots \in \{-1, 1\}$

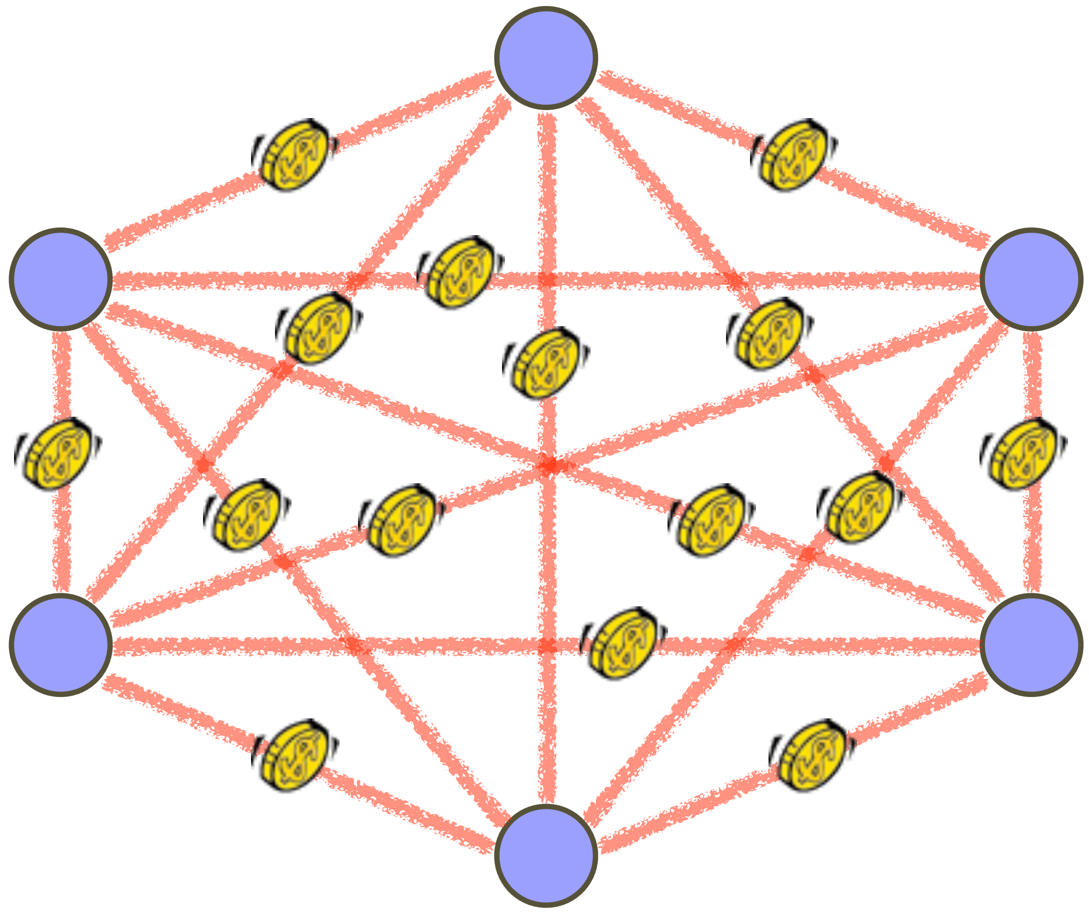
$X_0 = 0$ difference after *i* flips: $X_i = \sum_{j \le i} Z_j$

$$\begin{aligned}\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] - X_{i-1} &= \mathbf{E}[X_i - X_{i-1} \mid X_0, \ldots, X_{i-1}] \\ &= \mathbf{E}[Z_j \mid X_0, \ldots, X_{i-1}] \\ &= \mathbf{E}[Z_i] \quad = 0\end{aligned}$$

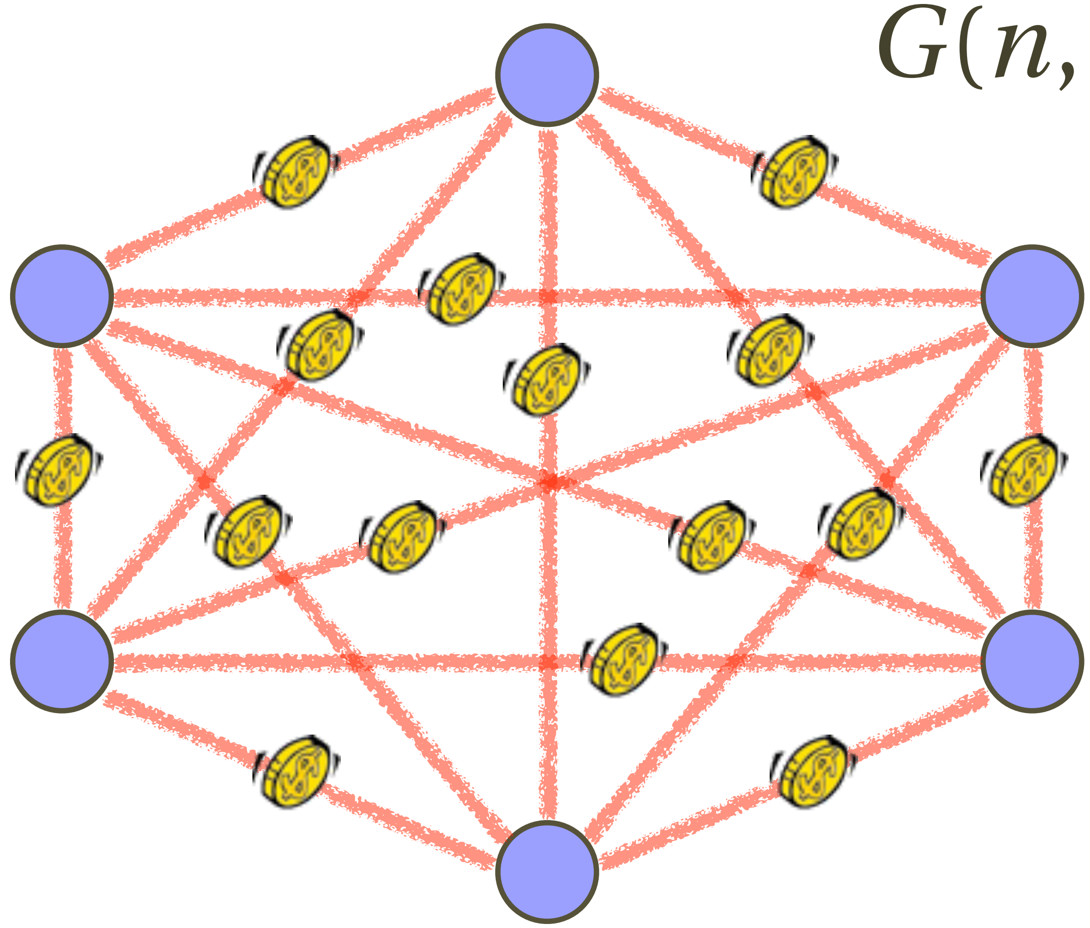
$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

Example II:
Chromatic number
of a random graph

Erdős–Rényi model for random graphs



$G(n, p)$



chromatic number:

$\chi(G)$

the smallest number of colors to properly color G

numbering all vertex-pairs: $1, 2, 3, \ldots, \binom{n}{2}$

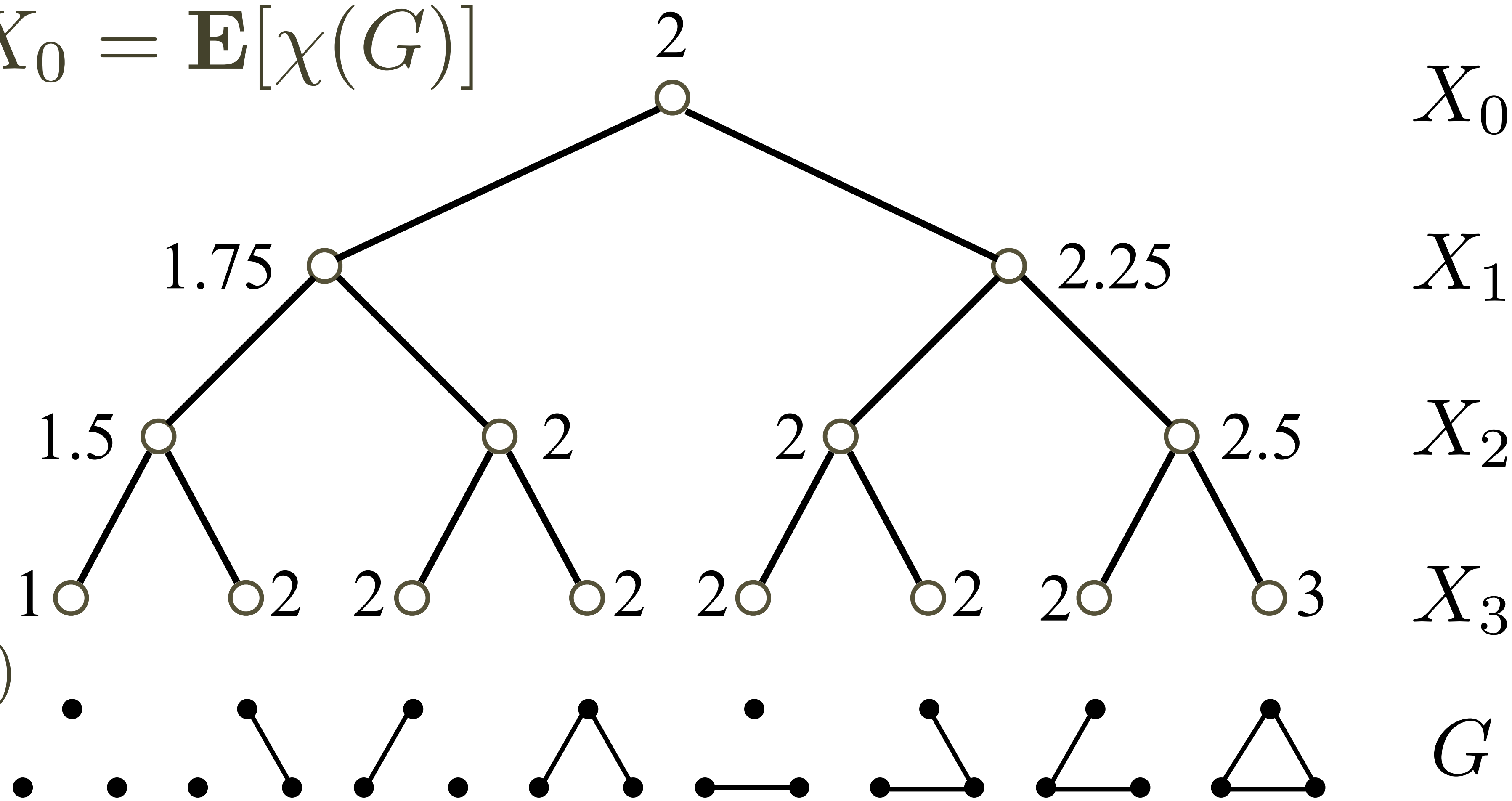
$$I_j = \begin{cases} 1 & \text{edge } j \in G \\ 0 & \text{edge } j \notin G \end{cases} \qquad X_i = \mathbf{E}[\chi(G) \mid I_1, \ldots, I_i]$$

$$X_0 = \mathbf{E}[\chi(G)] \dashrightarrow X_{\binom{n}{2}} = \chi(G)$$

The edge exposure martingale:

$$X_i = \mathbf{E}[\chi(G) \mid I_1, \ldots, I_i]$$

$$X_0 = \mathbf{E}[\chi(G)]$$



$$X_{\binom{n}{2}} = \chi(G)$$

$$\mathbf{E}[X_i \mid X_0, \ldots, X_{i-1}] = X_{i-1}$$

Martingale Tail Inequality

- For a martingale:

$$X_0, X_1, X_2, \ldots$$

- $\Pr[|X_n - X_0| \geq t] < ?$
    - deviation from initial value;
    - deviation from expectation.

Azuma's Inequality:

Let $X_0, X_1, \ldots$ be a martingale such that, for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c_k,$$

Then

$$\Pr\left[|X_n - X_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

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- For a sequence, if in each step:
  - averagely no change to the current value (martingale),
  - no big jump,
- the final does not deviate far from the initial.

Azuma's Inequality:

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Then

$$\Pr\left[|X_n - X_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

Corollary:

Let $X_0, X_1, \ldots$ be a martingale such that, for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c,$$

Then

$$\Pr\left[|X_n - X_0| \geq ct\sqrt{n}\right] \leq 2e^{-t^2/2}.$$

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1. Represent total difference as sum of step-wise differences.

   Let $Y_i = X_i - X_{i-1}$. $\quad X_n - X_0 = \sum_{i=1}^{n} Y_i$

2. Apply Markov's inequality to the moment generating function.

   $$\Pr[\sum_{i=1}^{n} Y_i \geq t] = \Pr[e^{\lambda \sum_{i=1}^{n} Y_i} \geq e^{\lambda t}] \leq \frac{\mathbf{E}[e^{\lambda \sum_{i=1}^{n} Y_i}]}{e^{\lambda t}}$$

3. Bound the moment generating function.

   by martingale property & convexity of MGF

Azuma's Inequality:
Let $X_0, X_1, \dots$ be a martingale such that, for all $k \geq 1$,

$$|X_k - X_{k-1}| \leq c,$$

Then

$$\Pr\left[|X_n - X_0| \geq ct\sqrt{n}\right] \leq 2e^{-t^2/2}.$$

independent $Z_1, Z_2, \dots, Z_n \in \{-1, 1\}$

$$Z_j = \begin{cases} 1 & \text{with prob } \frac{1}{2} \\ -1 & \text{with prob } \frac{1}{2} \end{cases} \qquad X_0 = 0 \qquad X_i = \sum_{j \leq i} Z_j$$

a martingale: $X_0, X_1, X_2, \dots, X_n$

$$|X_i - X_{i-1}| = |Z_i| \leq 1$$

$$\Rightarrow \quad \Pr[|X_n| \geq t\sqrt{n}] \leq 2e^{-t^2/2}$$

Generalization

Definition:

$Y_0, Y_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$ if, for all $i \geq 0$,

- Y_i is a function of $X_0, X_1, \ldots, X_i$;
- $\mathbf{E}[Y_{i+1} \mid X_0, \ldots, X_i] = Y_i$.

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- $\mathbf{E}[Y_{i+1} \mid X_0, \ldots, X_i] = Y_i$.

- Betting on a fair game;
- X_i : win/loss of the *i*-th bet;
- Y_i : wealth after the *i*-th bet -- Martingale (fair game)

Azuma's Inequality (general version):

Let $Y_0, Y_1, \ldots$ be a martingale with respect to $X_0, X_1, \ldots$ such that, for all $k \geq 1$,

$$|Y_k - Y_{k-1}| \leq c_k,$$

Then

$$\Pr\left[|Y_n - Y_0| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} c_k^2}\right).$$

Doob Sequence

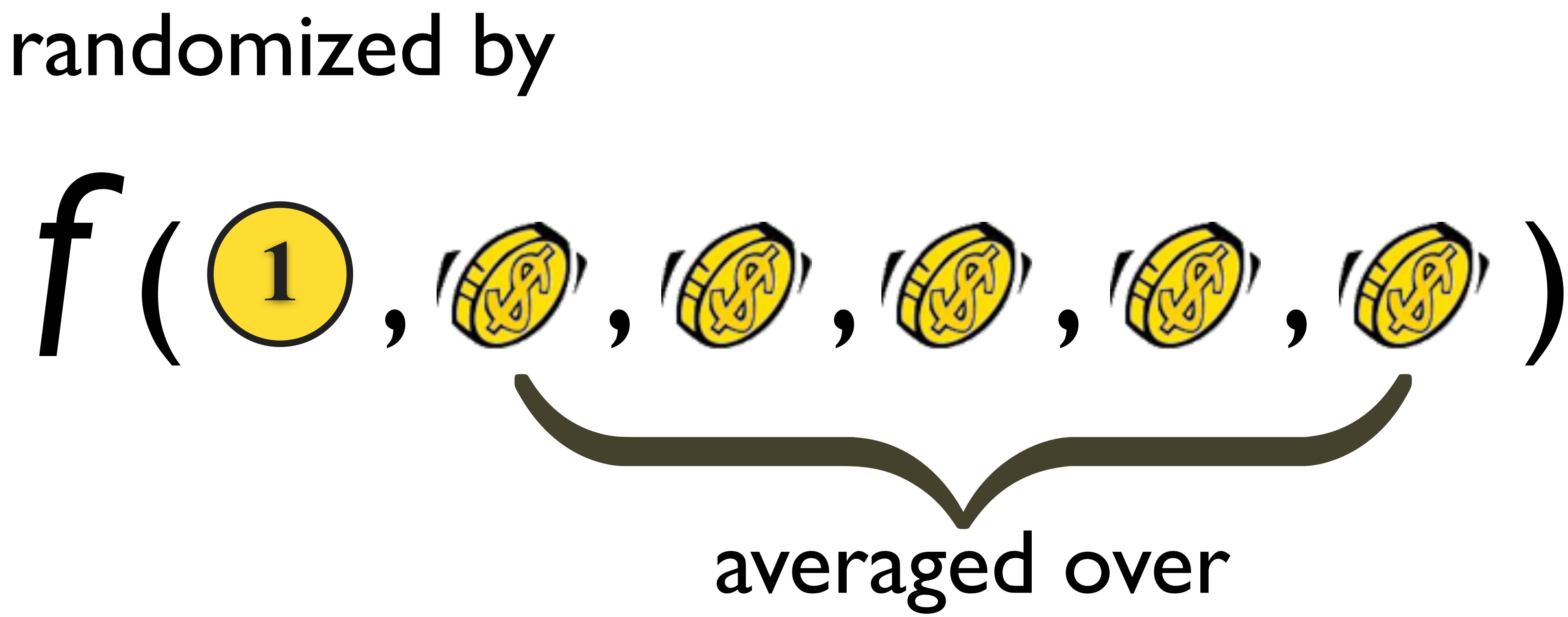
Definition (Doob sequence):

The Doob sequence of a function f with respect to a sequence $X_1, \ldots, X_n$ is

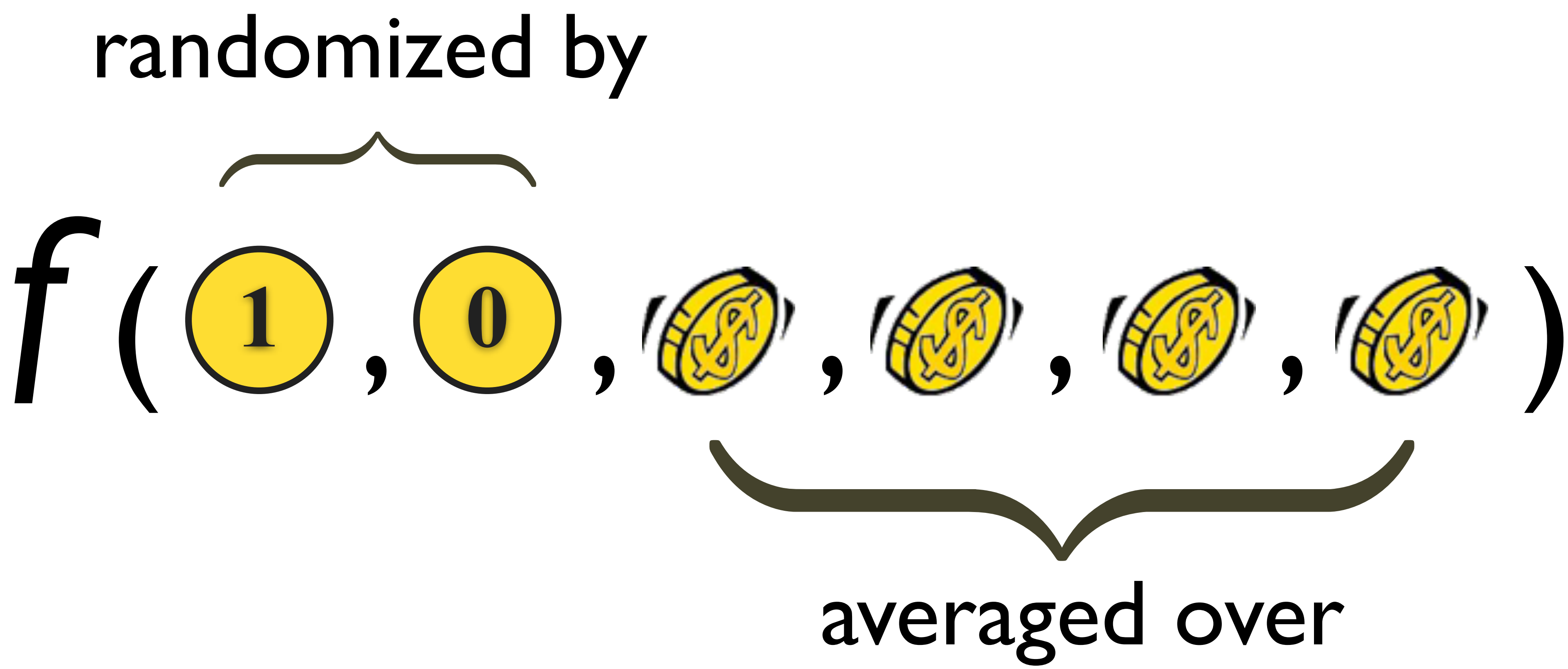
$$Y_i = \mathbf{E}[f(X_1, \ldots, X_n) \mid X_1, \ldots, X_i]$$

$$Y_0 = \mathbf{E}[f(X_1, \ldots, X_n)] \dashrightarrow Y_n = f(X_1, \ldots, X_n)$$

Doob Sequence



Doob Sequence



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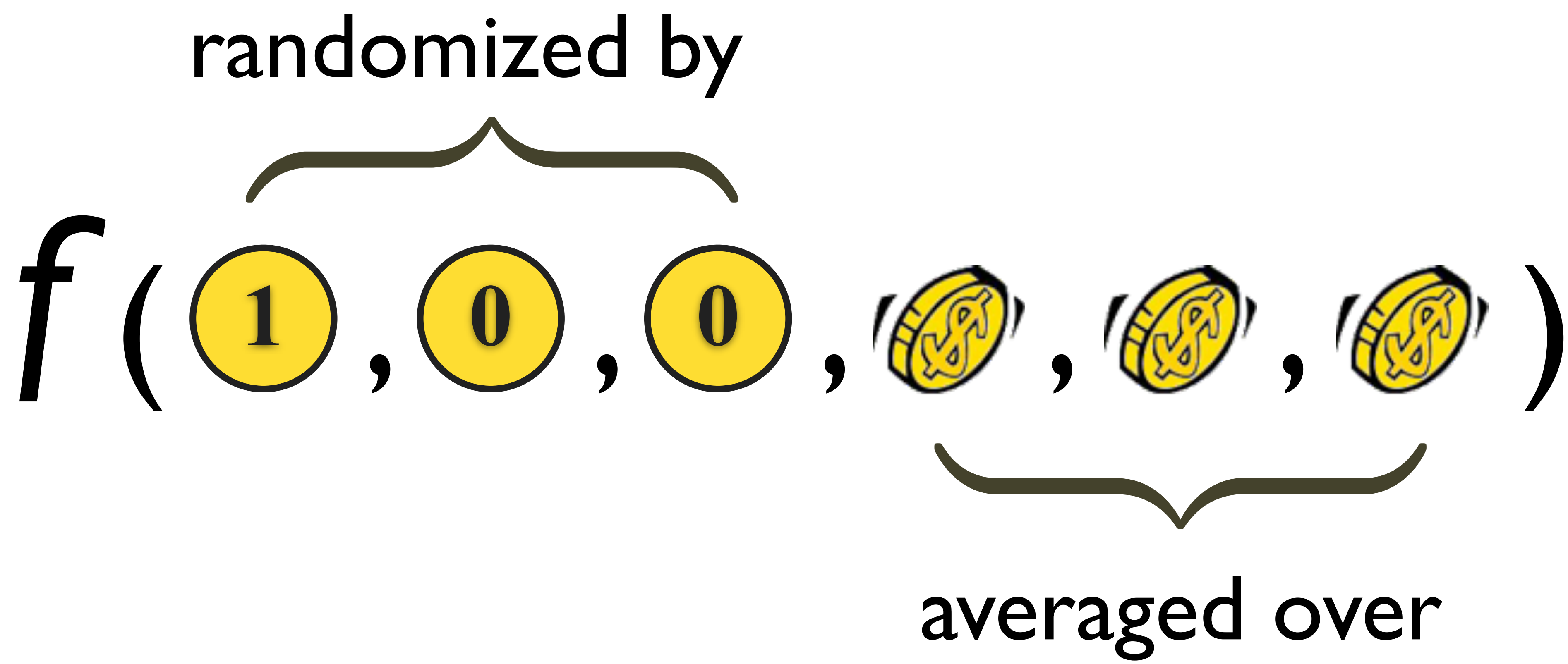

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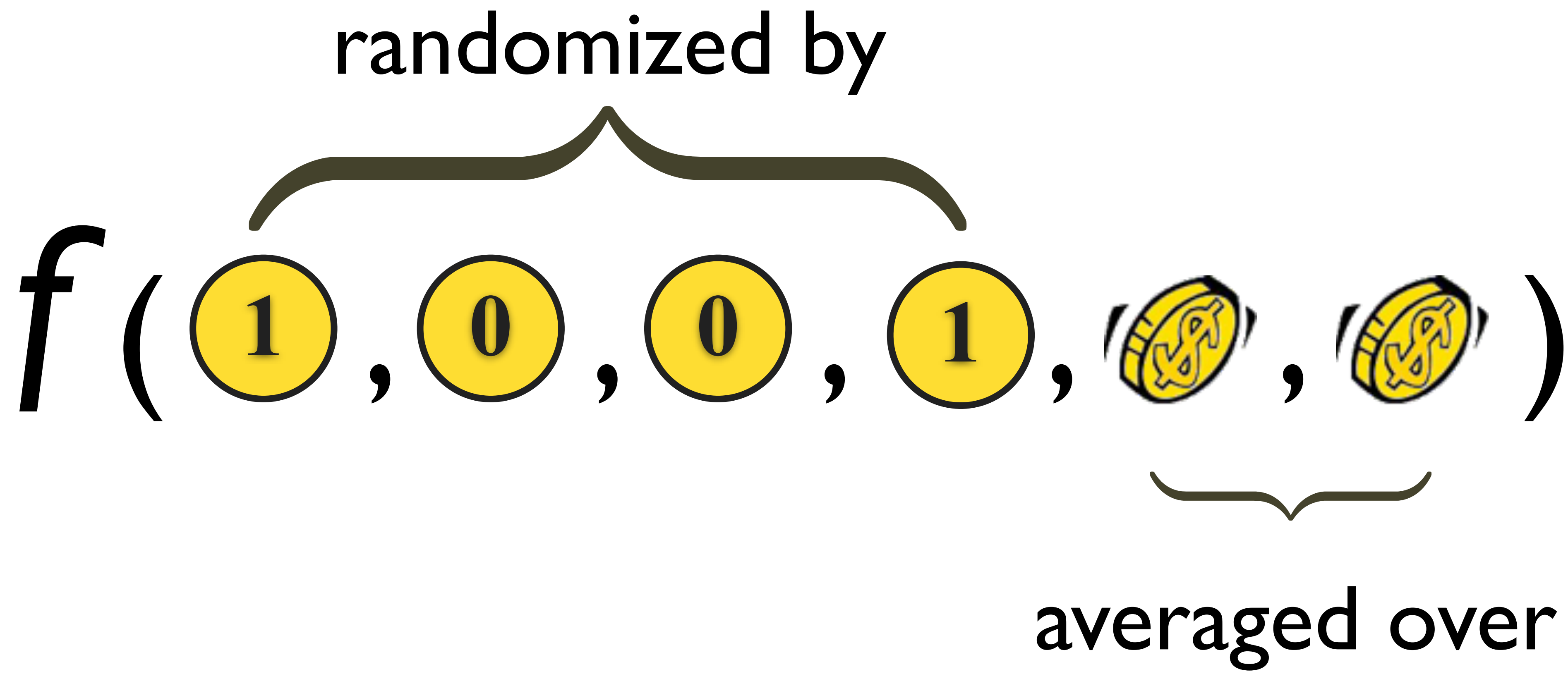

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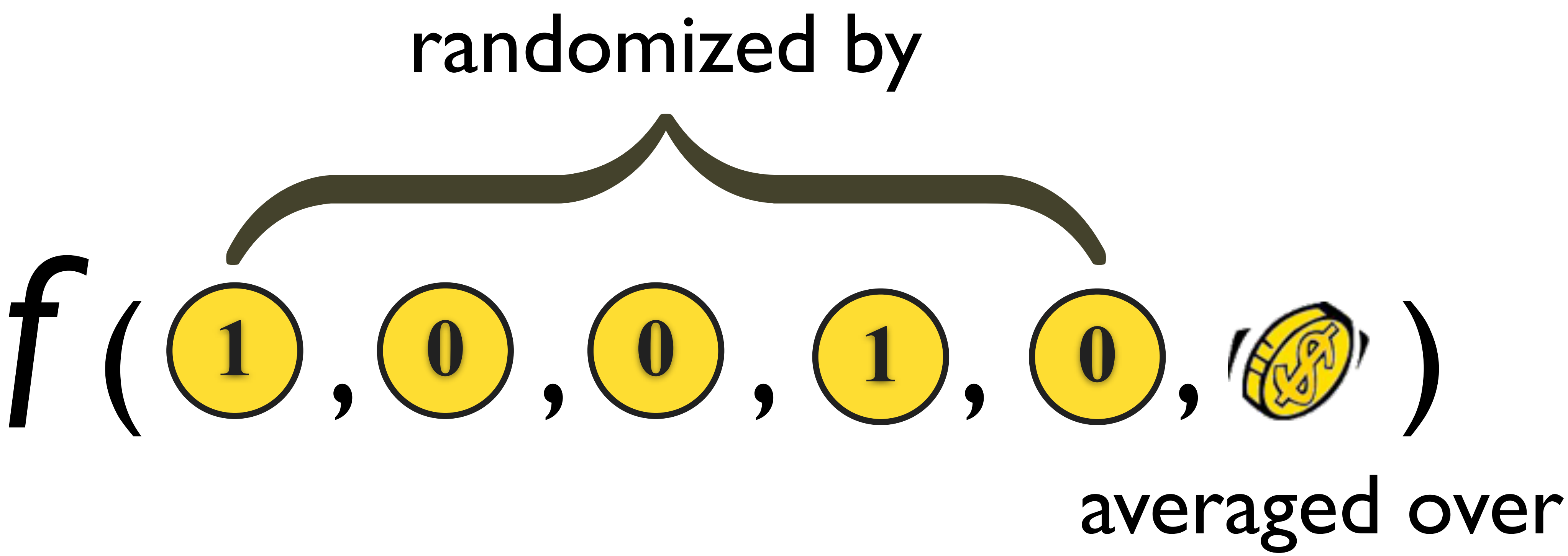


Doob Sequence

$$f(\overbrace{1, 0, 0, 1, 0, 1}^{\text{randomized by}})$$

Doob sequence:

$$Y_i = \mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_i]$$

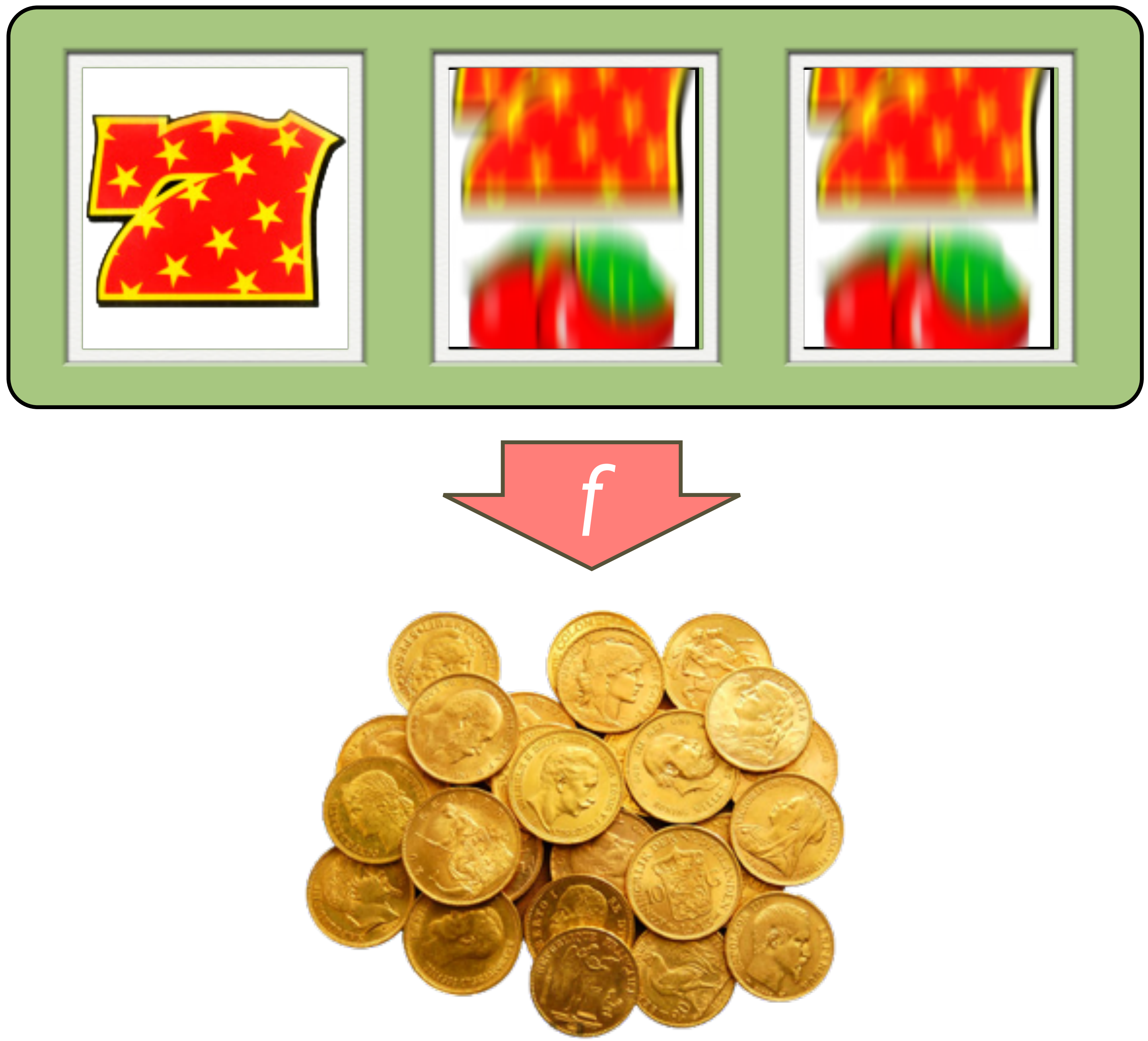
Doob sequence is a martingale:

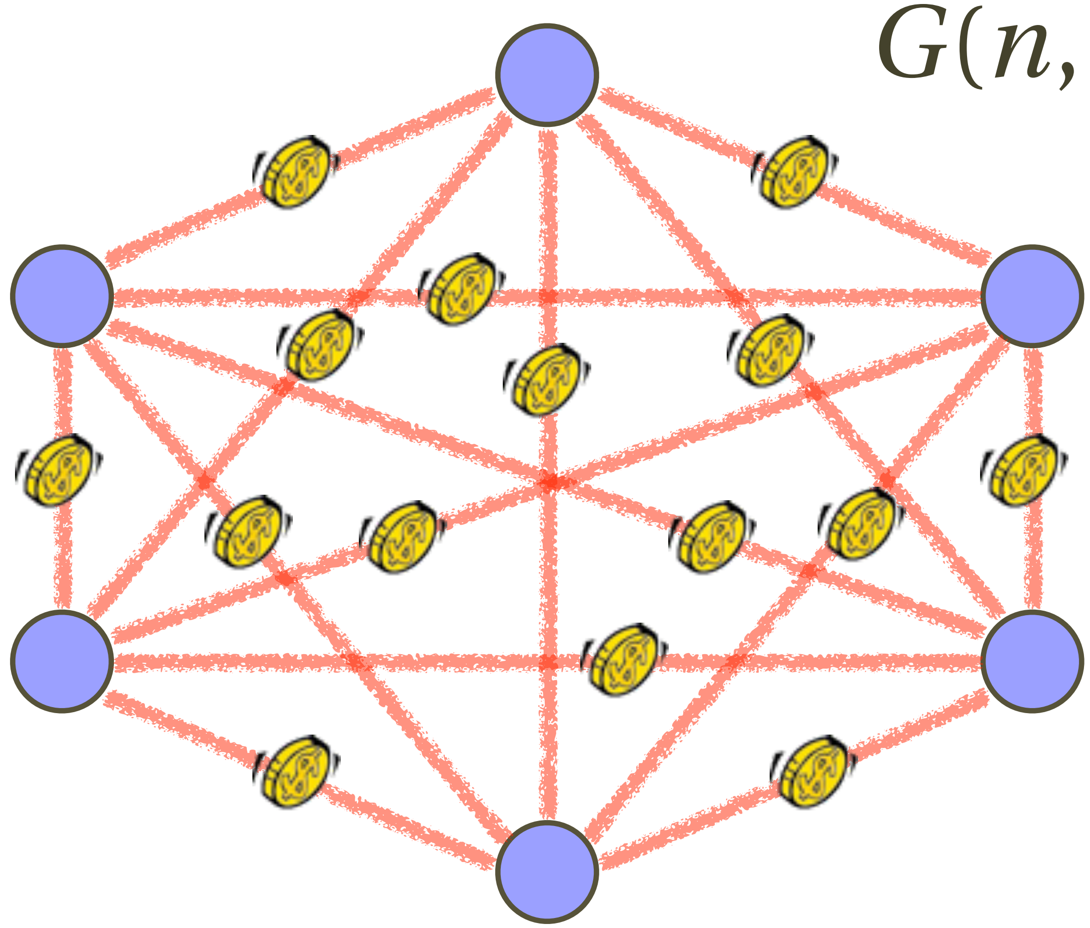
$$\mathbf{E}[Y_i \mid X_1,\ldots,X_{i-1}] = Y_{i-1}$$

Proof:

$$\begin{aligned}
&\mathbf{E}[Y_i \mid X_1,\ldots,X_{i-1}]\\
&= \mathbf{E}[\mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_i] \mid X_1,\ldots,X_{i-1}]\\
&= \mathbf{E}[f(X_1,\ldots,X_n) \mid X_1,\ldots,X_{i-1}]\\
&= Y_{i-1}
\end{aligned}$$

Doob Martingale





$G(n,p)$

Graph parameter:

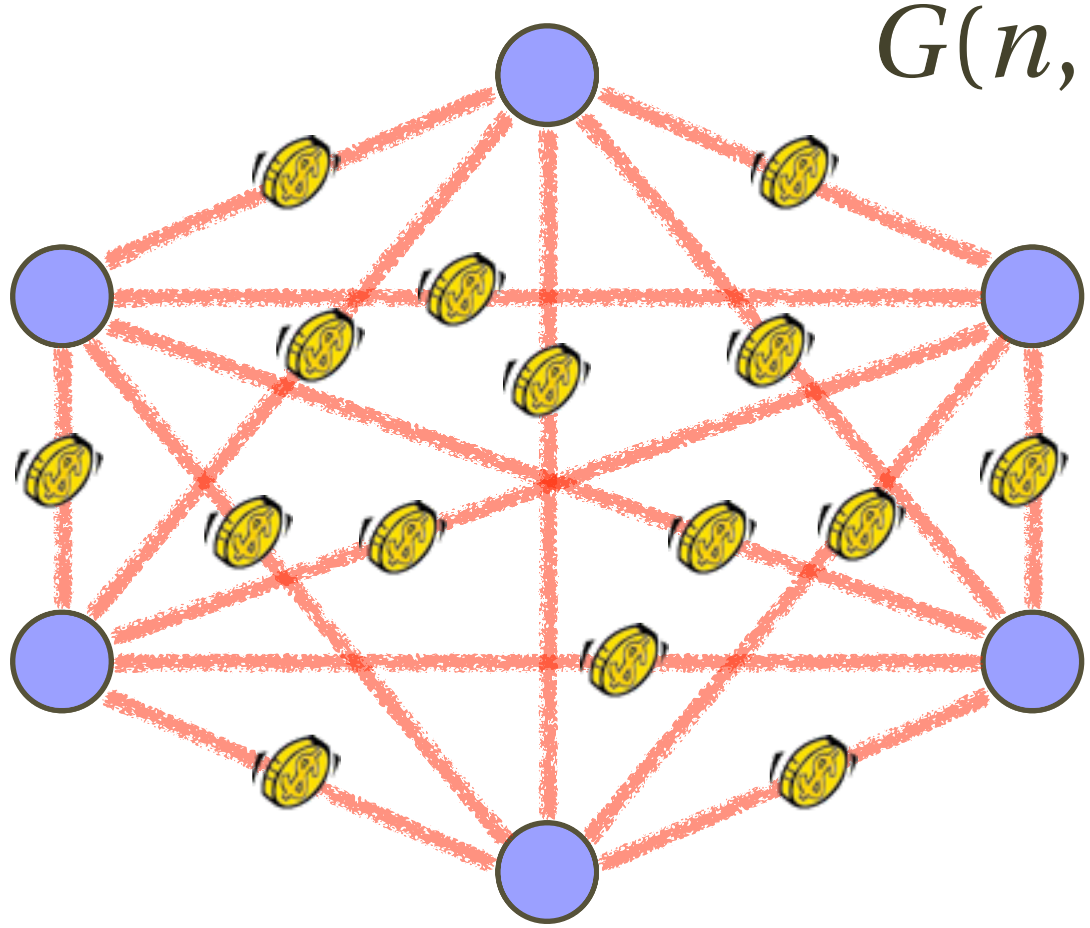
$$f(G)$$

example: chromatic #, components, diameter ...

numbering all vertex-pairs: $1, 2, 3, \ldots, \binom{n}{2}$

$$I_j = \begin{cases} 1 & \text{edge } j \in G \\ 0 & \text{edge } j \notin G \end{cases} \qquad Y_i = \mathbf{E}[f(G) \mid I_1, \ldots, I_i]$$

$$Y_0 = \mathbf{E}[f(G)] \dashrightarrow Y_{\binom{n}{2}} = f(G)$$



$G(n,p)$

Graph parameter:

$$f(G)$$

example: chromatic #, components, diameter ...

numbering all vertices: $1, 2, 3, \ldots, n$

X_i : subgraph of G induced by the first i vertices

$$Y_i = \mathbf{E}[f(G) \mid X_1, \ldots, X_i]$$

$$Y_0 = \mathbf{E}[f(G)] \dashrightarrow Y_n = f(G)$$

Martingales induced by a random graph

- Edge exposure martingale:

$$I_j \text{ indicates the } j\text{th edge}$$

$$Y_i = \mathbf{E}[f(G) \mid I_1, \ldots, I_i]$$

- Vertex exposure martingale:

$$X_i = G([i])$$

$$Y_i = \mathbf{E}[f(G) \mid X_1, \ldots, X_i]$$

martingale $X_0, X_1, X_2, \ldots$

$E[X_i \mid X_0, X_1, \ldots, X_{i-1}] = X_{i-1}$

generalization

martingale $Y_0, Y_1, Y_2, \ldots$
w.r.t. $X_0, X_1, X_2, \ldots$

$Y_i = f(X_0, X_1, \ldots, X_i)$

$E[Y_i \mid X_0, X_1, \ldots, X_{i-1}] = Y_{i-1}$

special case

Doob martingale

$Y_i = E[f(X_0, X_1, \ldots, X_i) \mid X_0, X_1, \ldots, X_{i-1}]$

special cases
in random graphs

edge-exposure martingale
vertex-exposure martingale