Randomized Algorithms

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Course Info

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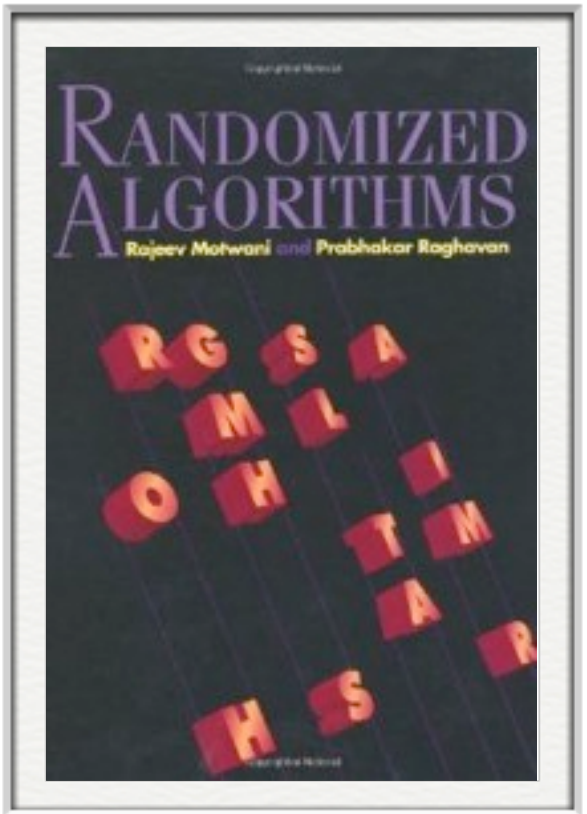
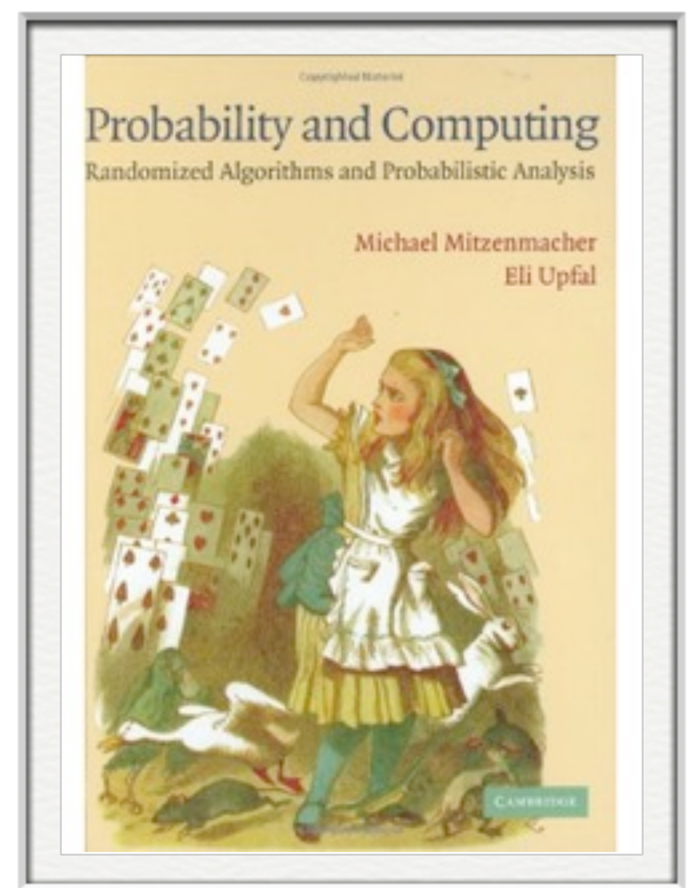
Textbooks

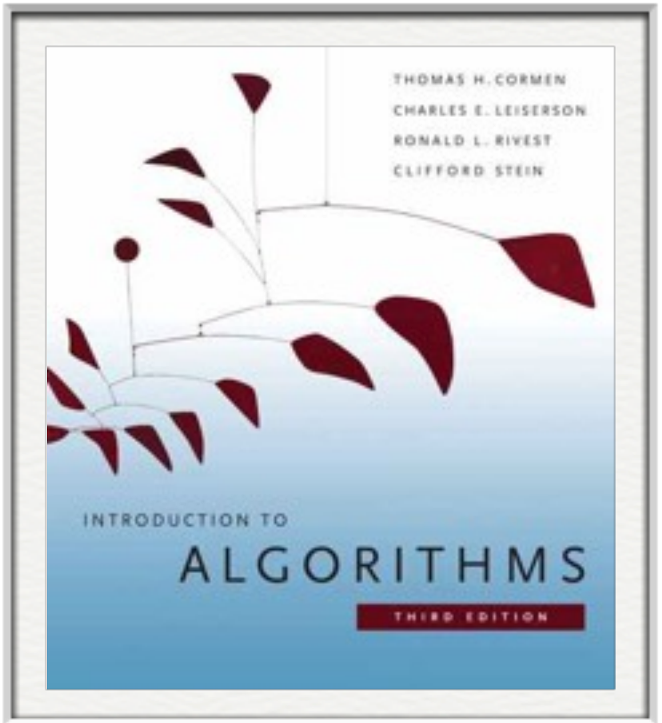


Rajeev Motwani and Prabhakar Raghavan.
Randomized Algorithms.
Cambridge University Press, 1995.

Michael Mitzenmacher and Eli Upfal.
Probability and Computing:
Randomized Algorithms and Probabilistic Analysis.
Cambridge University Press, 2005.

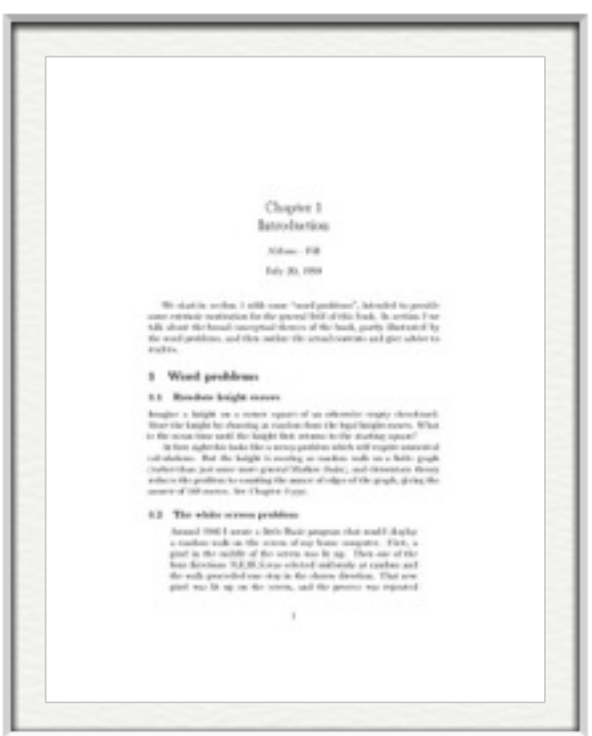


References



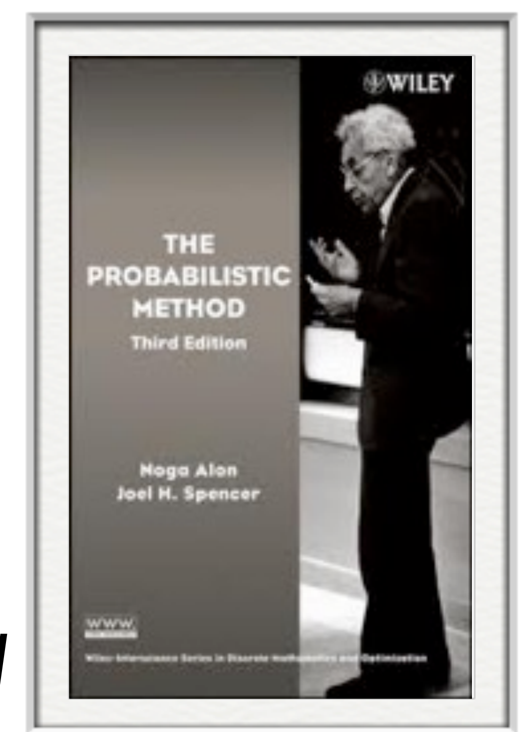
CLRS

Feller
An Introduction to Probability Theory and Its Applications



Aldous and Fill
Reversible Markov Chains and Random Walks on Graphs

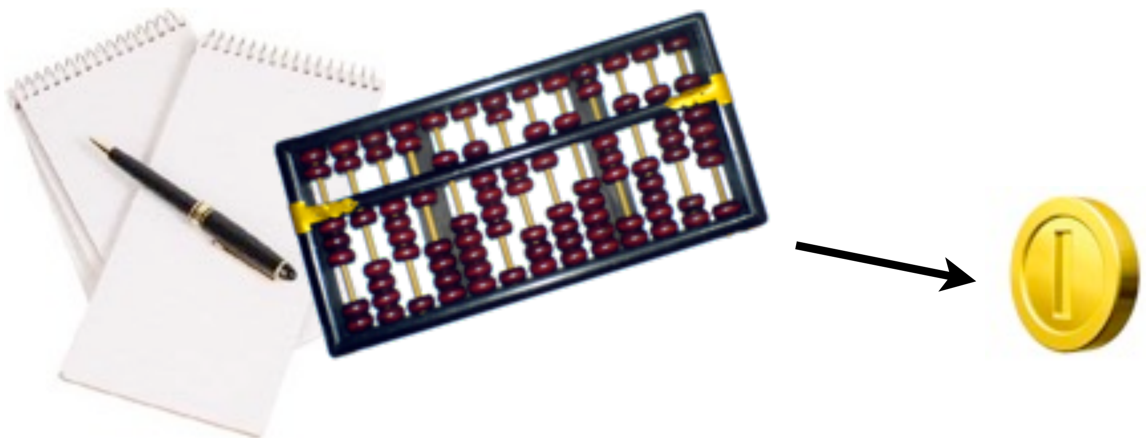
Alon and Spencer
The Probabilistic Method



Randomized Algorithms

“algorithms which use randomness in computation“

Turing Machine



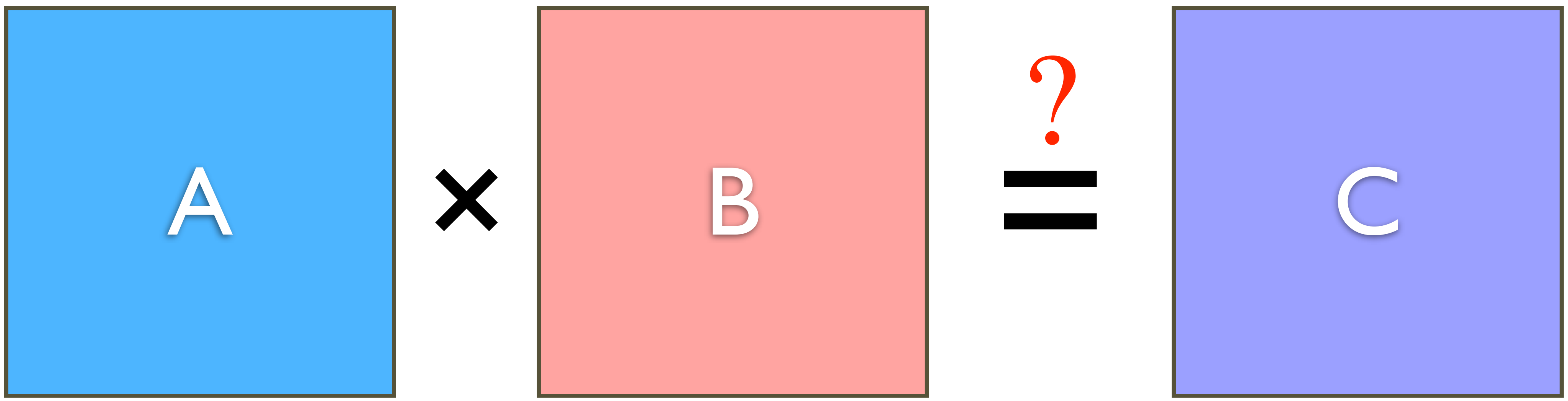
random coin

Why?

- Simpler.
- Faster.
- Can do impossibles.
- Can give us clever deterministic algorithms.
- Random input.
- Deterministic problem with random nature.
- ... ...

Checking Matrix Multiplication

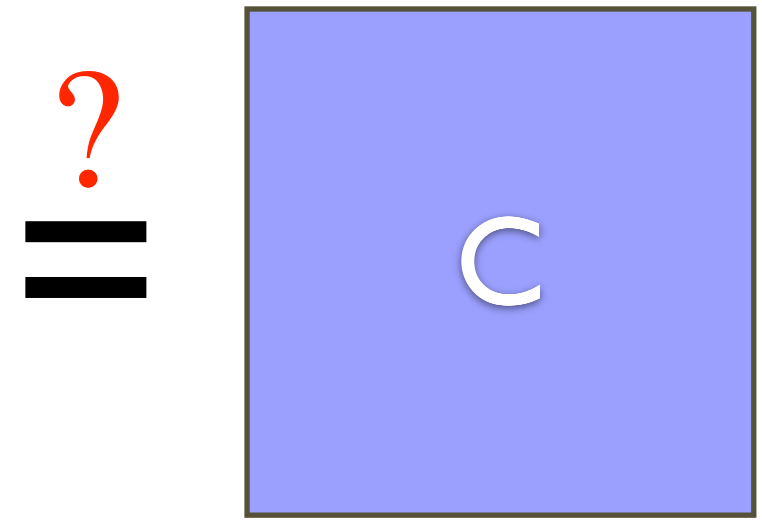
three $n \times n$ matrices *A*, *B*, *C*:

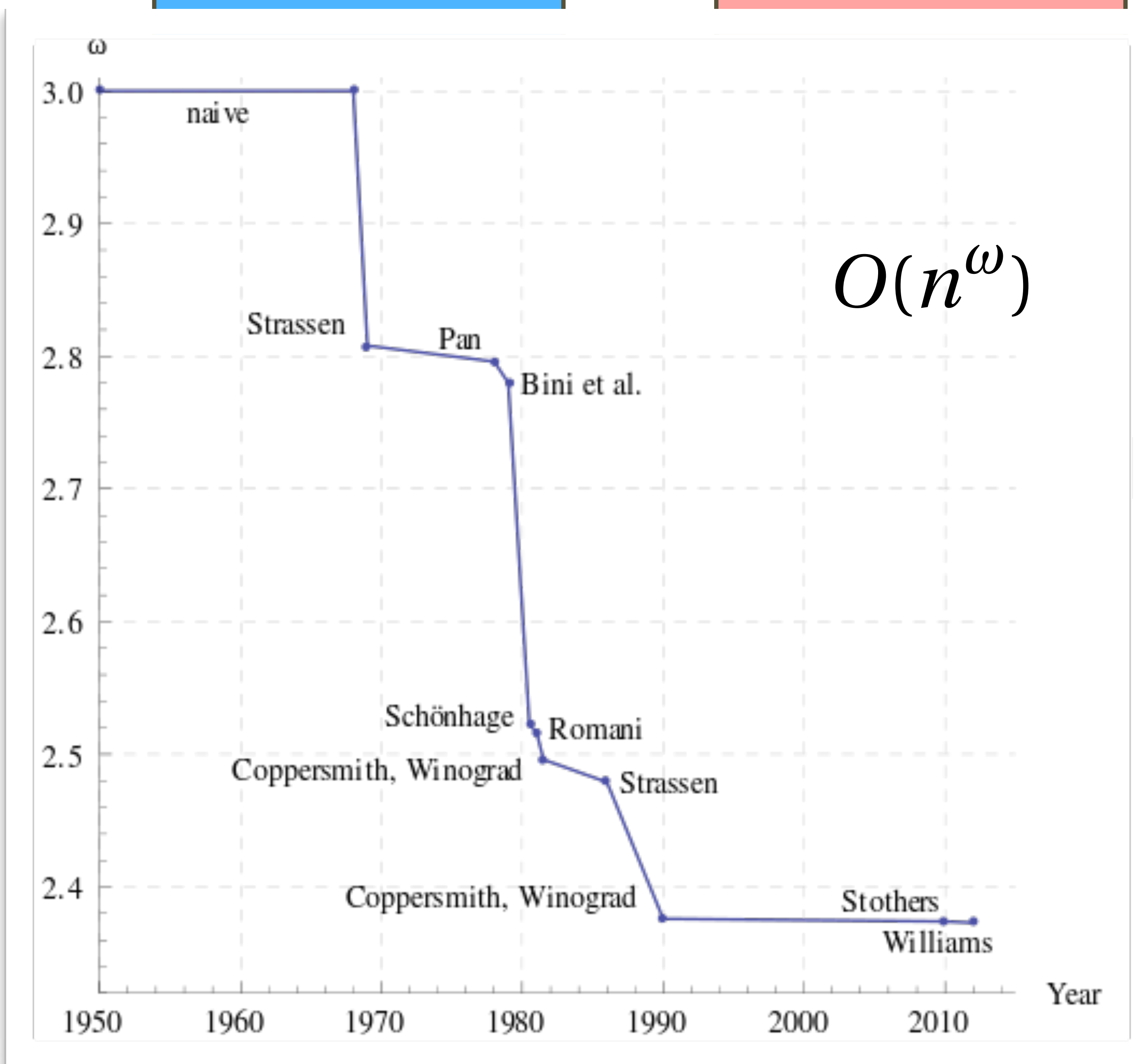


best matrix multiplication algorithm: $O(n^{2.373})$

Checking Matrix Multiplication

three $n \times n$ matrices *A*, *B*, *C*:





gorithm: $O(n^{2.373})$

Checking Matrix Multiplication

three $n \times n$ matrices *A*, *B*, *C*:

$$A \times B \stackrel{?}{=} C$$

best matrix multiplication algorithm: $O(n^{2.373})$

Freivald's Algorithm
pick a uniform random $r \in \{0,1\}^n$;
check whether $A(Br) = Cr$;

time: $\mathrm{O}(n^2)$ if $AB = C$, always correct

Freivald's Algorithm

pick a uniform random $r \in \{0,1\}^n$;

check whether $A(Br) = Cr$;

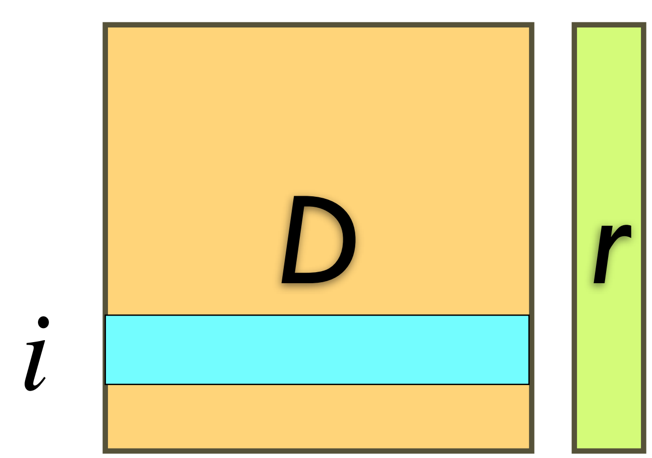
if $AB = C$, always correct

if $AB \neq C$, let $D = AB - C \neq \mathbf{0}_{n\times n}$

say $D_{ij} \neq 0$

$$\Pr[ABr = Cr] = \Pr[Dr = \mathbf{0}] \le \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

$$\Downarrow$$



$$(Dr)_i = \sum_{k=1}^{n} D_{ik} r_k = 0$$

$$\Rightarrow \quad r_j = -\frac{1}{D_{ij}} \sum_{k \neq j}^{n} D_{ik} r_k$$

Freivald's Algorithm
pick a uniform random $r \in \{0,1\}^n$;
check whether $A(Br) = Cr$;

if $AB = C$, always correct

Theorem (Freivald, 1979)

If $AB \neq C$, for a uniformly random $r \in \{0,1\}^n$,

$$\Pr[ABr = Cr] \leq \frac{1}{2}.$$

repeat independently for 100 times

time: $O(n^2)$ if $AB \neq C$, error probability $\leq$ 2^{-100}

Monte Carlo vs Las Vegas

Two types of randomized algorithms:

Monte Carlo	Las Vegas
	
running time is fixed correctness is random	always correct running time is random

Polynomial Identity Testing (PIT)

Input: two polynomials $f, g \in \mathbb{F}[x]$ of degree d

Output: $f \equiv g?$

$f \in \mathbb{F}[x]$ of degree d : $\quad f(x) = \sum_{i=0}^{d} a_i x^i$ for $a_i \in \mathbb{F}$

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d

Output: $f \equiv 0?$

f is given as black-box

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d

Output: $f \equiv 0?$

simple deterministic algorithm:

check whether $f(x)=0$ for all $x \in \{1, 2, \ldots, d+1\}$

Fundamental Theorem of Algebra:
A degree d polynomial has at most d roots.

Randomized Algorithm
pick a uniform random $r \in S$;
check whether $f(r) = 0$;

$S \subseteq \mathbb{F}$

Randomized Algorithm

pick a uniform random $r \in S$;

check whether $f(r) = 0$;

$S \subseteq \mathbb{F}$

$|S| = 2d$

if $f \not\equiv 0$

$$\Pr[f(r) = 0] \leq \frac{d}{|S|} = \frac{1}{2}$$

Fundamental Theorem of Algebra:

A degree d polynomial has at most d roots.

Checking Identity

北京

database 1

Are they identical?

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database 2

Communication Complexity

(Yao 1979)

$f(a,b)$

$\uparrow$

a

Han Meimei

\# of bits communicated

$\rightleftarrows$

Li Lei

b

$$\mathrm{EQ} : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$$

Theorem (Yao, 1979)

There is no deterministic communication protocol solving EQ with less than *n* bits in the worst-case.

Communication Complexity

$$f = \sum_{i=0}^{n-1} a_i x^i$$

$f(r)=g(r)$?

$$g = \sum_{i=0}^{n-1} b_i x^i$$

$a \in \{0,1\}^n$

Han Meimei

$r, g(r)$

Li Lei

$b \in \{0,1\}^n$

by PIT:

one-sided error $\leq \frac{1}{2}$

pick uniform random $r \in [2n]$

# of bit communicated: too large!

Communication Complexity

$$f = \sum_{i=0}^{n-1} a_i x^i$$

$f(r)=g(r)$?

$$g = \sum_{i=0}^{n-1} b_i x^i$$

$a \in \{0,1\}^n$

Han Meimei

$r, g(r)$

O(log n) bits

Li Lei

$b \in \{0,1\}^n$

pick uniform random $r \in [p]$

$k = \lceil \log_2(2n) \rceil$

choose a prime $p \in [2^k, 2^{k+1}]$

let $f, g \in \mathbb{Z}_p[x]$

Polynomial Identity Testing (PIT)

Input: $f, g \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv g$?

$\mathbb{F}[x_1, x_2, \ldots, x_n]$: ring of *n*-variate polynomials over field $\mathbb{F}$

$f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i_1, i_2, \ldots, i_n \geq 0} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

degree of f: maximum $i_1 + i_2 + \cdots + i_n$ with $a_{i_1, i_2, \ldots, i_n} \neq 0$

Input: $f, g \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv g?$

equivalently:

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0?$

$$f(x_1, x_2, \ldots, x_n) = \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n \leq d}} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

Input: $f, g \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv g?$

equivalently:

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0?$

f is given as block-box: given any $\vec{x} = (x_1, x_2, \ldots, x_n)$ returns $f(\vec{x})$

or as product from: e.g. Vandermonde determinant

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{bmatrix} \qquad f(\vec{x}) = \det(M) = \prod_{j<i}(x_i - x_j)$$

PIT: Polynomial Identity Testing

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0?$

f is given as block-box or product from

if ∃ a poly-time deterministic algorithm for PIT:

either: **NEXP ≠ P/poly**

or: **#P ≠ FP**

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0?$

fix an arbitrary $S \subseteq \mathbb{F}$

pick random $r_1, r_2, \ldots, r_n \in S$
uniformly and *independently* at random;

check whether $f(r_1, r_2, \ldots, r_n) = 0$;

$$f \equiv 0 \Rightarrow f(r_1, r_2, \ldots, r_n) = 0$$

Input: a polynomial $f \in \mathbb{F}[x]$ of degree d

Output: $f \equiv 0?$

fix an arbitrary $S \subseteq \mathbb{F}$

pick a *uniform* random $r \in S$;

check whether $f(r) = 0$;

$$f \equiv 0 \Rightarrow f(r) = 0$$

Fundamental Theorem of Algebra:

A degree d polynomial has at most d roots.

$$f \not\equiv 0 \Rightarrow \Pr[f(r) = 0] \leq \frac{d}{|S|}$$

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0$?

fix an arbitrary $S \subseteq \mathbb{F}$

pick random $r_1, r_2, \ldots, r_n \in S$
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check whether $f(r_1, r_2, \ldots, r_n) = 0$;

$$f \equiv 0 \Rightarrow f(r_1, r_2, \ldots, r_n) = 0$$

Schwartz-Zippel Theorem

$$f \not\equiv 0 \Rightarrow \Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|S|}$$

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$$f(x_1, x_2, \ldots, x_n) = \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \cdots + i_n \leq d}} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

f can be treated as a single-variate polynomial of x_n:

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{d} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$$

$$= g_{x_1, x_2, \ldots, x_{n-1}}(x_n)$$

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0] = \Pr[g_{r_1, r_2, \ldots, r_{n-1}}(r_n) = 0]$$

$$g_{r_1, r_2, \ldots, r_{n-1}} \not\equiv 0?$$

done?

Schwartz-Zippel Theorem

$$f \neq 0 \Rightarrow \Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|S|}$$

induction on n :

basis: $n=1$ single-variate case, proved by the *fundamental Theorem of algebra*

I.H.: Schwartz-Zippel Thm is true for all smaller n

Schwartz-Zippel Theorem

$$f \not\equiv 0 \Rightarrow \Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|S|}$$

induction step:

k: highest power of x_n in f $\Rightarrow \begin{cases} f_k \not\equiv 0 \\ \text{degree of } f_k \leq d - k \end{cases}$

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{k} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$$

$$= x_n^k f_k(x_1, x_2, \ldots, x_{n-1}) + \bar{f}(x_1, x_2, \ldots, x_n)$$

where $\bar{f}(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{k-1} x_n^i f_i(x_1, x_2, \ldots, x_{n-1})$

highest power of x_n in $\bar{f}$ $< k$

Schwartz-Zippel Theorem

$$f \not\equiv 0 \Rightarrow \Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|S|}$$

$$f(x_1, x_2, \ldots, x_n) = x_n^k f_k(x_1, x_2, \ldots, x_{n-1}) + \bar{f}(x_1, x_2, \ldots, x_n)$$

$$\begin{cases} f_k \not\equiv 0 \\ \text{degree of } f_k \leq d-k \end{cases}$$

highest power of x_n in $\bar{f}$ $< k$

law of total probability:

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0]$$

I.H. $\Rightarrow$ $\leq \frac{d-k}{|S|}$

$$= \Pr[f(\vec{r}) = 0 \mid f_k(r_1, \ldots, r_{n-1}) = 0] \cdot \Pr[f_k(r_1, \ldots, r_{n-1}) = 0]$$

$$+ \Pr[f(\vec{r}) = 0 \mid f_k(r_1, \ldots, r_{n-1}) \neq 0] \cdot \Pr[f_k(r_1, \ldots, r_{n-1}) \neq 0]$$

$$= \Pr[g_{r_1, \ldots, r_{n-1}}(r_n) = 0 \mid f_k(r_1, \ldots, r_{n-1}) \neq 0] \leq \frac{k}{|S|}$$

where $g_{x_1, \ldots, x_{n-1}}(x_n) = f(x_1, \ldots, x_n)$

Schwartz-Zippel Theorem

$$f \not\equiv 0 \quad \Rightarrow \quad \Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d}{|S|}$$

$$\Pr[f(r_1, r_2, \ldots, r_n) = 0] \leq \frac{d-k}{|S|} + \frac{k}{|S|} = \frac{d}{|S|}$$

Input: $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ of degree d

Output: $f \equiv 0$?

fix an arbitrary $S \subseteq \mathbb{F}$

pick random $r_1, r_2, \ldots, r_n \in S$
uniformly and *independently* at random;
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Schwartz-Zippel Theorem

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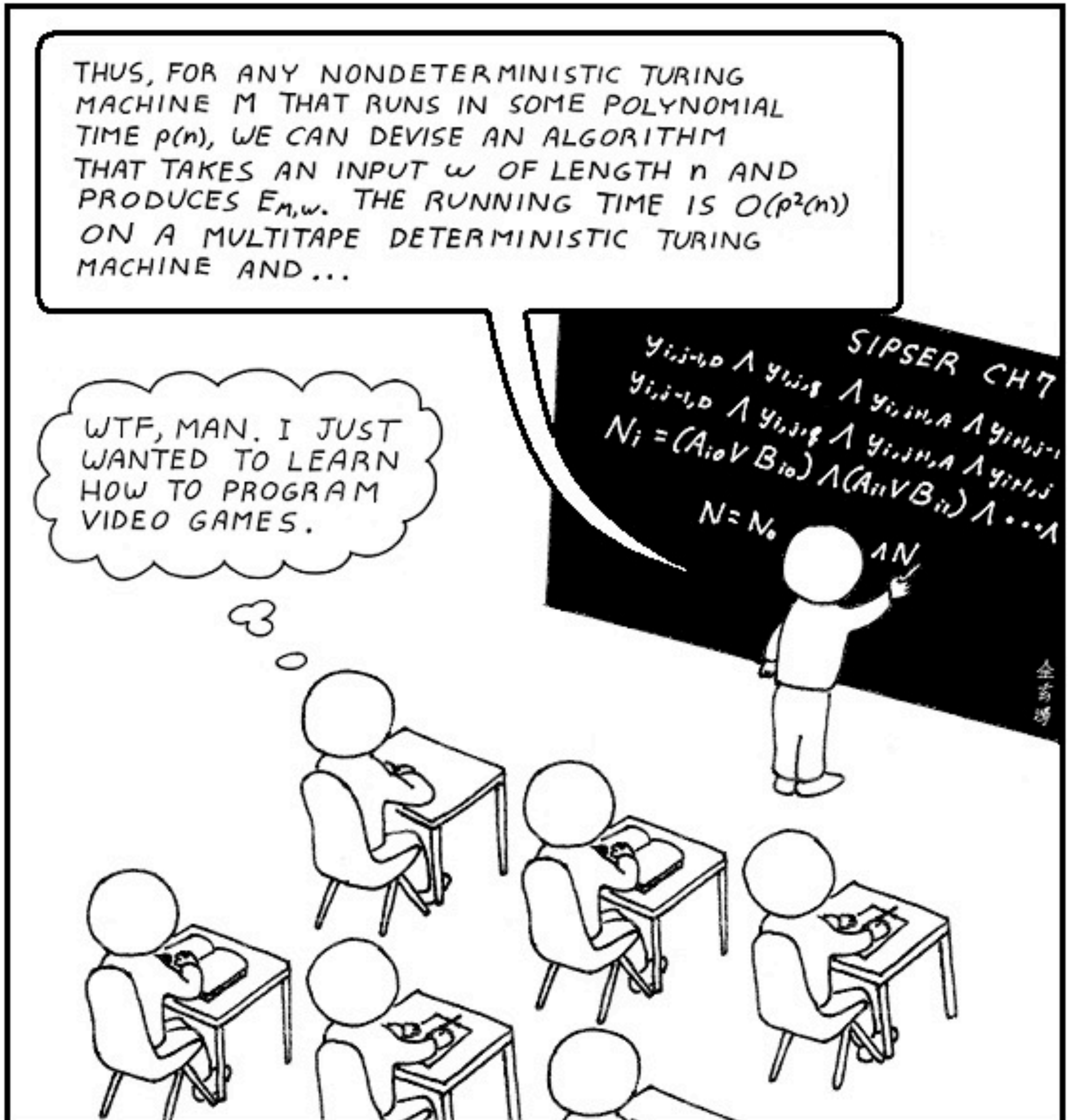
THUS, FOR ANY NONDETERMINISTIC TURING MACHINE M THAT RUNS IN SOME POLYNOMIAL TIME $p(n)$, WE CAN DEVISE AN ALGORITHM THAT TAKES AN INPUT w OF LENGTH n AND PRODUCES $E_{M,w}$. THE RUNNING TIME IS $O(p^2(n))$ ON A MULTITAPE DETERMINISTIC TURING MACHINE AND...
WTF, MAN. I JUST WANTED TO LEARN HOW TO PROGRAM VIDEO GAMES.
SIPSER CH7