Randomized Algorithms

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MAX-SAT

- Boolean variables:
  $x_1, x_2, \ldots, x_n$
- literal: $x_i, \neg x_i$
- clause: OR of literals
- MAX-SAT
  - NP-hard

Instance: a set of clauses

$$C_1 = (x_1 \vee \neg x_2 \vee \neg x_3)$$

$$C_2 = (\neg x_1 \vee \neg x_3)$$

$$C_3 = (x_1 \vee x_2 \vee x_4)$$

$$C_4 = (x_4 \vee \neg x_3)$$

$$C_5 = (x_4 \vee \neg x_1)$$

Solution: a truth assignment maximizing # of satisfied clauses

Approximation Algorithms

maximization problems

Fix a problem instance (input) I:

- optimal solution: OPT(I)
- solution returned by Alg: S(I)

$$\forall I, \quad S(I) \geq \alpha \cdot \mathrm{OPT}(I)$$

α-approximation algorithm

Approximation Algorithms

α-approximation algorithm

maximization problems

$$\forall I, \quad S(I) \geq \alpha \cdot \mathrm{OPT}(I) \qquad 0 < \alpha < 1$$

minimization problems

$$\forall I, \quad S(I) \leq \alpha \cdot \mathrm{OPT}(I) \qquad \alpha > 1$$

Randomized Approximation

Instance: a set of m clauses

Solution: a truth assignment

maximizing # of satisfied clauses

assign each variable with true or false
uniformly and independently at random

a clause $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$ $\qquad \ell_j \in \{x_i, \neg x_i\}$

$$\Pr[C \text{ is satisfied}] = 1 - 2^{-k} \geq \frac{1}{2}$$

linearity of expectation

$$\mathbf{E}[\ \#\text{ satisfied clauses}\ ] \geq \frac{m}{2}$$

Randomized Approximation

Instance: a set of m clauses

Solution: a truth assignment

maximizing # of satisfied clauses

assign each variable with true or false uniformly and independently at random

$$\mathbf{E}[\text{ \# satisfied clauses }] \geq \frac{m}{2} \geq \frac{1}{2}\mathrm{OPT}$$

$$\mathrm{OPT} \leq m$$

Integer Programming

boolean variables: $x_1, x_2, \ldots, x_n$

$$y_i = \begin{cases} 1 & \text{if } x_i = \texttt{true} \\ 0 & \text{if } x_i = \texttt{false} \end{cases}$$

truth assignment $\Leftrightarrow$ $y \in \{0,1\}^n$

a clause $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$ $\qquad \ell_j \in \{x_i, \neg x_i\}$

C^+ : set of i that x_i is in C

C^- : set of i that $\neg x_i$ is in C

C is satisfied $\Leftrightarrow$ $\displaystyle\sum_{i \in C^+} y_i + \sum_{i \in C^-} (1 - y_i) \geq 1$

Integer Programming

boolean variables: $x_1, x_2, \ldots, x_n$

clauses: $C_1, C_2, \ldots, C_m$

$$\text{maximize} \quad \sum_{j=1}^{m} z_j$$

$$\text{subject to} \quad \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j, \quad \forall 1 \leq j \leq m$$

$$y_i \in \{0, 1\}, \quad \forall 1 \leq i \leq n$$

$$z_j \in \{0, 1\}, \quad \forall 1 \leq j \leq m$$

integral solution

Linear Programming

boolean variables: $x_1, x_2, \ldots, x_n$

clauses: $C_1, C_2, \ldots, C_m$

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{m} z_j \\
\text{subject to} \quad & \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j, & \forall 1 \leq j \leq m \\
& y_i \in [0, 1], & \forall 1 \leq i \leq n \\
& z_j \in [0, 1], & \forall 1 \leq j \leq m
\end{aligned}
$$

LP-relaxation

Rounding LP Relaxation

- represent the problem as an IP;
- relax the IP to an LP;
- solve the LP fractionally in poly-time;
- round the fractional optimal solution to an integer feasible solution.

Randomized rounding!

LP Relaxation

$$\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{m} z_j \\
\text{subject to} \quad & \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j, \quad && \forall 1 \leq j \leq m \\
& 0 \leq y_i \leq 1, && \forall 1 \leq i \leq n \\
& 0 \leq z_j \leq 1, && \forall 1 \leq j \leq m
\end{aligned}$$

y_i^*, z_j^* : optimal fractional solution $\in [0,1]$

$$\text{OPT} \leq \sum_{j=1}^{m} z_j^*$$

max $\sum_{j=1}^{m} z_j$ s.t.: $\sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j$

y_i^*, z_j^* : optimal fractional solution $\in [0,1]$

randomized rounding: independently

$$x_i = \begin{cases} \texttt{true} & \text{with probability } y_i^*. \\ \texttt{false} & \text{with probability } 1 - y_i^*. \end{cases}$$

$C_j = x_1 \vee x_2 \vee \cdots \vee x_k$ is satisfied with prob:

$$1 - \prod_{i=1}^{k} (1 - y_i^*)$$

max $\sum_{j=1}^{m} z_j$ s.t.: $\sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j$

in general

$C_j = x_1 \vee x_2 \vee \cdots \vee x_k$ is satisfied with prob:

$$1 - \prod_{i=1}^{k} (1 - y_i^*)$$

$$\geq 1 - (1 - z_j^*/k)^k \quad \text{concave}$$

$$\geq (1 - (1 - 1/k)^k) z_j^*$$

$$\geq (1 - 1/e) z_j^*$$

$$\mathbf{E}[\text{ \# satisfied clauses }] \geq (1 - 1/e) \sum_{j=1}^{m} z_j^*$$

$$\text{OPT} \leq \sum_{j=1}^{m} z_j^* \qquad \geq (1 - 1/e) \cdot \text{OPT}$$

LP relaxation

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{m} z_j & \\ \text{subject to} & \sum_{i \in C_j^+} y_i + \sum_{i \in C_j^-} (1 - y_i) \geq z_j, & \forall 1 \leq j \leq m \\ & 0 \leq y_i \leq 1, & \forall 1 \leq i \leq n \\ & 0 \leq z_j \leq 1, & \forall 1 \leq j \leq m \end{array}$$

optimal fractional solution:

$$y_i^*, z_j^*$$

$$\text{OPT} \leq \sum_{j=1}^{m} z_j^*$$

$$x_i = \begin{cases} \texttt{true} & \text{with probability } y_i^*. \\ \texttt{false} & \text{with probability } 1 - y_i^*. \end{cases}$$

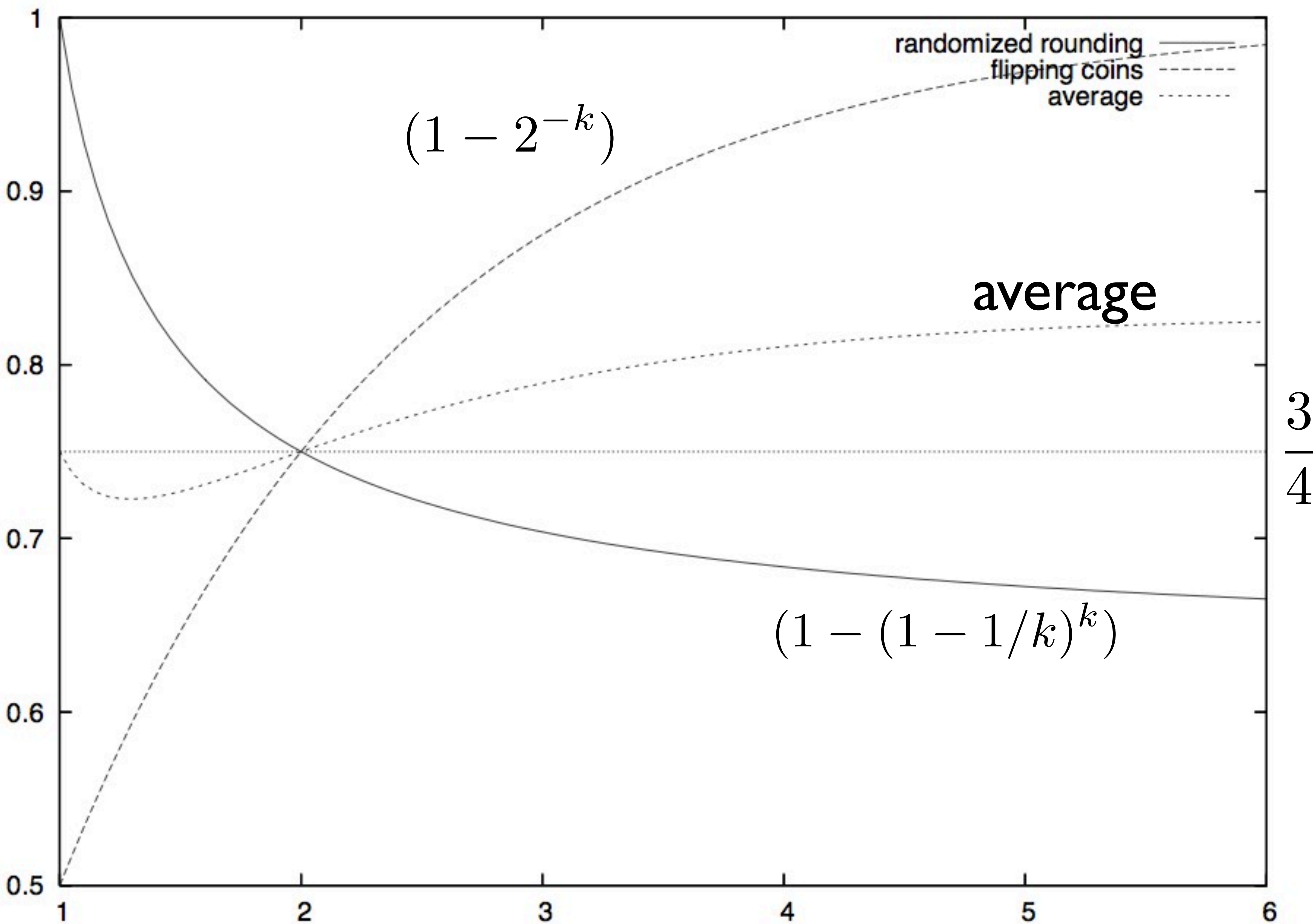
$$\mathbf{E}[\text{ \# satisfied clauses }] \geq (1 - 1/e) \sum_{j=1}^{m} z_j^*$$

$$\geq (1 - 1/e) \cdot \text{OPT}$$

Put two algorithms together

- uniform & independent assignment:
  - $1/2$-approximation
  - k-clause C_j : satisfied with prob $1-2^{-k}$
    $\geq (1-2^{-k})z_j^*$
- LP relaxation + randomized rounding:
  - $(1-1/e)$-approximation
  - k-clause C_j : satisfied with prob
    $\geq (1-(1-1/k)^k)z_j^*$

$$\mathrm{OPT} \leq \sum_{j=1}^{m} z_j^*$$


randomized rounding
flipping coins
average
$(1 - 2^{-k})$
average
$\frac{3}{4}$
$(1 - (1 - 1/k)^k)$
1
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6

run coin-flipping algorithm, satisfy m_1 clauses

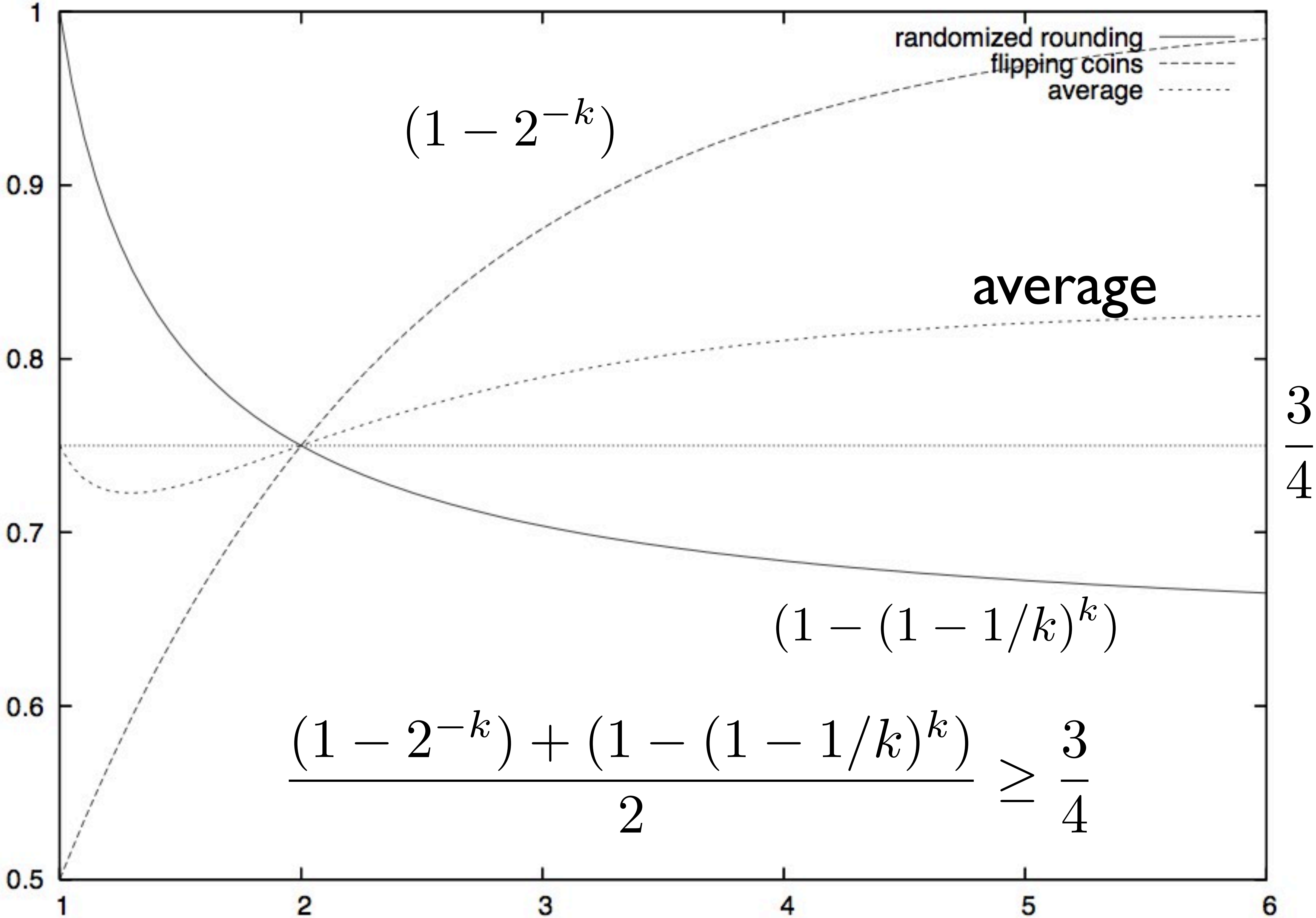
run LP-relaxation algorithm, satisfy m_2 clauses

$$\mathbf{E}[\max\{m_1, m_2\}] \geq \mathbf{E}\left[\frac{m_1 + m_2}{2}\right] \qquad \text{OPT} \leq \sum_{j=1}^{m} z_j^*$$

S_k : set of k-clauses

$$m_1 = \sum_{k=1}^{n} \sum_{C_j \in S_k} (1 - 2^{-k}) \geq \sum_{k=1}^{n} \sum_{C_j \in S_k} (1 - 2^{-k}) z_j^*$$

$$m_2 \geq \sum_{k=1}^{n} \sum_{C_j \in S_k} (1 - (1 - 1/k)^k) z_j^*$$


randomized rounding
flipping coins
average
$(1-2^{-k})$
average
$\frac{3}{4}$
$(1-(1-1/k)^k)$
$\frac{(1-2^{-k})+(1-(1-1/k)^k)}{2} \geq \frac{3}{4}$
1
0.9
0.8
0.7
0.6
0.5
1
2
3
4
5
6

run coin-flipping algorithm, satisfy m_1 clauses

run LP-relaxation algorithm, satisfy m_2 clauses

$$\mathbf{E}[\max\{m_1, m_2\}] \geq \mathbf{E}\left[\frac{m_1+m_2}{2}\right] \geq \frac{3}{4}\,\mathrm{OPT} \leq \sum_{j=1}^{m} z_j^*$$

S_k : set of k-clauses

$$m_1 = \sum_{k=1}^{n}\sum_{C_j \in S_k}(1-2^{-k}) \geq \sum_{k=1}^{n}\sum_{C_j \in S_k}(1-2^{-k})z_j^*$$

$$m_2 \geq \sum_{k=1}^{n}\sum_{C_j \in S_k}(1-(1-1/k)^k)z_j^*$$

$$\frac{m_1+m_2}{2} \geq \sum_{j=1}^{m}\frac{3}{4}z_j^*$$