Approximate Counting *via* Correlation Decay *in* Spin Systems

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Joint work with
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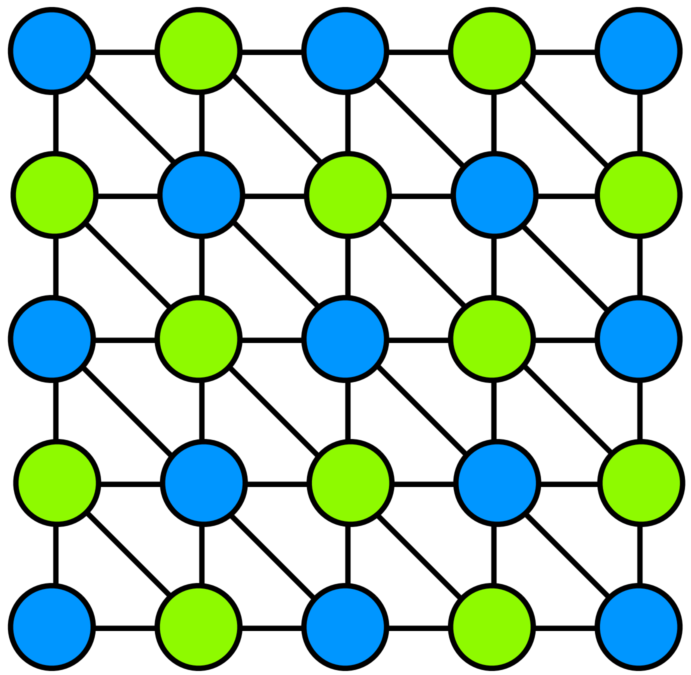
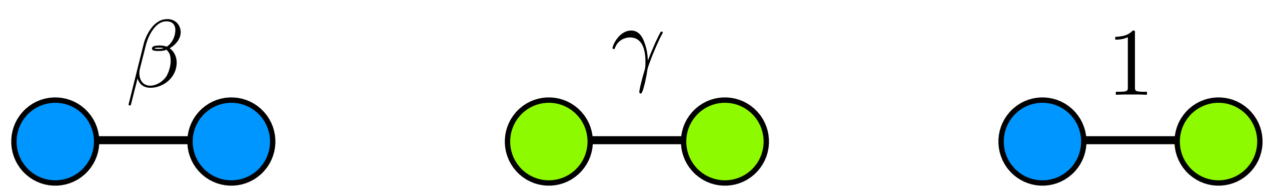
Two-State Spin System



graph $G=(V,E)$ 2 states $\{0,1\}$

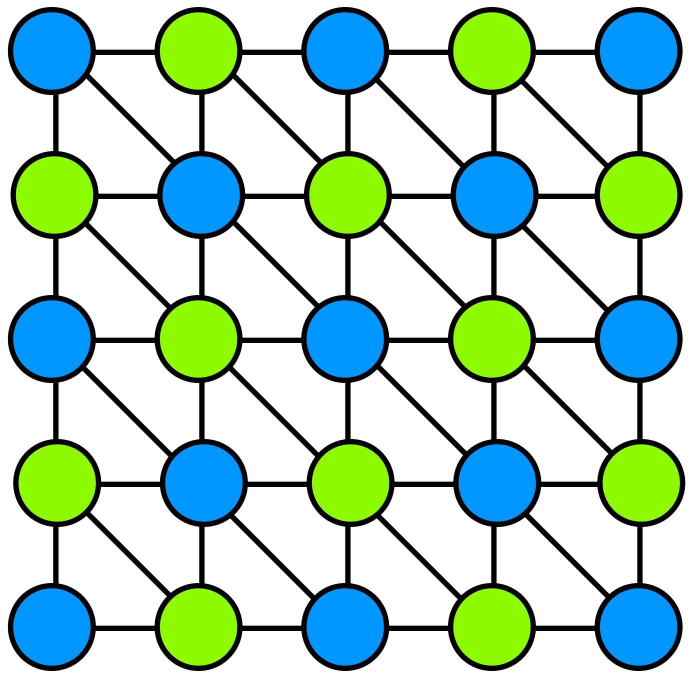
configuration $\sigma : V \to \{0,1\}$

contributions of local interactions:



weight: $$w(\sigma) = \prod_{e \in E} \text{contribution}(e)$$

Two-State Spin System

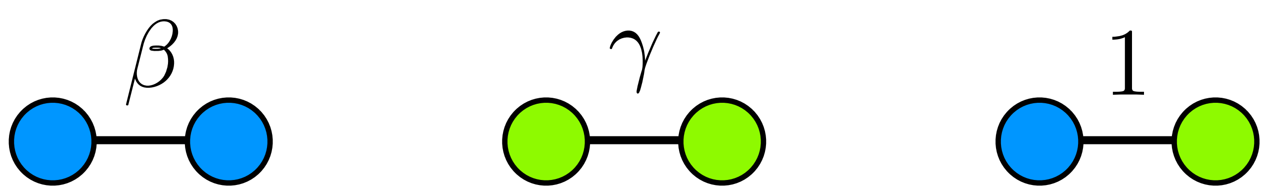


graph $G=(V,E)$ 2 states $\{0,1\}$

configuration $\sigma : V \to \{0,1\}$

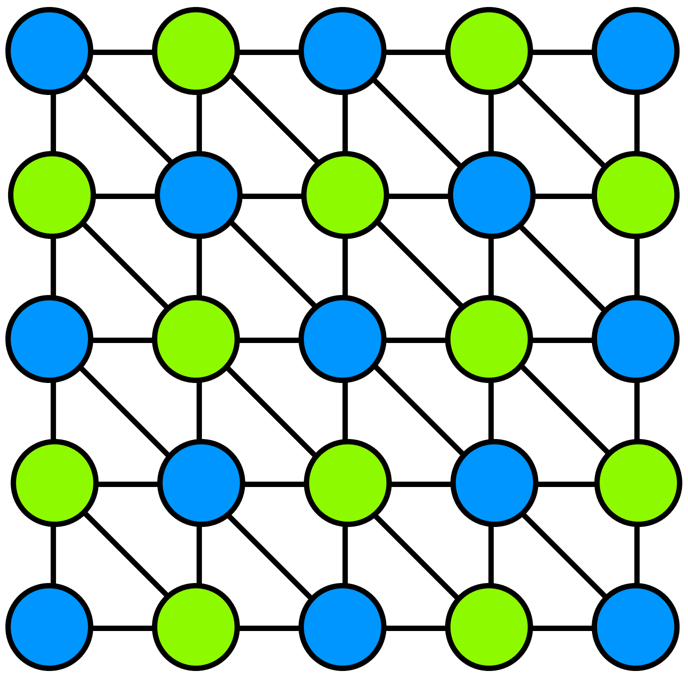
$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

contributions of local interactions:



weight: $$w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)}$$

Two-State Spin System



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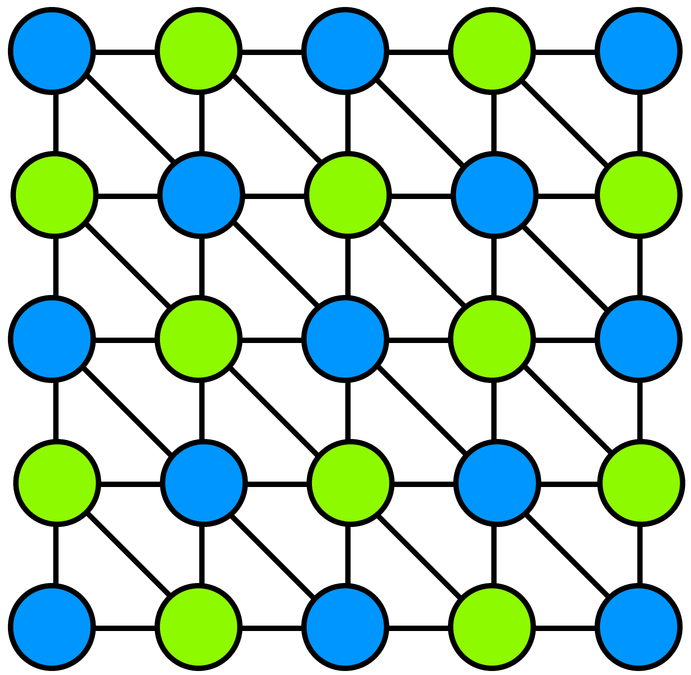
$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

weight: $w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)}$

Gibbs measure: $\mu(\sigma) = \dfrac{w(\sigma)}{Z_A(G)}$

partition function: $Z_A(G) = \sum_{\sigma\in\{0,1\}^V} w(\sigma)$

Partition Function



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$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$$

partition function:

$$Z_A(G) = \sum_{\sigma \in \{0,1\}^V} \prod_{(u,v) \in E} A_{\sigma(u),\sigma(v)}$$

$\beta = 0, \gamma = 1$ # independent set # vertex cover

weighted Boolean CSP with one symmetric relation

Approximate Counting

fix $A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$

partition function:

$$Z_A(G) = \sum_{\sigma \in \{0,1\}^V} \prod_{(u,v) \in E} A_{\sigma(u),\sigma(v)}$$

is a well-define computational problem

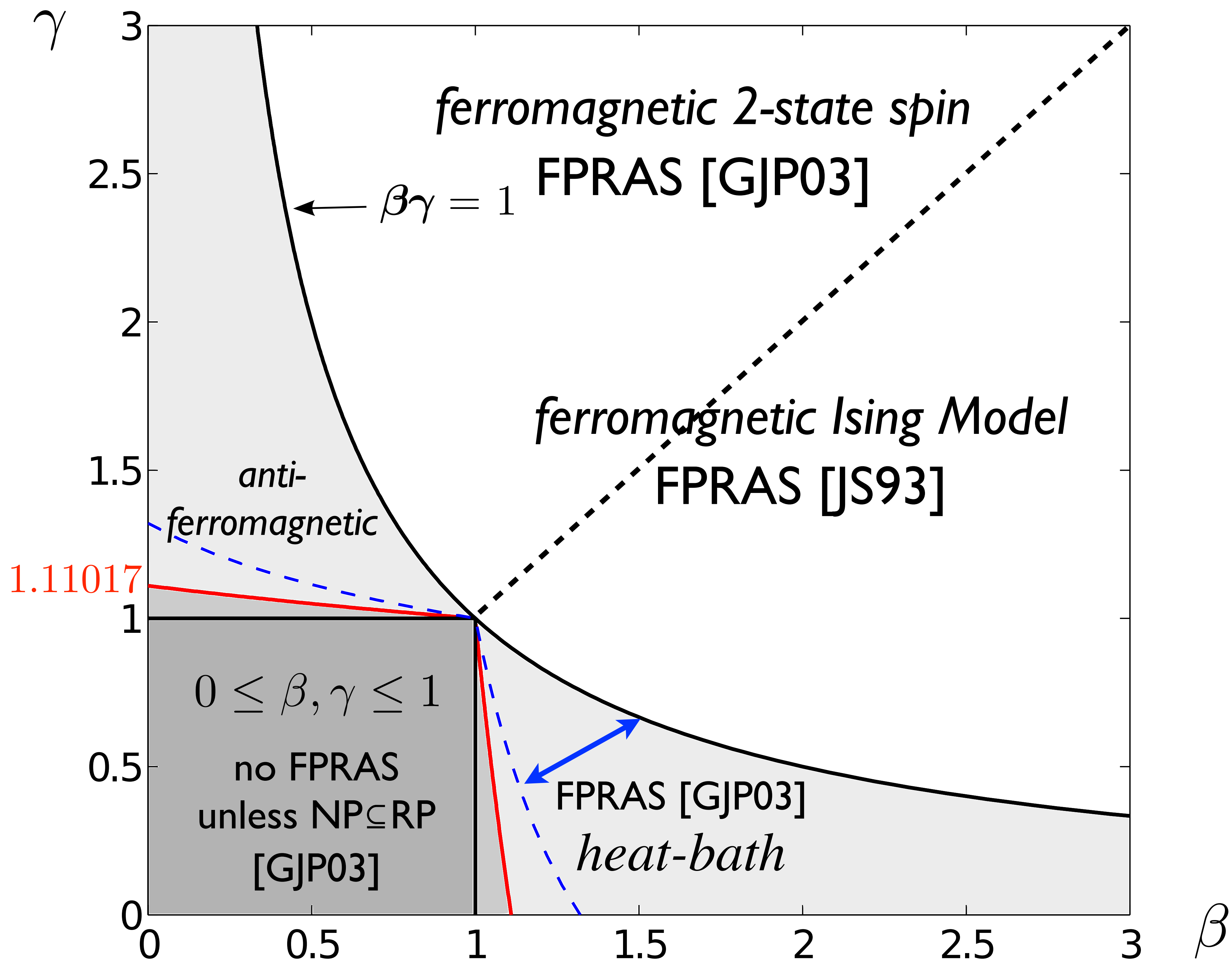
poly-time computable if $\beta\gamma = 1$ or $(\beta, \gamma) = (0, 0)$

#P-hard if otherwise

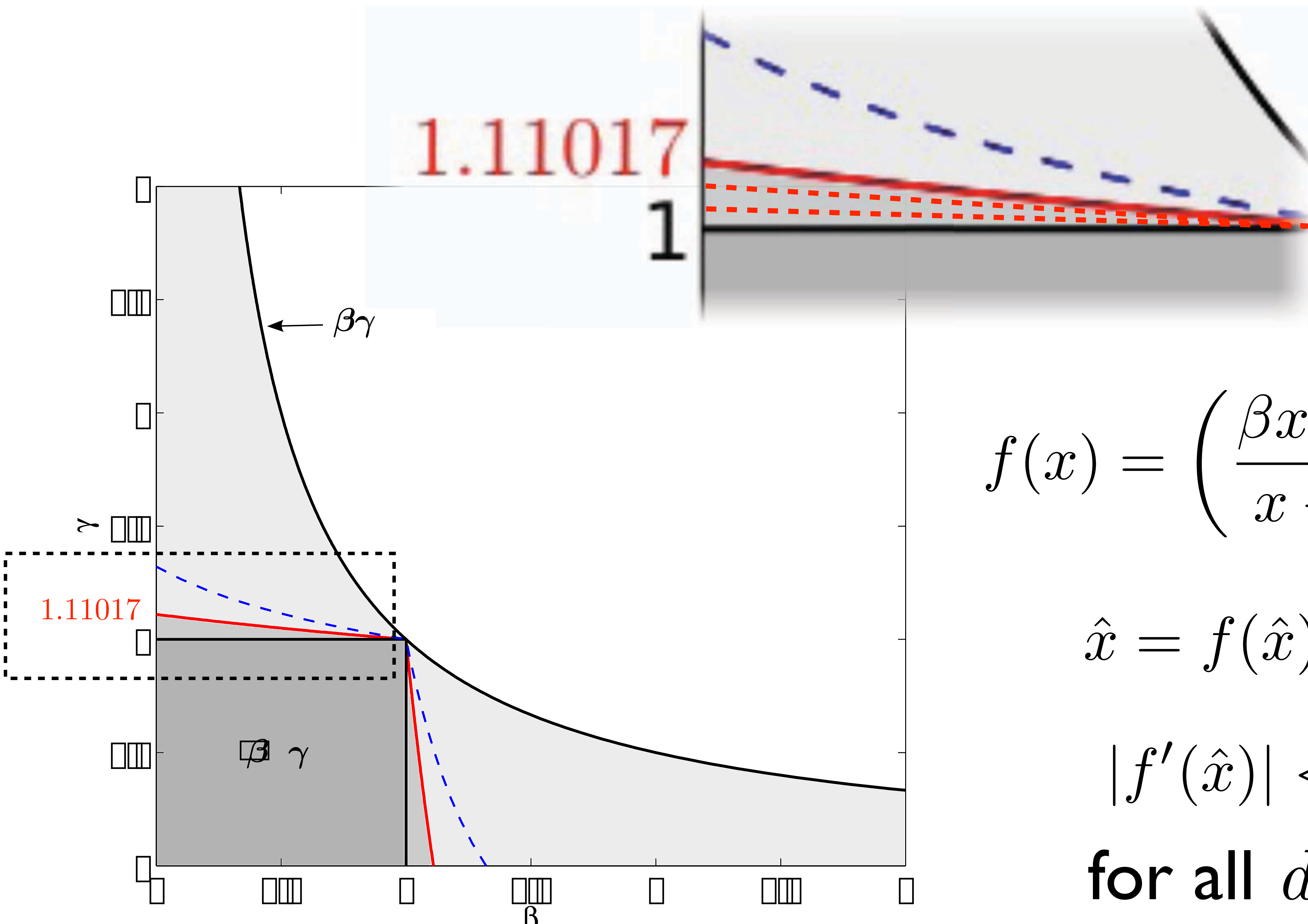
Approximation!

[JS93] Jerrum-Sinclair'93

[GJP03] Goldberg-Jerrum-Paterson'03



Uniqueness Threshold



$0 \leq \beta < 1 < \gamma$

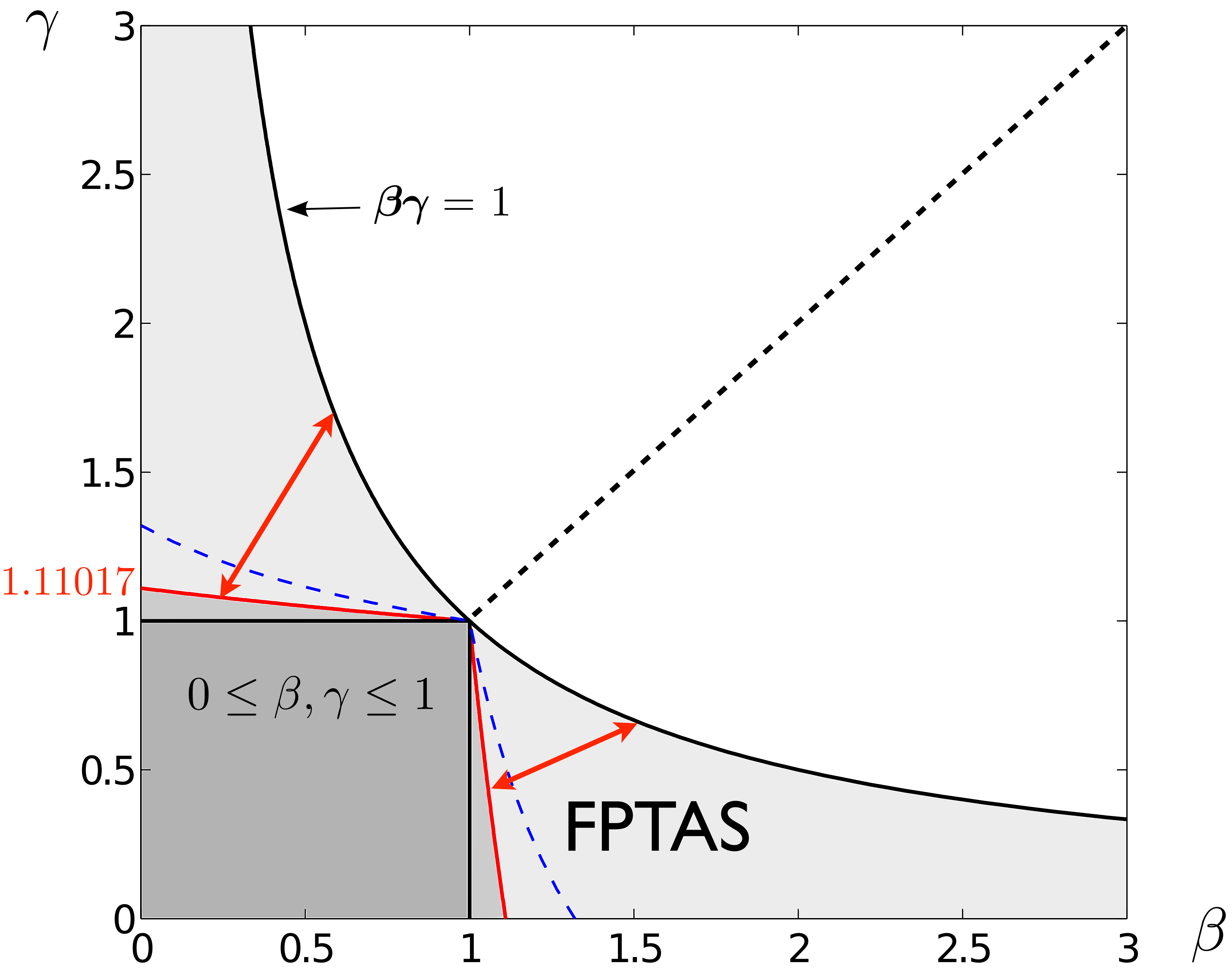
$$f(x) = \left(\frac{\beta x + 1}{x + \gamma}\right)^d$$

$$\hat{x} = f(\hat{x})$$

$$|f'(\hat{x})| < 1$$

for all d

Our Result



γ
3
2.5
2
1.5
1.11017
1
0.5
0
0
0.5
1
1.5
2
2.5
3
β
$\beta\gamma = 1$
$0 \le \beta, \gamma \le 1$
FPTAS

Marginal Distribution

weight: $$w(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)}$$

Gibbs measure: $$\mu(\sigma) = \frac{w(\sigma)}{Z_A(G)}$$

$$Z_A(G) = \sum_{\sigma\in\{0,1\}^V} \prod_{(u,v)\in E} A_{\sigma(u),\sigma(v)}$$

marginal distributions at vertex v:

$$p_v = \Pr_{\sigma\sim\mu}[\sigma(v) = 0]$$

$\Lambda \subset V$ $\sigma_\Lambda \in \{0, 1\}^\Lambda$ fixed $v \in \Lambda$ free $v \notin \Lambda$

$$p_v^{\sigma_\Lambda} = \Pr_{\sigma\sim\mu}[\sigma(v) = 0 \mid \sigma(\Lambda) = \sigma_\Lambda]$$

Self-reduction

(Jerrum-Valiant-Vazirani)

$\Lambda \subset V \quad \sigma_\Lambda \in \{0, 1\}^\Lambda$ $\quad p_v^{\sigma_\Lambda} = \Pr_\sigma[\sigma(v) = 0 \mid \sigma(\Lambda) = \sigma_\Lambda]$

$V = \{v_1, v_2, \ldots, v_n\} \qquad \sigma_i : v_1, v_2, \ldots, v_i \mapsto 1$

$$\begin{aligned}
\Pr(\sigma_n) &= \Pr_\sigma[\sigma : v_1, \ldots, v_n \mapsto 1] \\
&= \prod_{i=1}^{n} \Pr_\sigma[\sigma(v_i) = 1 \mid \sigma : v_1, \ldots, v_{i-1} \mapsto 1] \\
&= \prod_{i=1}^{n} (1 - p_{v_i}^{\sigma_{i-1}})
\end{aligned}$$

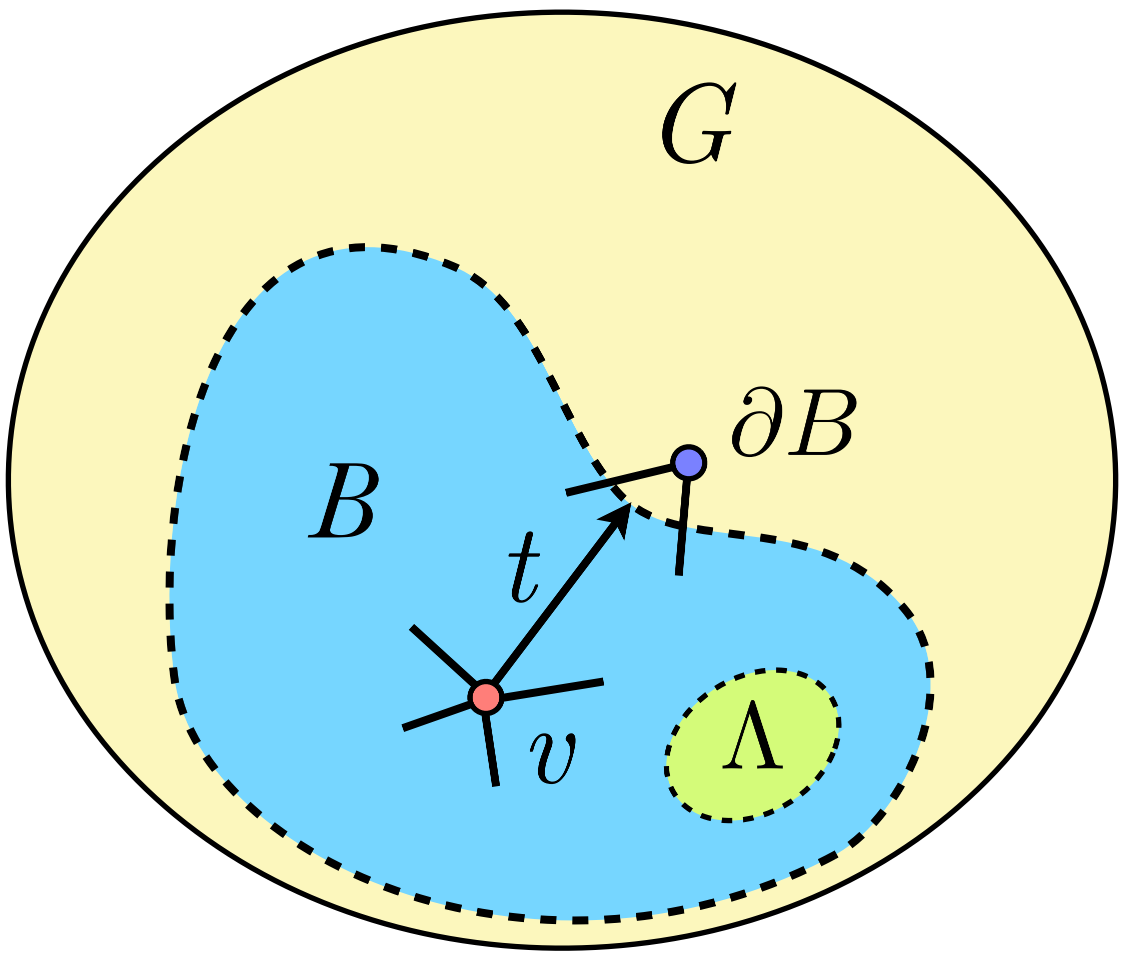
$$\Pr(\sigma_n) = \frac{w(\sigma_n)}{Z(G)} = \frac{\gamma^{|E|}}{Z(G)} \Rightarrow Z(G) = \frac{\gamma^{|E|}}{\prod_{i=1}^{n}(1 - p_{v_i}^{\sigma_{i-1}})}$$

Correlation Decay

$$\forall \sigma_{\partial B}, \tau_{\partial B} \in \{0,1\}^{\partial B}$$

$$\Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}]$$

$$p_v^{\sigma_\Lambda} \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \sigma_{\partial B}, \sigma_\Lambda] \approx \Pr_{\sigma}[\sigma(v) = 0 \mid \tau_{\partial B}, \sigma_\Lambda]$$

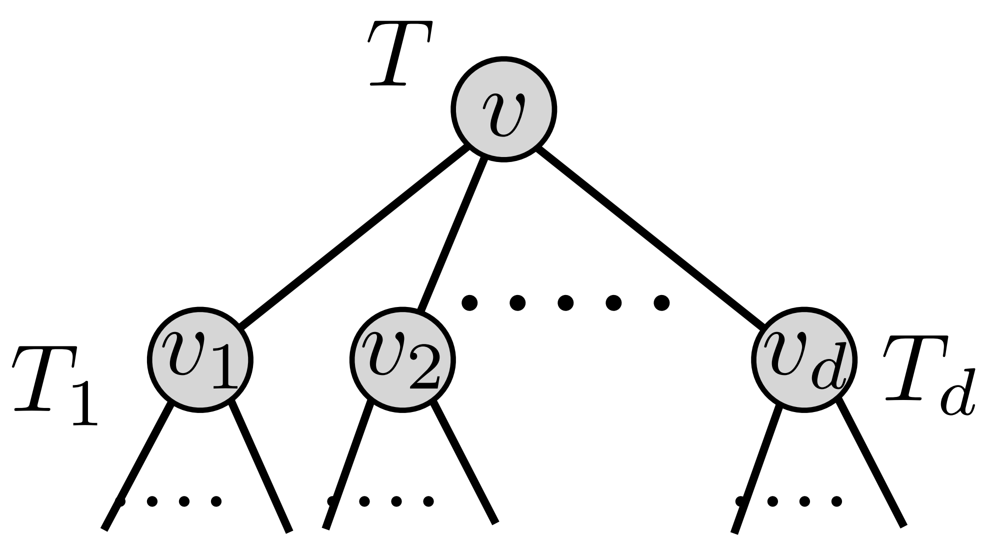


error < exp (-t)

exponential correlation decay

"**strong spatial mixing**" in [Weitz'06]

Recursion for Tree



$$\Lambda \subset V \qquad \sigma_\Lambda \in \{0, 1\}^\Lambda$$

$$R_T^{\sigma_\Lambda} = \frac{p_v^{\sigma_\Lambda}}{1 - p_v^{\sigma_\Lambda}} = \frac{\Pr_{\sigma \sim \mu | \sigma_\Lambda}[\sigma(v) = 0]}{\Pr_{\sigma \sim \mu | \sigma_\Lambda}[\sigma(v) = 1]}$$

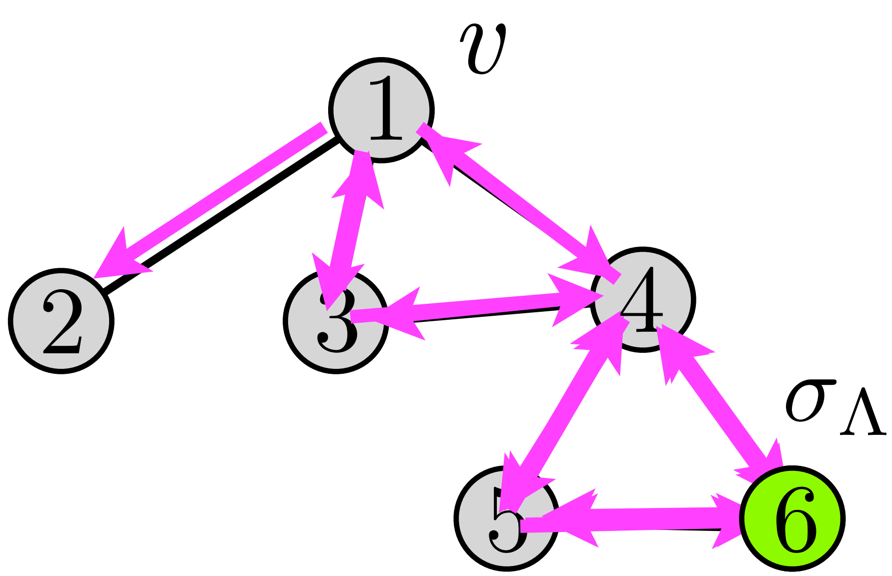
$$R_T^{\sigma_\Lambda} = \prod_{i=1}^{d} \frac{\beta R_{T_i}^{\sigma_\Lambda} + 1}{R_{T_i}^{\sigma_\Lambda} + \gamma}$$

$$\frac{w(\sigma_T : v \mapsto 0)}{w(\sigma_T : v \mapsto 1)} = \frac{\prod_{i=1}^{d} (\beta w(\sigma_{T_i} : v_i \mapsto 0) + w(\sigma_{T_i} : v_i \mapsto 1))}{\prod_{i=1}^{d} (w(\sigma_{T_i} : v_i \mapsto 0) + \gamma w(\sigma_{T_i} : v_i \mapsto 1))}$$

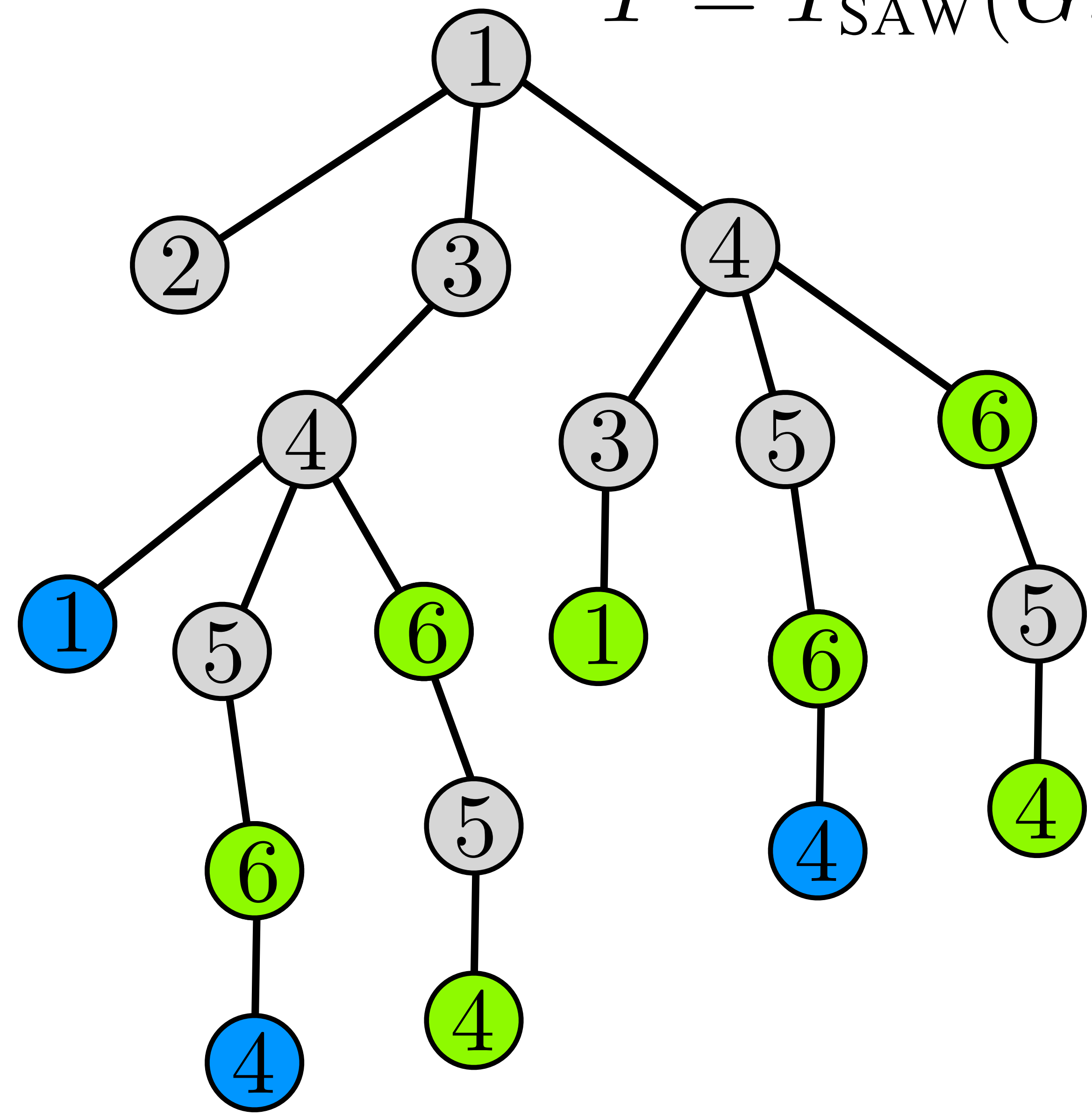
Self-Avoiding Walk Tree

due to Weitz (2006)

$G=(V,E)$



$T = T_{\mathrm{SAW}}(G,v)$



$$R_{G,v}^{\sigma_\Lambda} = \frac{p_v^{\sigma_\Lambda}}{1 - p_v^{\sigma_\Lambda}}$$

Weitz (2006)

$$R_{G,v}^{\sigma_\Lambda} = R_T^{\sigma_\Lambda}$$

- (blue) if cycle closing > cycle starting
- (green) if cycle closing < cycle starting

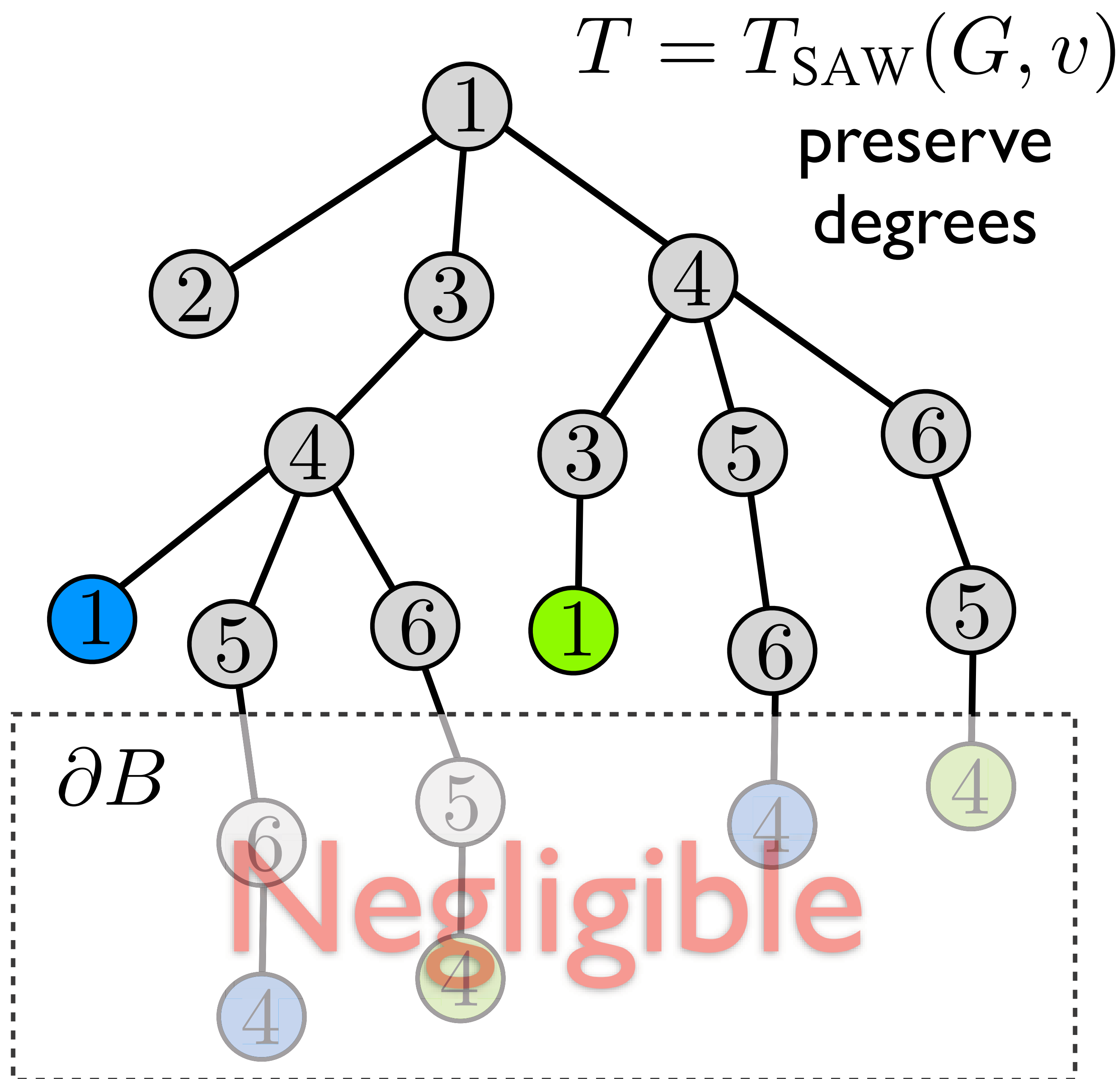
Approximation Algorithm

exponential
correlation decay:

error= $R^{\sigma_{\Lambda \cap B}}_{\text{upper}} - R^{\sigma_{\Lambda \cap B}}_{\text{lower}}$
$= \exp(-\text{depth to } \partial B)$

error decreases
exponentially in depth

poly-time on
$\mathrm{O}(1)$-degree graphs



Correlation Decay!

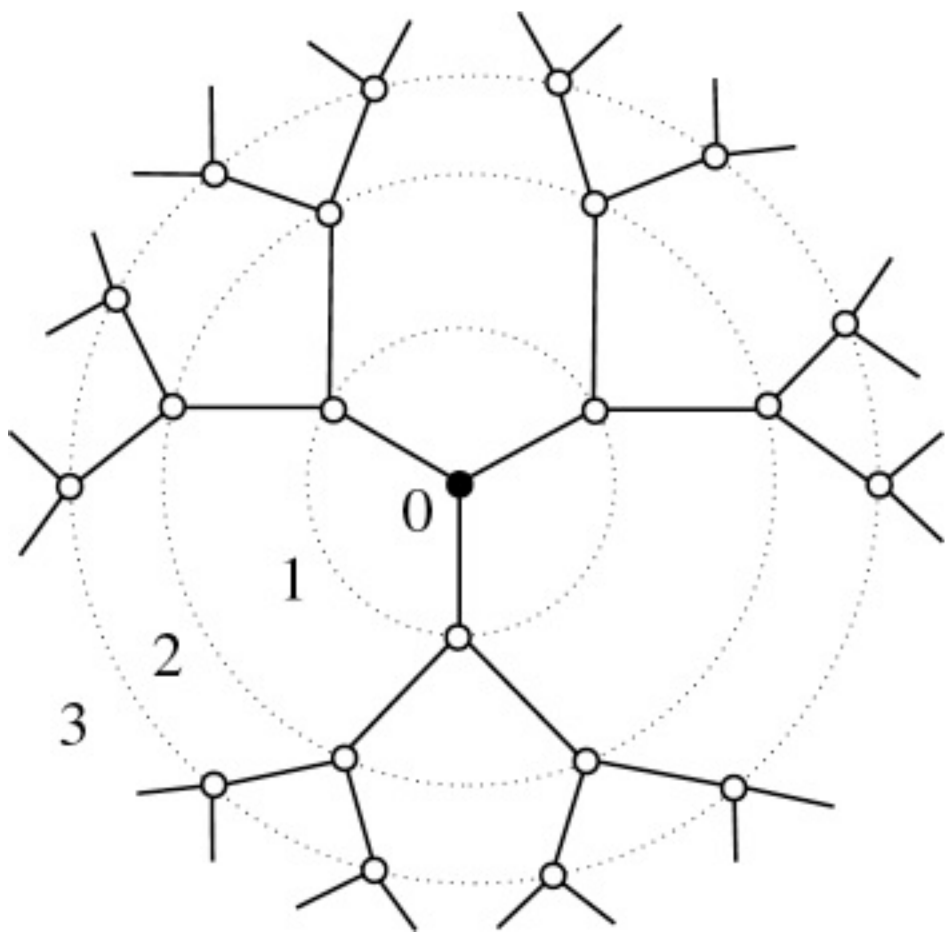
Technique

- *amortized* analysis of decay:
  - the potential method;
- *Computationally Efficient Correlation Decay*: dealing with unbounded-degree graphs;

Uniqueness Threshold

$\widehat{\mathbb{T}}_d$ infinite $(d+1)$-regular tree

(Bethe lattice, Cayley tree)



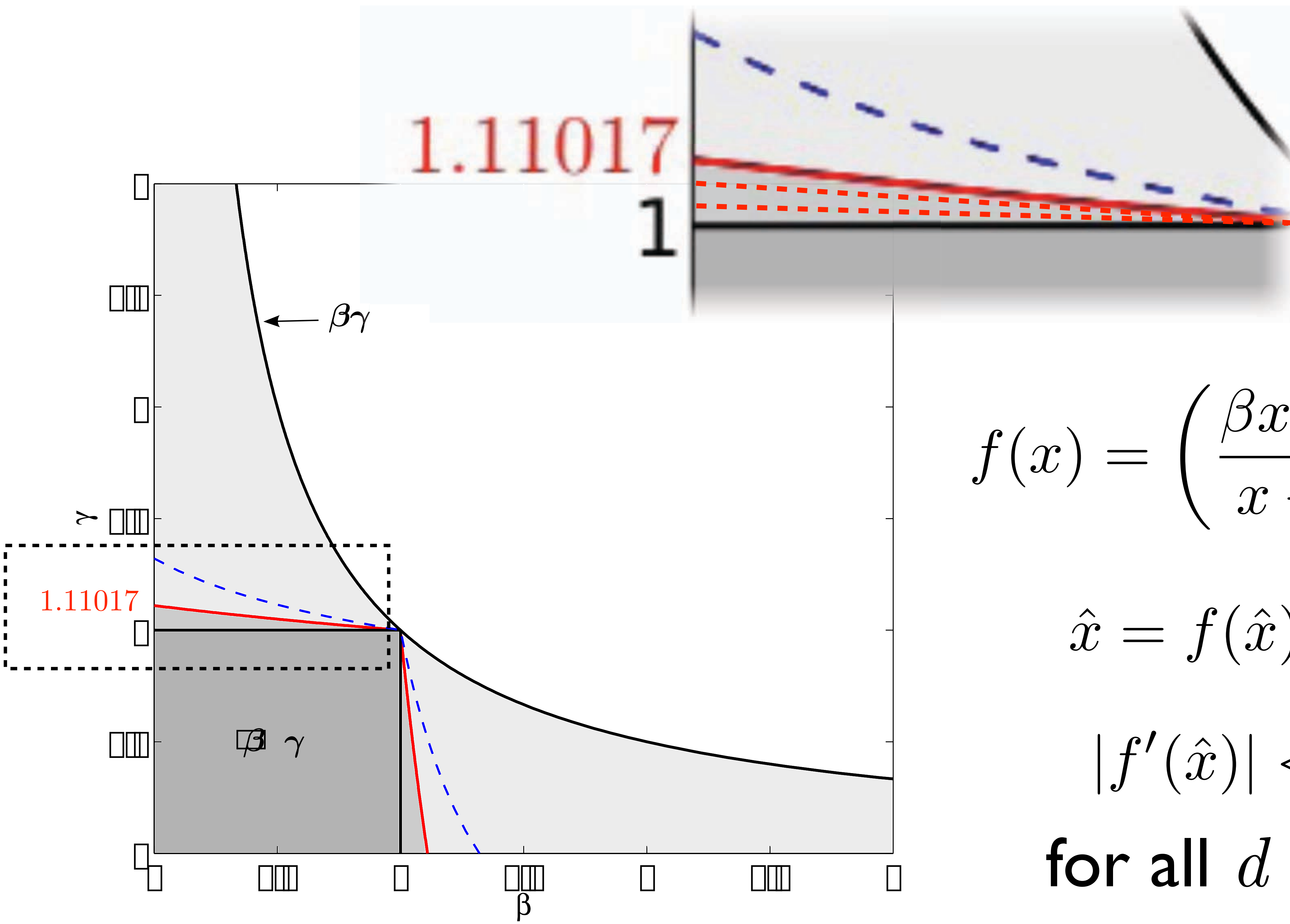
Uniqueness of Gibbs measure

$$f(x) = \left(\frac{\beta x + 1}{x + \gamma}\right)^d$$

$$\hat{x} = f(\hat{x})$$

$$|f'(\hat{x})| < 1$$

Uniqueness Threshold



$$f(x) = \left(\frac{\beta x + 1}{x + \gamma}\right)^d$$

$$\hat{x} = f(\hat{x})$$

$$|f'(\hat{x})| < 1$$

for all d

$$0 \leq \beta < 1 < \gamma$$

Correlation Decay

anti-ferromagnetic $\beta\gamma < 1$

$$R_T^{\sigma_\Lambda} = f(R_{T_1}^{\sigma_\Lambda}, \ldots, R_{T_d}^{\sigma_\Lambda}) = \prod_{i=1}^{d} \frac{\beta R_{T_i}^{\sigma_\Lambda} + 1}{R_{T_i}^{\sigma_\Lambda} + \gamma}$$

monotonically decreasing

upper bound $=$ f(lower bounds)

lower bound $=$ f(upper bounds)

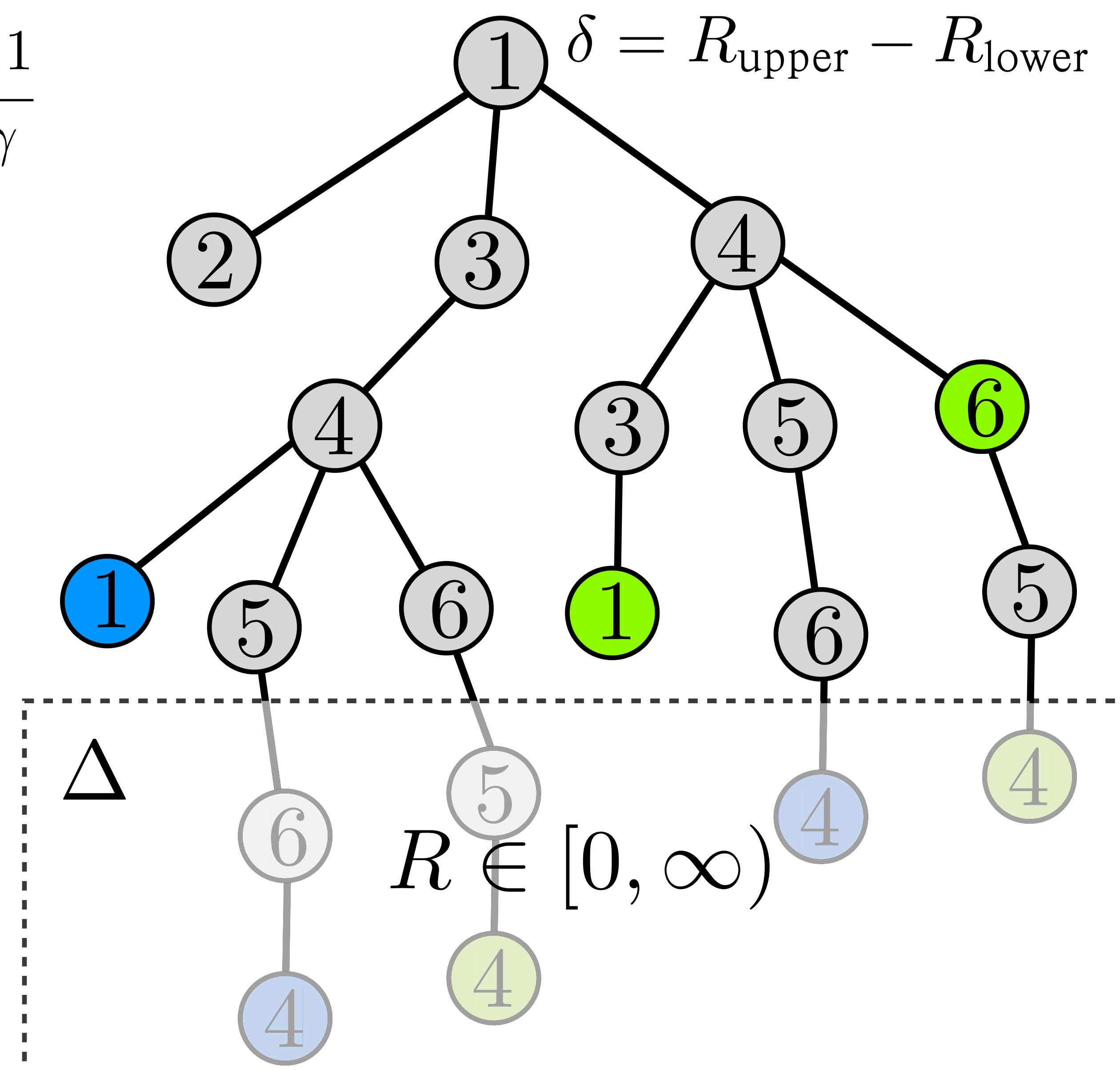
$v \in \Lambda$ fixed to be 0

lower=upper= ∞

$v \in \Lambda$ fixed to be 1

lower=upper= 0

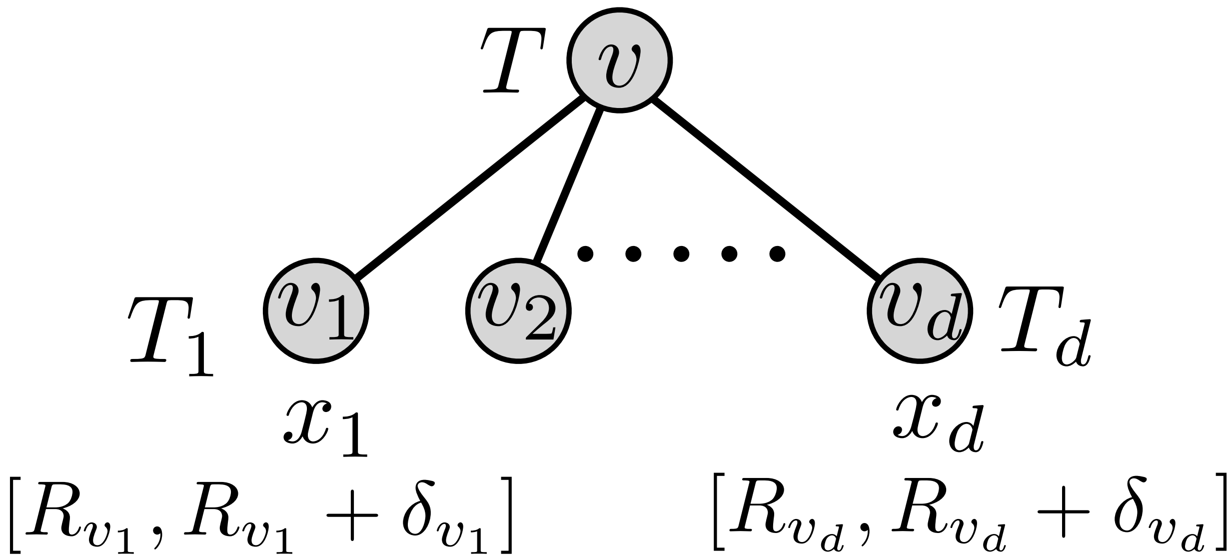
$T = T_{\text{SAW}}(G, v)$



Goal: $\delta = \exp(-\Omega(\text{depth to } \Delta))$

$$R_v \leq R_T^{\sigma\Lambda} \leq R_v + \delta_v$$

$$x \in [R_v, R_v + \delta_v]$$



$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma}$$

$$\frac{\delta_{\text{parent}}}{\Phi(x_{\text{parent}})} \leq \alpha \cdot \frac{\delta_{\text{child}}}{\Phi(x_{\text{child}})}$$

Cheating:

$$f(x) = \left(\frac{\beta x + 1}{x + \gamma}\right)^d$$

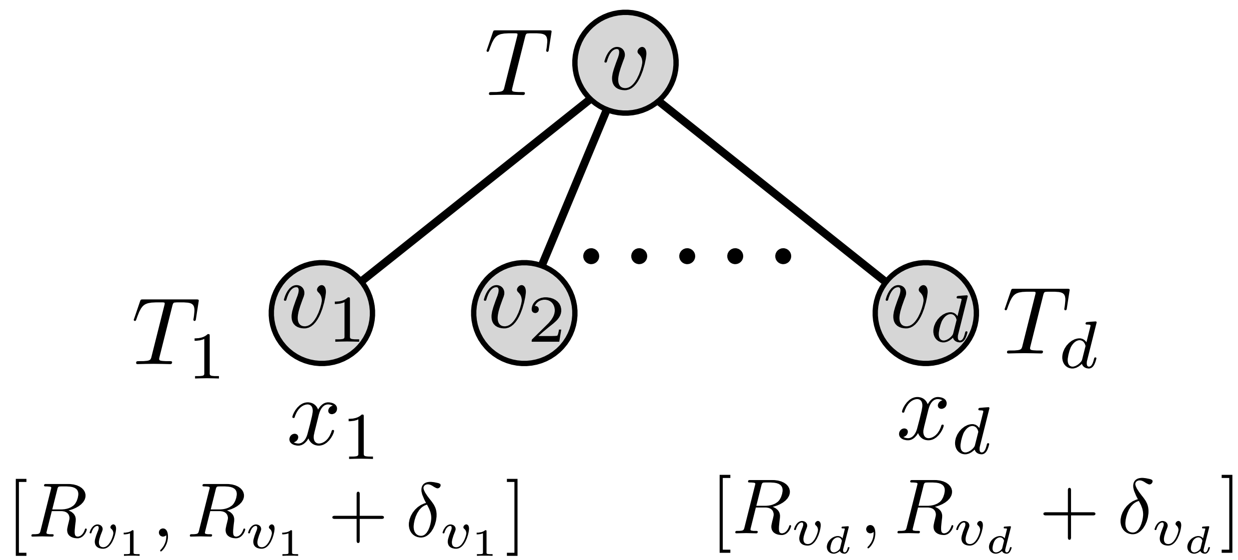
$$\delta_{\text{parent}} = f(R_{\text{child}}) - f(R_{\text{child}} + \delta_{\text{child}})$$
$$= |f'(x)| \cdot \delta_{\text{child}}$$
$$x \in [R_{\text{child}}, R_{\text{child}} + \delta_{\text{child}}]$$

we do not always have
$|f'(x)| < 1$

$$\alpha = \frac{|f'(x)|\Phi(x)}{\Phi(f(x))} < 1$$

$$\Phi(x) = x^{\frac{D+1}{2D}}(\beta x + 1)$$

$$x \in [R_v, R_v + \delta_v]$$



$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma}$$

$$\Phi(x) = x^{\frac{D+1}{2D}} (\beta x + 1)$$

$$\frac{\delta_v}{\Phi(x)}$$

$$= \frac{f(R_{v_1}, \ldots, R_{v_d}) - f(R_{v_1} + \delta_{v_1}, \ldots, \ldots, R_{v_d} + \delta_{v_1})}{\Phi(x)}$$

$\cdots\cdots$ Mean Value Thms

$$\leq \alpha(d; x_1, \ldots, x_d) \cdot \max_{1 \leq i \leq d} \frac{\delta_{v_i}}{\Phi(x_i)}$$

$$\alpha(d; x_1, \ldots, x_d) = \frac{(1 - \beta\gamma) \left(\prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} \right)^{\frac{D-1}{2D}}}{\beta \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} + 1} \cdot \sum_{i=1}^{d} \frac{x_i^{\frac{D+1}{2D}}}{x_i + \gamma}$$

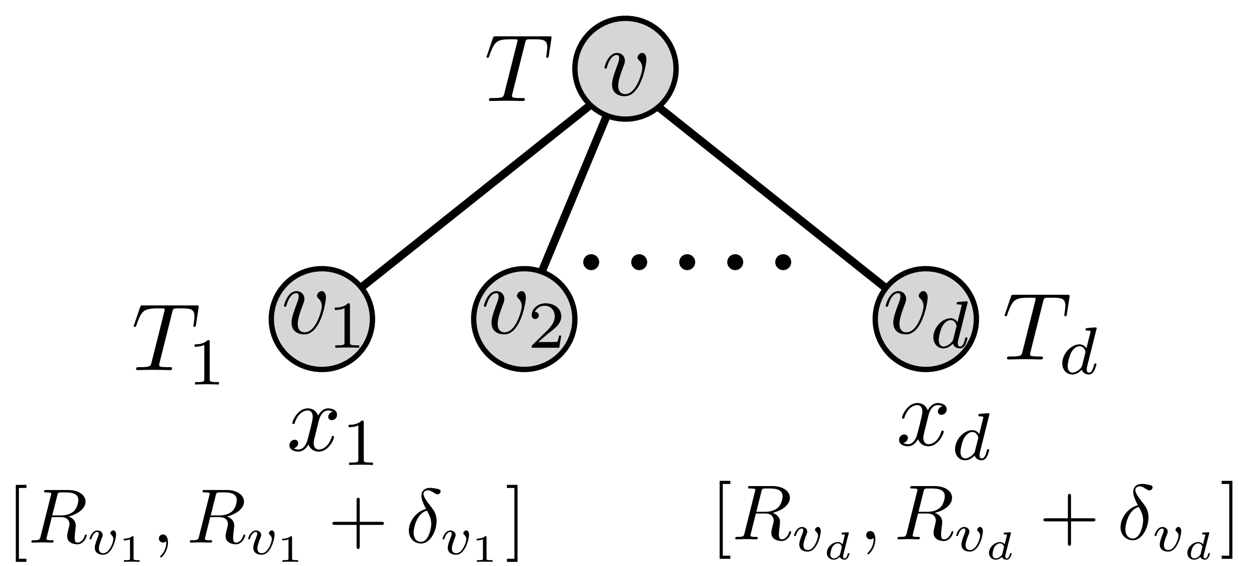
$$\alpha(d; x_1, \ldots, x_d) = \frac{(1-\beta\gamma)\left(\prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma}\right)^{\frac{D-1}{2D}}}{\beta \prod_{i=1}^{d} \frac{\beta x_i + 1}{x_i + \gamma} + 1} \cdot \sum_{i=1}^{d} \frac{x_i^{\frac{D+1}{2D}}}{x_i + \gamma}$$

Jensen's Inequality

$$\leq \alpha(d, x) = \frac{d(1-\beta\gamma) x^{\frac{D+1}{2D}} (\beta x + 1)^{\frac{d(D-1)}{2D}}}{(x+\gamma)^{1+\frac{d(D-1)}{2D}} \left(\beta\left(\frac{\beta x + 1}{x+\gamma}\right)^{d} + 1\right)}$$

$$= \boxed{\frac{\Phi(x)}{\Phi(f(x))} |f'(x)|}$$

$x \in [R_v, R_v + \delta_v]$

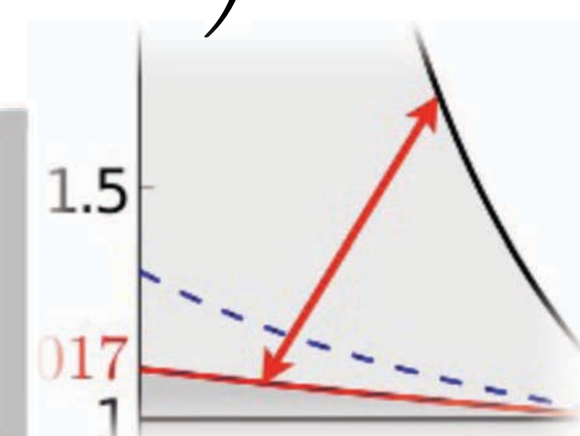


$$\Phi(x) = x^{\frac{D+1}{2D}} (\beta x + 1)$$

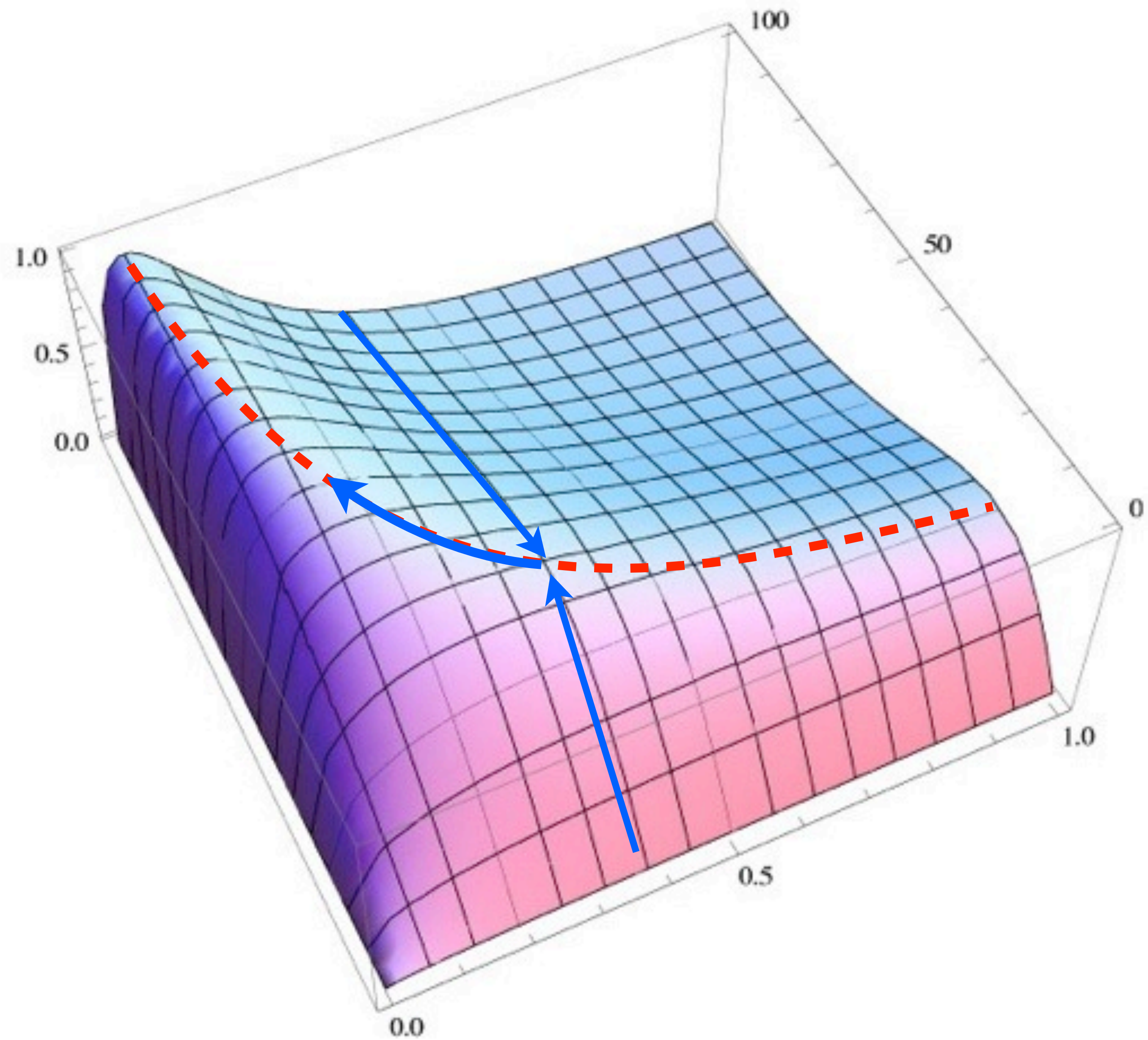
$$\frac{\delta_v}{\Phi(x)} \leq \alpha(d, x) \cdot \max_{1 \leq i \leq d} \frac{\delta_{v_i}}{\Phi(x_i)}$$

$$\alpha(d, x) = \frac{d(1 - \beta\gamma) x^{\frac{D+1}{2D}} (\beta x + 1)^{\frac{d(D-1)}{2D}}}{(x + \gamma)^{1 + \frac{d(D-1)}{2D}} \left(\beta \left(\frac{\beta x + 1}{x + \gamma} \right)^d + 1 \right)}$$

$$\alpha(d, x) \leq 1 \quad \textbf{if} \quad \Gamma \leq \gamma < \frac{1}{\beta}$$



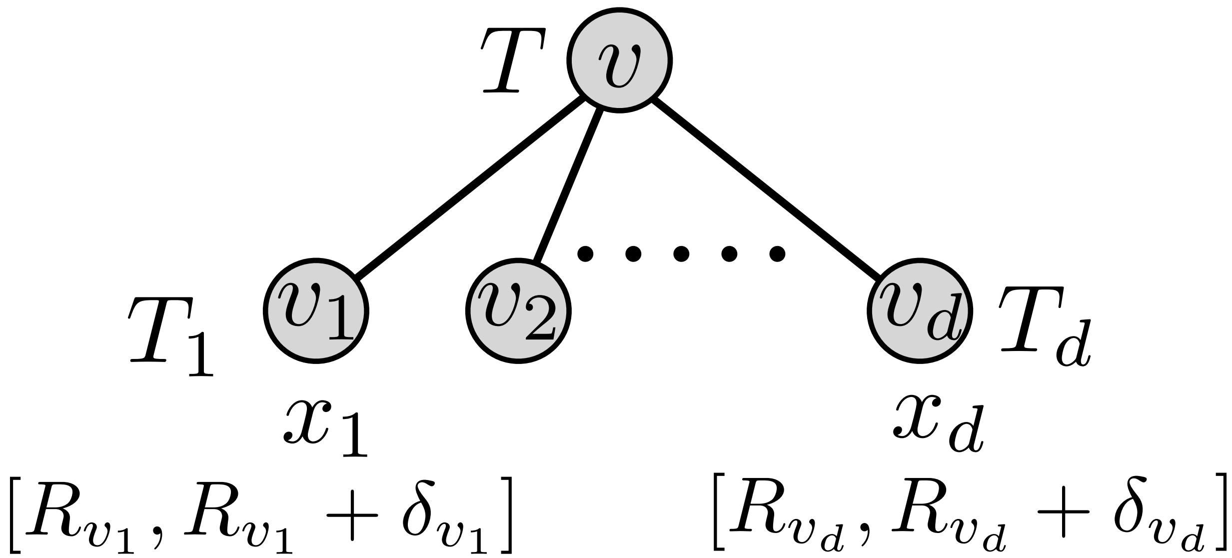
$$\alpha(d,x) = \frac{d(1-\beta\gamma)x^{\frac{D+1}{2D}}(\beta x+1)^{\frac{d(D-1)}{2D}}}{(x+\gamma)^{1+\frac{d(D-1)}{2D}}\left(\beta\left(\frac{\beta x+1}{x+\gamma}\right)^{d}+1\right)}$$

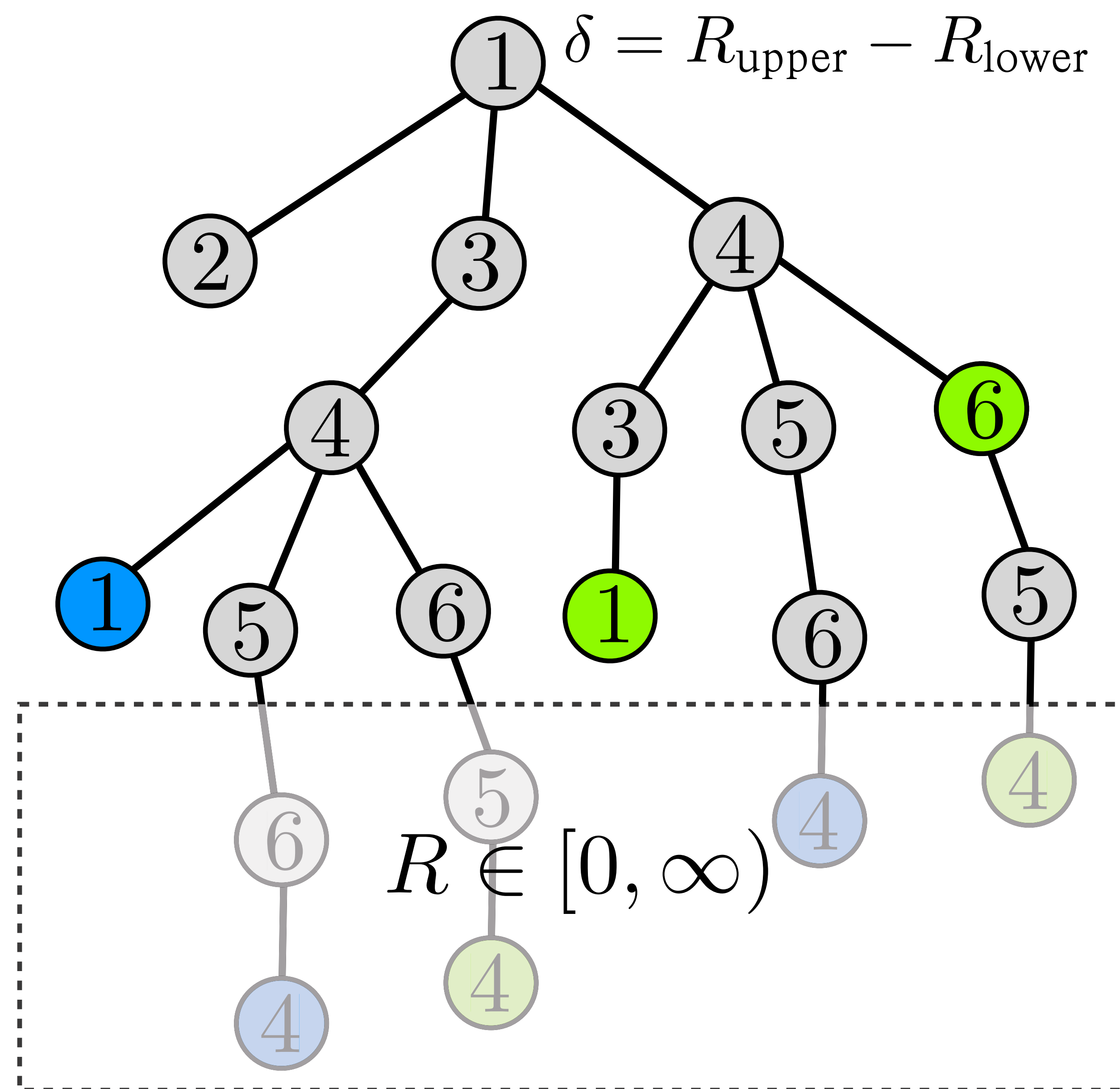


$$R_v \le R_T^{\sigma_\Lambda} \le R_v + \delta_v$$

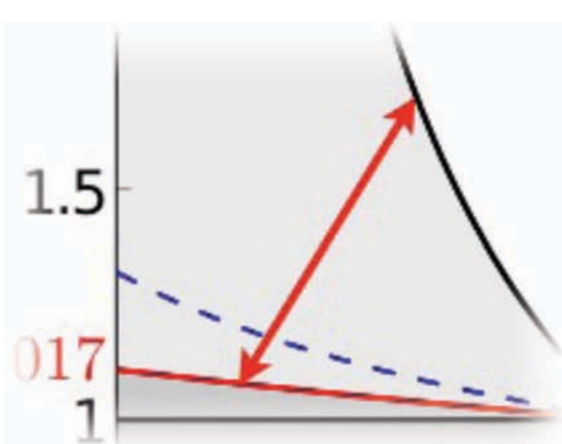
$$T = T_{\mathrm{SAW}}(G, v)$$

$$x \in [R_v, R_v + \delta_v]$$





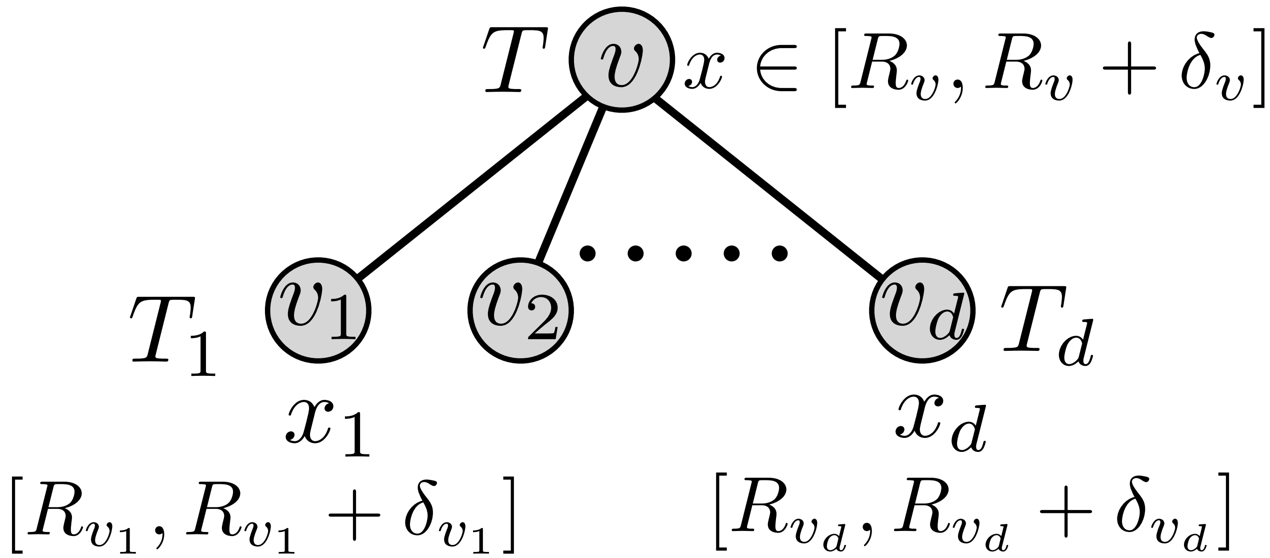
$$\frac{\delta_v}{\Phi(x)} \le \alpha(d, x) \cdot \max_{1 \le i \le d} \frac{\delta_{v_i}}{\Phi(x_i)}$$



$$\alpha(d, x) < 1$$

$$\delta = \exp(-\Omega(\text{depth to } \Delta))$$

Computationally Efficient Correlation Decay



$$\frac{\delta_v}{\Phi(x)} \leq \alpha(d, x) \cdot \max_{1 \leq i \leq d} \frac{\delta_{v_i}}{\Phi(x_i)}$$

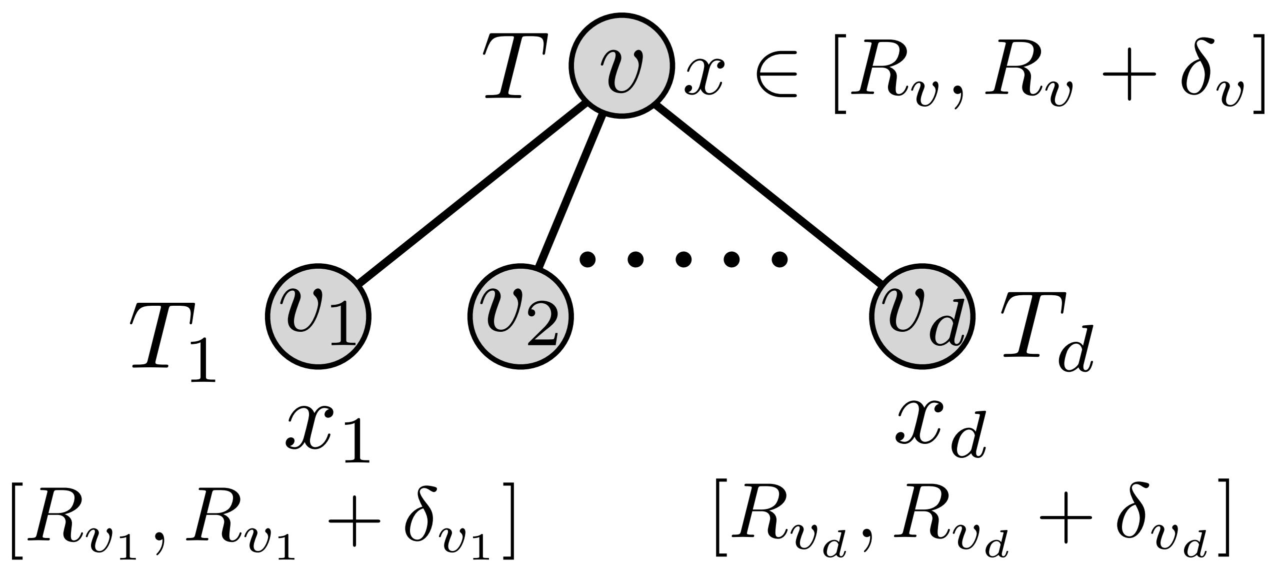
$$\alpha(d, x) = \frac{d(1 - \beta\gamma) x^{\frac{D+1}{2D}} (\beta x + 1)^{\frac{d(D-1)}{2D}}}{(x + \gamma)^{1 + \frac{d(D-1)}{2D}} \left(\beta \left(\frac{\beta x + 1}{x + \gamma} \right)^d + 1 \right)}$$

$$\leq \frac{d}{\gamma^{d \frac{D-1}{2D}}} \quad \leq \alpha^{\lceil \log_M (d+1) \rceil}$$

for some $\alpha < 1$, $M > 1$

$\gamma > 1$

Computationally Efficient Correlation Decay



$$\frac{\delta_v}{\Phi(x)} \le \alpha(d,x) \cdot \max_{1 \le i \le d} \frac{\delta_{v_i}}{\Phi(x_i)}$$

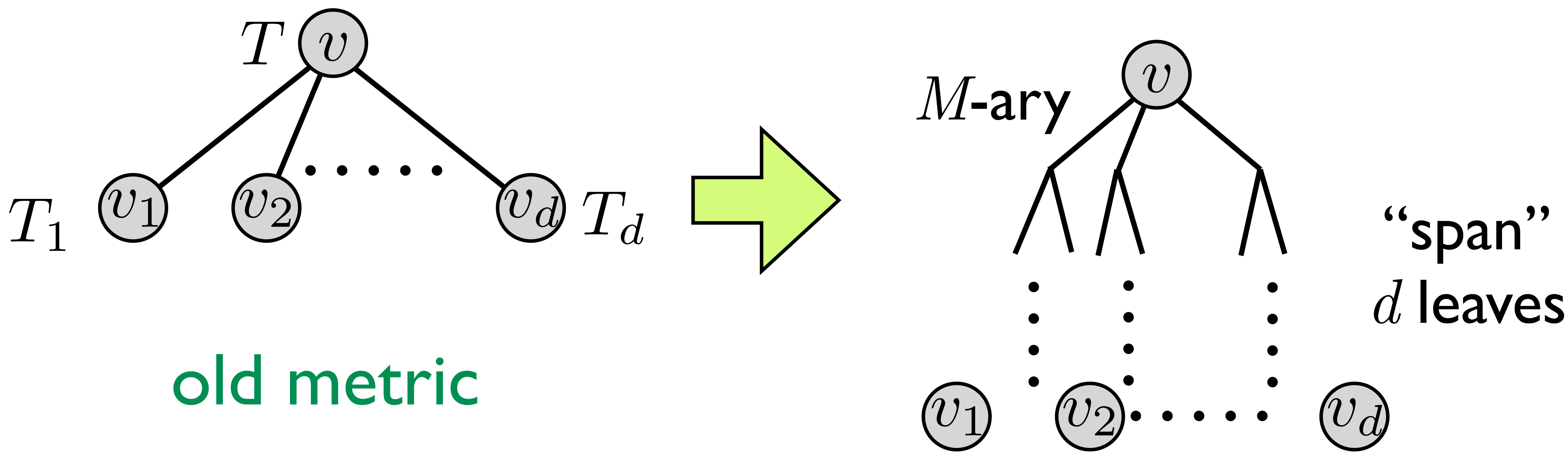
$\alpha(d,x) \le \alpha^{\lceil \log_M (d+1) \rceil}$ for some $\alpha < 1$ $M > 1$

for small $d < M$ one-step recursion decays α

for large $d \ge M$ one-step recursion decays $\alpha^{\lceil \log_M (d+1) \rceil}$

behaves like $\lceil \log_M (d+1) \rceil$ steps!

Computationally Efficient Correlation Decay



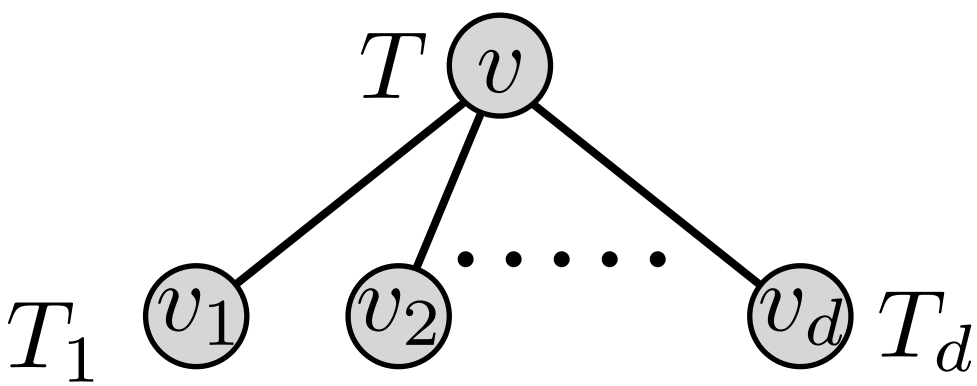
exponential decay in new metric

for small $d < M$ one-step recursion decays α

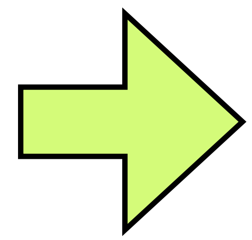
for large $d \geq M$ one-step recursion decays $\alpha^{\lceil \log_M (d+1) \rceil}$

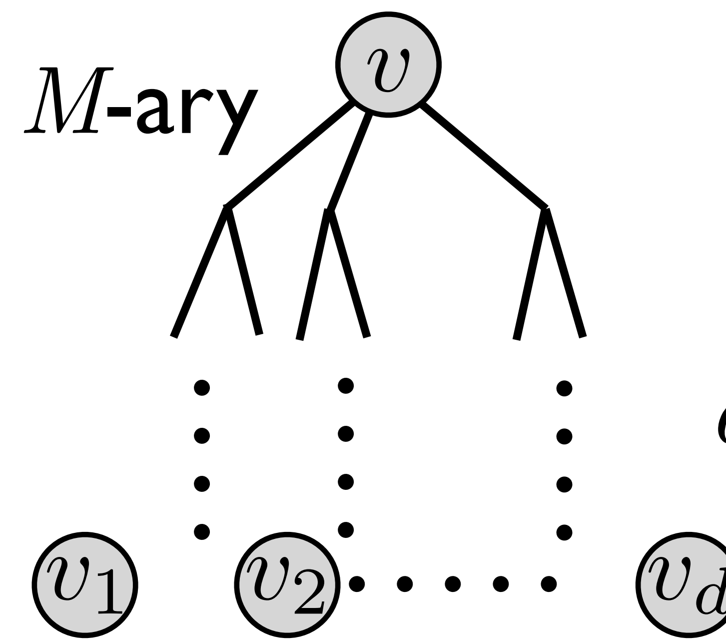
behaves like $\lceil \log_M (d+1) \rceil$ steps!

Computationally Efficient Correlation Decay



old metric





“span” d leaves

exponential decay in new metric α^{distance}

only moderately grows M^{distance}

distance = $\mathrm{O}(\log\ n)$ 1/poly-precision in poly-time

Conclusion

- FPTAS for 2-state spin system up to uniqueness threshold (conjectured to be the boundary of approximability).
- Computationally Efficient Correlation Decay (the first time that Correlation Decay is used to deal with arbitrary graphs).

Thank You!