Introduction to the Correlation Decay Method

Yitong Yin
Nanjing University

Introduction to Partition Functions Workshop, July 20, 2018
Computational Aspects of Partition Functions, CIB@EPFL

- July 20: Introduction to the correlation decay method (Yitong)
- July 23: Correlation decay for distributed counting (Yitong)
- July 25: Beyond bounded degree graphs (Piyush)
- July 26: Correlation decay, zeros of polynomials, and the Lovász local lemma (Piyush)

Counting Independent Set

hardcore model: undirected graph $G(V,E)$ fugacity $\lambda > 0$

$\forall I \subseteq V$: $$w(I) = \begin{cases} \lambda^{|I|} & I \text{ is an independent set in } G \\ 0 & \text{otherwise} \end{cases}$$

partition function: $$Z = Z_G(\lambda) = \sum_{I \subseteq V} w(I)$$

uniqueness threshold: $$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \approx \frac{\mathrm{e}}{\Delta-2}$$

Computing $Z_G(\lambda)$ in graphs with constant max-degree $\leq \Delta$

- $\lambda < \lambda_c(\Delta) \Rightarrow$ FPTAS [Weitz 06] $\Longleftarrow$ strong spatial mixing
- $\lambda > \lambda_c(\Delta) \Rightarrow$ no FPRAS unless NP=RP

[Galanis Štefankovič Vigoda 12] [Sly Sun 12]

Spin System

undirected graph $G = (V, E)$

finite integer $q \geq 2$

configuration $\sigma \in [q]^V$

weight: $w(\sigma) = \prod_{\{u,v\}\in E} A(\sigma_u, \sigma_v) \prod_{v \in V} b(\sigma_v)$

$A : [q] \times [q] \to \mathbb{R}_{\geq 0}$ symmetric q×q matrix (symmetric binary constraint)

$b : [q] \to \mathbb{R}_{\geq 0}$ q-vector (unary constraint)

partition function: $Z_G = \sum_{\sigma \in [q]^V} w(\sigma)$

Gibbs distribution: $\mu_G(\sigma) = \frac{w(\sigma)}{Z_G}$

undirected graph $G = (V, E)$ finite integer $q \geq 2$

configuration $\sigma \in [q]^V$

weight: $w(\sigma) = \prod_{\{u,v\}\in E} A(\sigma_u, \sigma_v) \prod_{v \in V} b(\sigma_v)$

- 2-spin system: $q = 2$, $\sigma \in \{0,1\}^V$

  symmetric $A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$ $b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$

  - hardcore model: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$
  - Ising model: $A = \begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$ $b = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

- multi-spin system: general $q \geq 2$
  - Potts model:
  - q-coloring: $\beta = 0$

$$A = \begin{bmatrix} \beta & & 1 \\ & \beta & \\ 1 & \ddots & \\ & & \beta \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Marginal Probability

undirected graph $G = (V, E)$ finite integer $q \geq 2$

Gibbs distribution μ_G over all configurations in $[q]^V$

$\forall$ possible boundary condition $\sigma \in [q]^\Lambda$

specified on an arbitrary subset $\Lambda \subset V$

marginal distribution μ_v^σ at vertex $v \in V$:

$$\forall x \in [q]: \quad \mu_v^\sigma(x) = \Pr_{X \sim \mu_G}[X_v = x \mid X_\Lambda = \sigma]$$

$$\frac{w(\sigma)}{Z_G} = \mu(\sigma) = \prod_{i=1}^{n} \mu_{v_i}^{\sigma_1, \ldots, \sigma_{i-1}}(\sigma_i)$$

Marginal Probability

undirected graph $G=(V,E)$ finite integer $q\geq 2$

Gibbs distribution μ_G over all configurations in $[q]^V$

$\forall$ possible boundary condition $\sigma\in[q]^\Lambda$

specified on an arbitrary subset $\Lambda\subset V$

marginal distribution μ_v^σ at vertex $v\in V$:

$$\forall x\in[q]:\qquad \mu_v^\sigma(x)=\Pr_{X\sim\mu_G}[X_v=x\mid X_\Lambda=\sigma]$$

approximately computing μ_v^σ

approximately computing Z_G

approx. inference

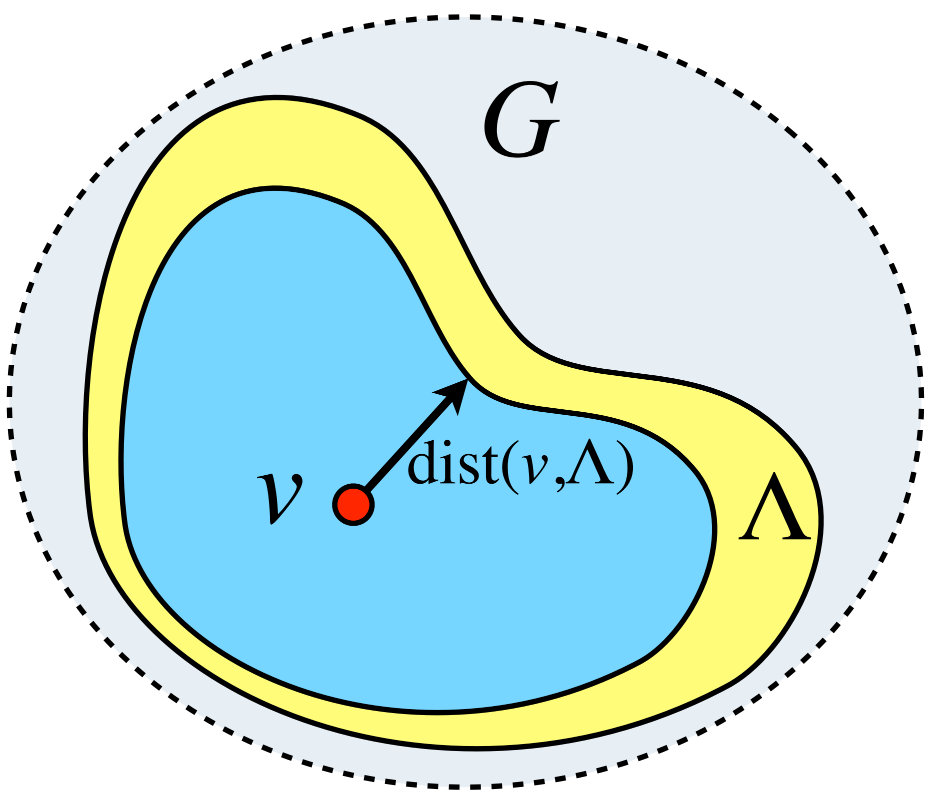
approx. counting

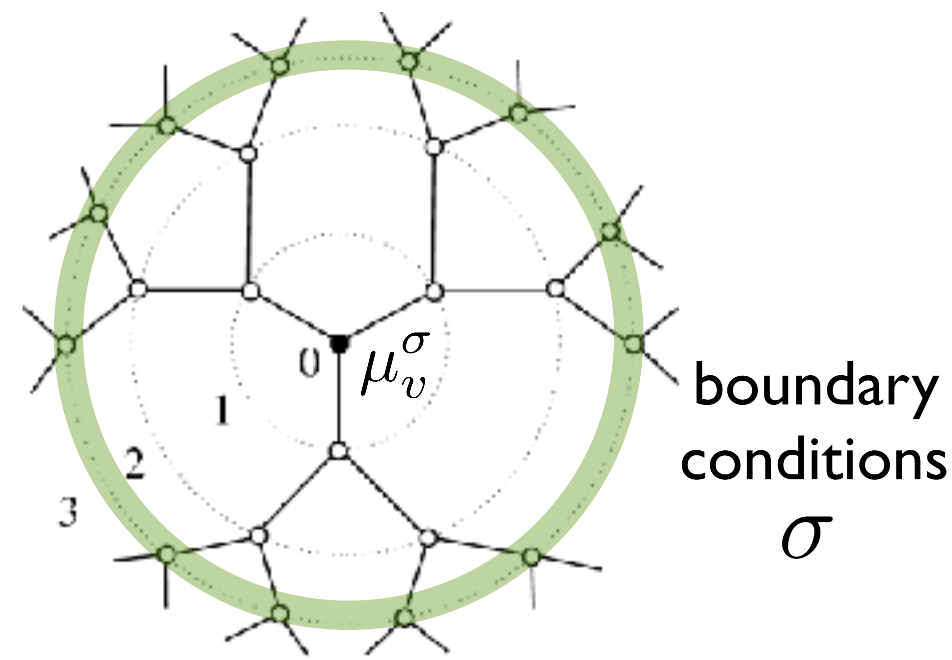
Spatial Mixing (Decay of Correlation)

μ_v^σ : marginal distribution at vertex v conditioning on σ

weak spatial mixing (WSM) at rate $\delta(\ )$:

$$\forall \sigma, \tau \in [q]^\Lambda : \qquad \|\mu_v^\sigma - \mu_v^\tau\|_{TV} \leq \delta(\mathrm{dist}_G(v, \Lambda))$$





on infinite graphs:

WSM $\Leftrightarrow$ uniqueness of infinite-volume Gibbs measure

$\lambda \leq \lambda_c(\Delta)$ $\Leftrightarrow$ WSM of hardcore model on infinite Δ-regular tree

Spatial Mixing (Decay of Correlation)

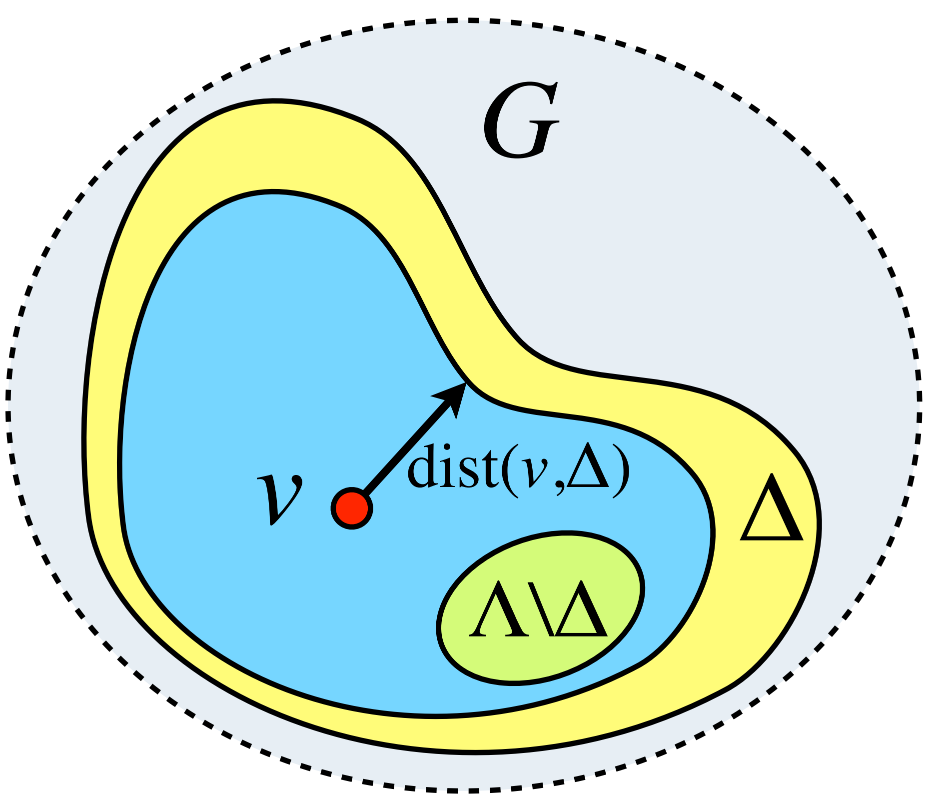
μ_v^σ : marginal distribution at vertex v conditioning on σ

weak spatial mixing (WSM) at rate $\delta(\ )$:

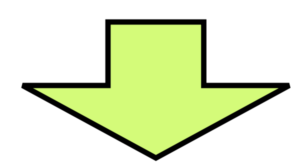
$$\forall \sigma, \tau \in [q]^\Lambda : \qquad \|\mu_v^\sigma - \mu_v^\tau\|_{TV} \leq \delta(\mathrm{dist}_G(v, \Lambda))$$

strong spatial mixing (SSM) at rate $\delta(\ )$:

$\forall \sigma, \tau \in [q]^\Lambda$ that differ on Δ: $\|\mu_v^\sigma - \mu_v^\tau\|_{TV} \leq \delta(\mathrm{dist}_G(v, \Delta))$



SSM

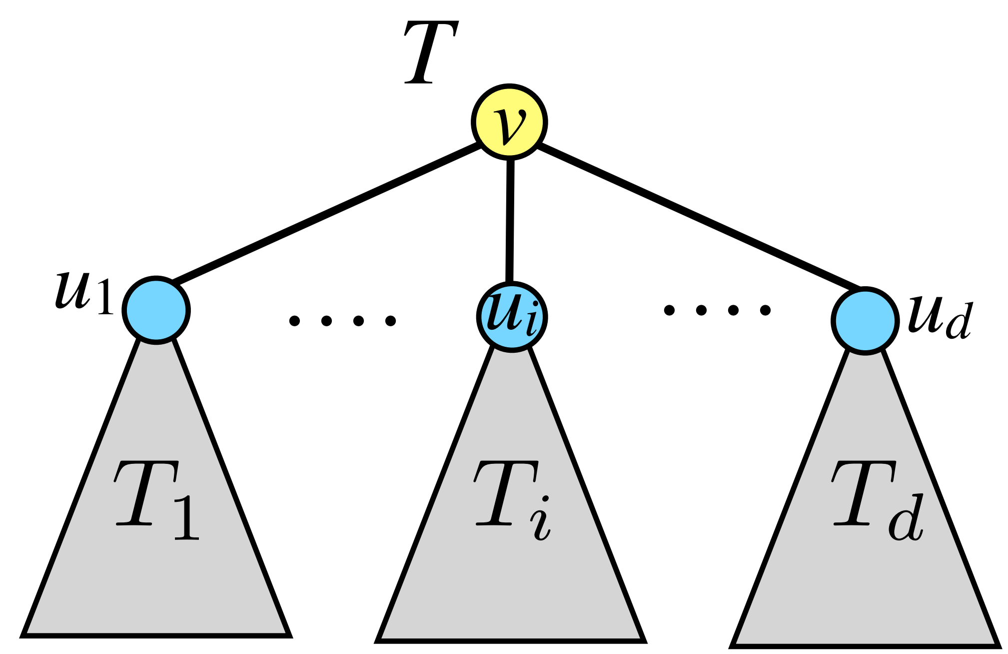


marginal probabilities are well approximated by the *local information*

Tree Recurrence

hardcore model: independent set I in T $\mu_T(I) \propto \lambda^{|I|}$

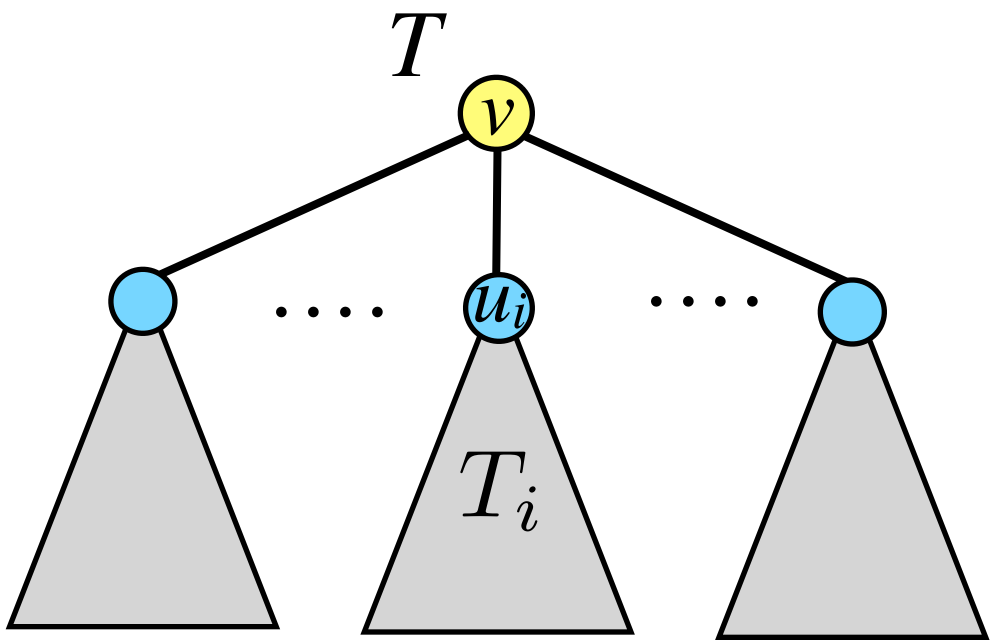
$$p_T \triangleq \Pr_{I \sim \mu_T}[v \text{ is unoccupied by } I]$$



$$p_{T_i} \triangleq \Pr_{I \sim \mu_{T_i}}[u_i \text{ is unoccupied by } I]$$

$$p_T \triangleq \Pr_{I \sim \mu_T}[v \text{ is unoccupied by } I\,]$$

$$= \frac{Z_T(v \text{ is unoccupied})}{Z_T(v \text{ is unoccupied}) + Z_T(v \text{ is occupied})}$$

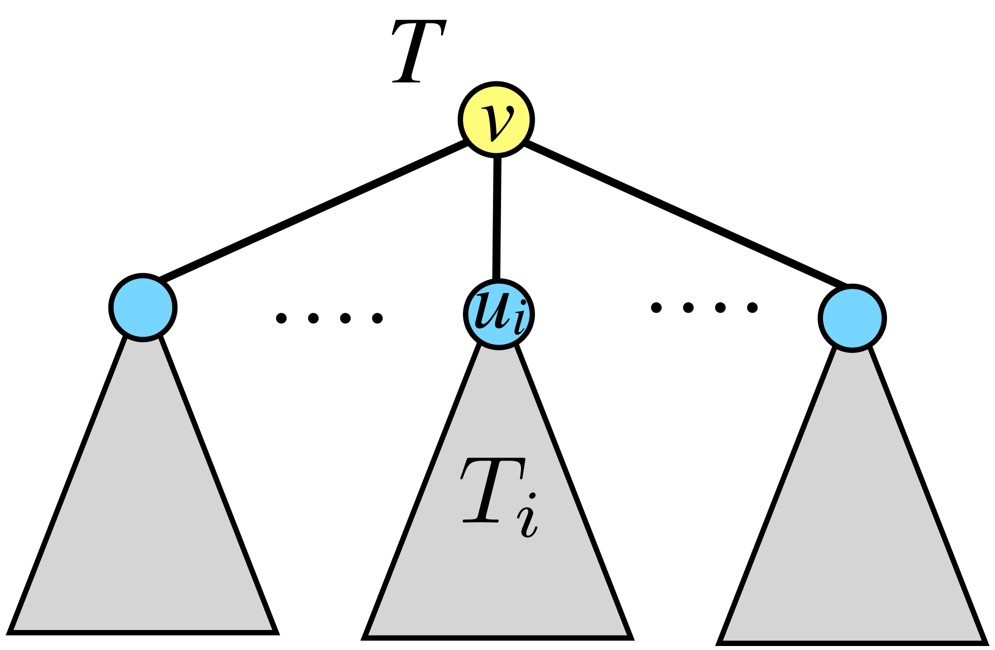


where

$$Z_T(\text{event } A) \triangleq \sum_{I:\, A \text{ holds}} w(I)$$

$$p_T \triangleq \Pr_{I \sim \mu_T}[v \text{ is unoccupied by } I\,]$$

$$= \frac{Z_T(v \text{ is unoccupied})}{Z_T(v \text{ is unoccupied}) + Z_T(v \text{ is occupied})}$$



Product rule:

$$Z_T(v \text{ is unoccupied}) = \prod_{i=1}^{d} Z_{T_i}$$

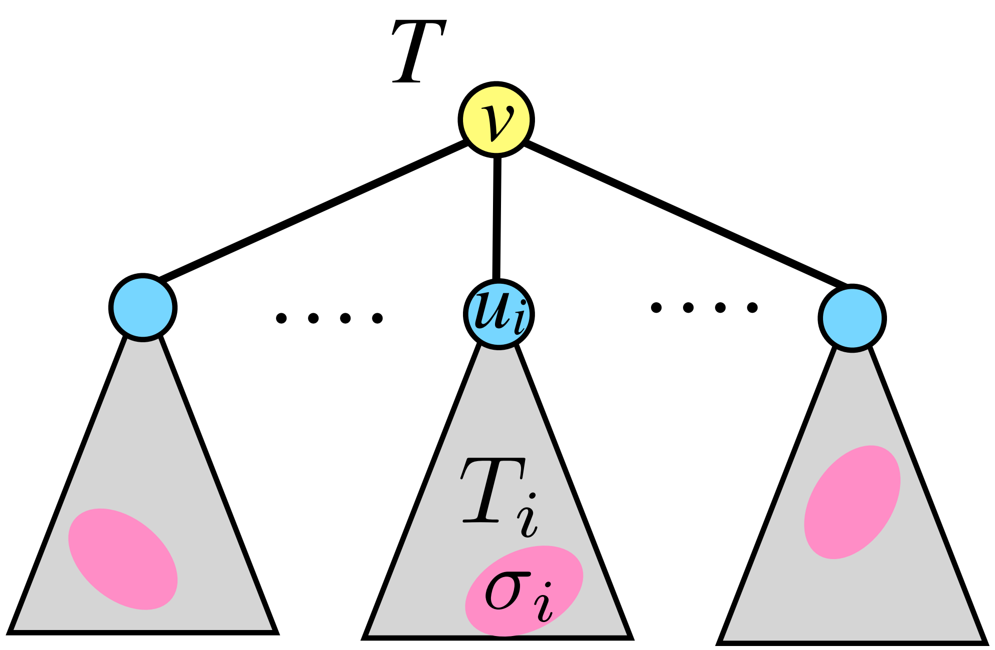
$$Z_T(v \text{ is occupied}) = \lambda \prod_{i=1}^{d} Z_{T_i}(u_i \text{ is unoccupied})$$

$$= \frac{\prod_{i=1}^{d} Z_{T_i}}{\prod_{I=1}^{d} Z_{T_i} + \lambda \prod_{i=1}^{d} Z_{T_i}(u_i \text{ is unoccupied})}$$

$$= \frac{1}{1 + \lambda \prod_{i=1}^{d} p_{T_i}}$$

$$p_T^{\sigma} \triangleq \Pr_{I \sim \mu_T}[v \text{ is unoccupied by } I \mid \sigma]$$

$$= \frac{Z_T(v \text{ is unoccupied} \wedge \sigma)}{Z_T(v \text{ is unoccupied} \wedge \sigma) + Z_T(v \text{ is occupied} \wedge \sigma)}$$



Product rule:

$$Z_T(v \text{ is unoccupied} \wedge \sigma) = \prod_{i=1}^{d} Z_{T_i}(\sigma_i)$$

$$Z_T(v \text{ is occupied} \wedge \sigma) = \lambda \prod_{i=1}^{d} Z_{T_i}(u_i \text{ is unoccupied} \wedge \sigma_i)$$

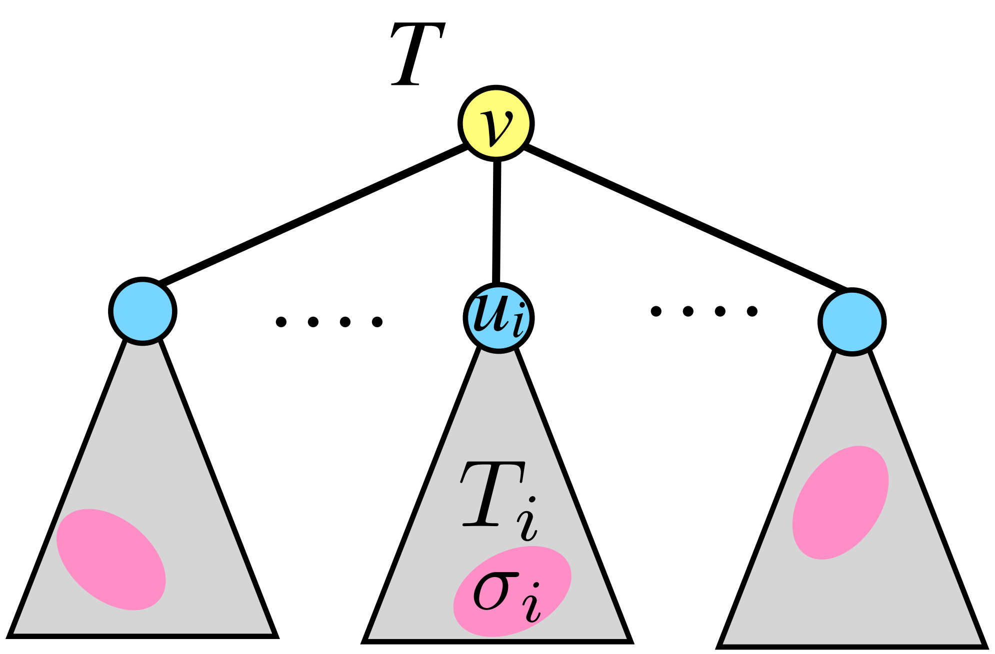
$$= \frac{\prod_{i=1}^{d} Z_{T_i}(\sigma_i)}{\prod_{I=1}^{d} Z_{T_i} + \lambda \prod_{i=1}^{d} Z_{T_i}(\sigma_i)(u_i \text{ is unoccupied} \wedge \sigma_i)}$$

$$= \frac{1}{1 + \lambda \prod_{i=1}^{d} p_{T_i}^{\sigma_i}}$$

Tree Recurrence

hardcore model: independent set I in T $\mu_T(I) \propto \lambda^{|I|}$

$$p_T^\sigma \triangleq \Pr_{I\sim\mu_T}[v \text{ is unoccupied by } I \mid \sigma]$$



Occupancy ratio:

$$R_T^\sigma \triangleq \frac{\Pr_T[v \text{ is occupied} \mid \sigma]}{\Pr_T[v \text{ is unoccupied} \mid \sigma]}$$

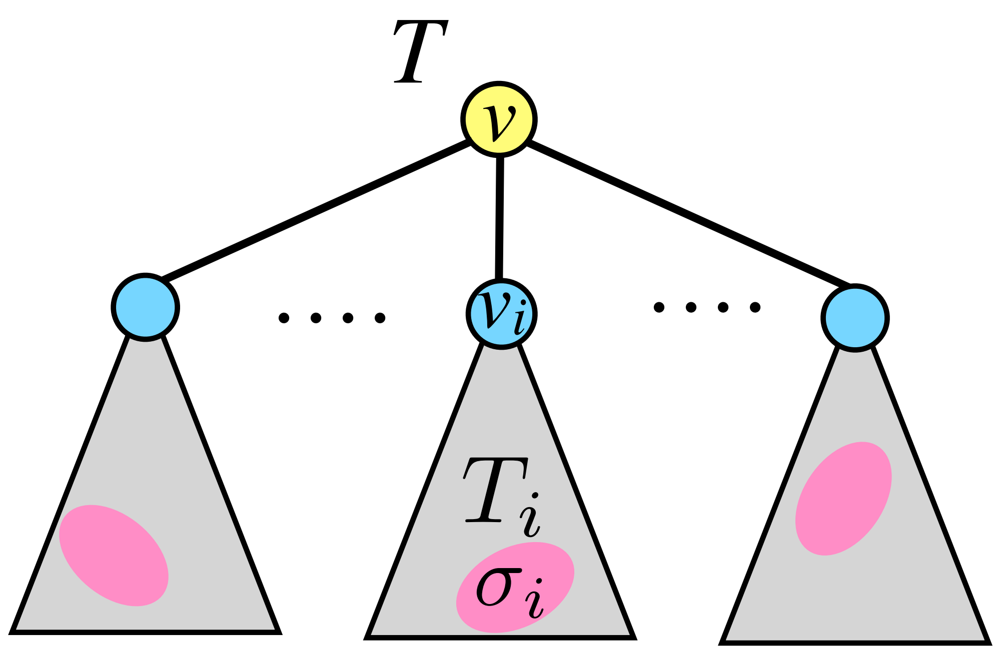
$$= (1 - p_T^\sigma)/p_T^\sigma$$

$$p_T^\sigma = \frac{1}{1 + \lambda \prod_{i=1}^{d} p_{T_i}^{\sigma_i}}$$

$$R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{R_{T_i}^{\sigma_i} + 1}$$

Tree Recurrence

2-spin system: $A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$ $b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ $\mu_T(\sigma) \propto \prod_{uv \in E} a_{\sigma(u),\sigma(v)} \prod_{v \in V} b_{\sigma(v)}$



Occupancy ratio:

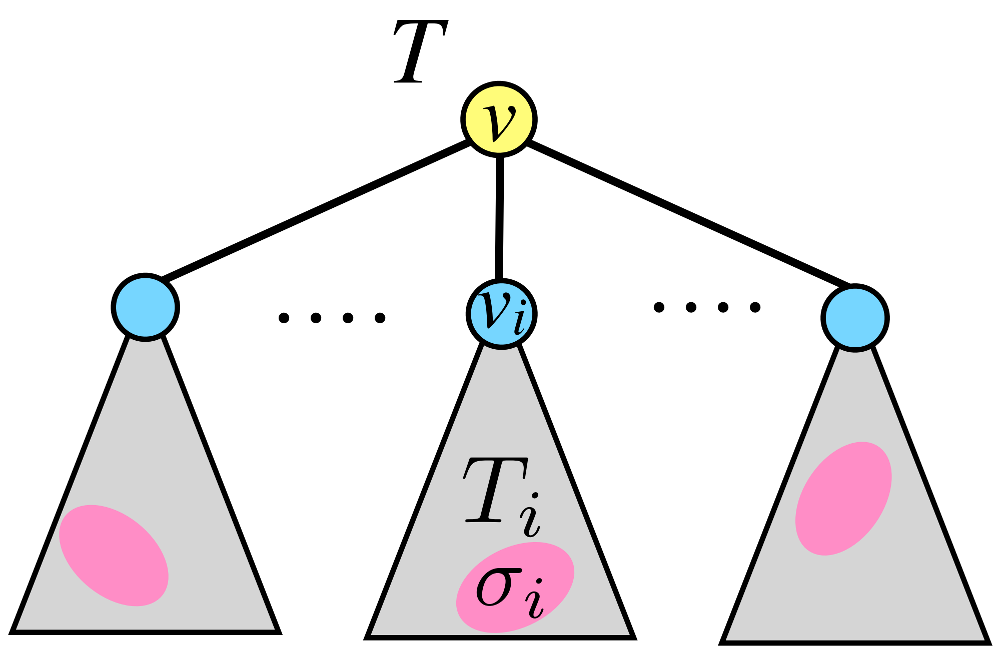
$$R_T^\sigma \triangleq \frac{\Pr_{X\sim\mu_T}[X_v = 0 \mid \sigma]}{\Pr_{X\sim\mu_T}[X_v = 1 \mid \sigma]}$$

$$R_T^\sigma = \frac{b_0}{b_1} \prod_{i=1}^{d} \frac{a_{00} R_{T_i}^{\sigma_i} + a_{01}}{a_{10} R_{T_i}^{\sigma_i} + a_{11}}$$

a Möbius transformation

Tree Recurrence

hardcore model: $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$

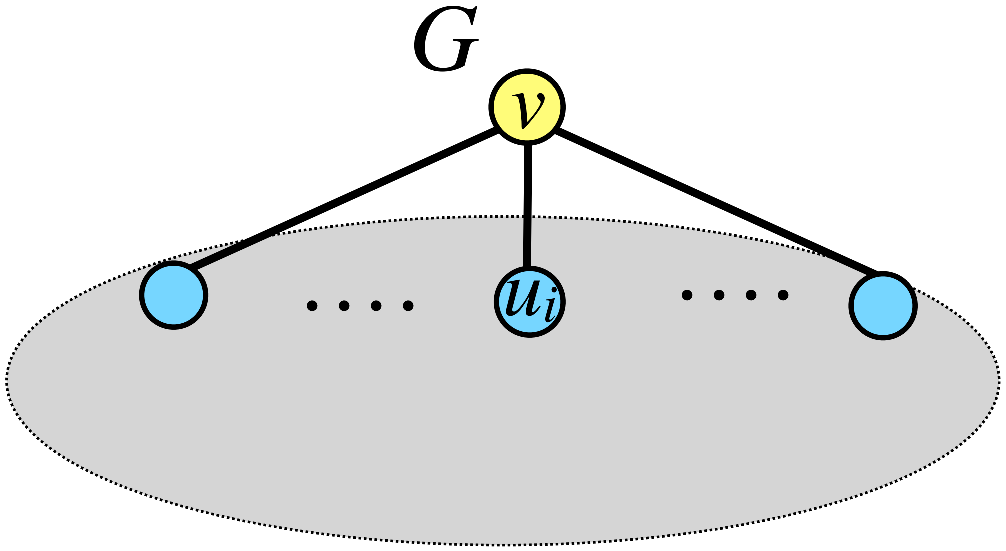


Occupancy ratio:

$$R_T^\sigma \triangleq \frac{\Pr_{X\sim\mu_T}[X_v = 0 \mid \sigma]}{\Pr_{X\sim\mu_T}[X_v = 1 \mid \sigma]}$$

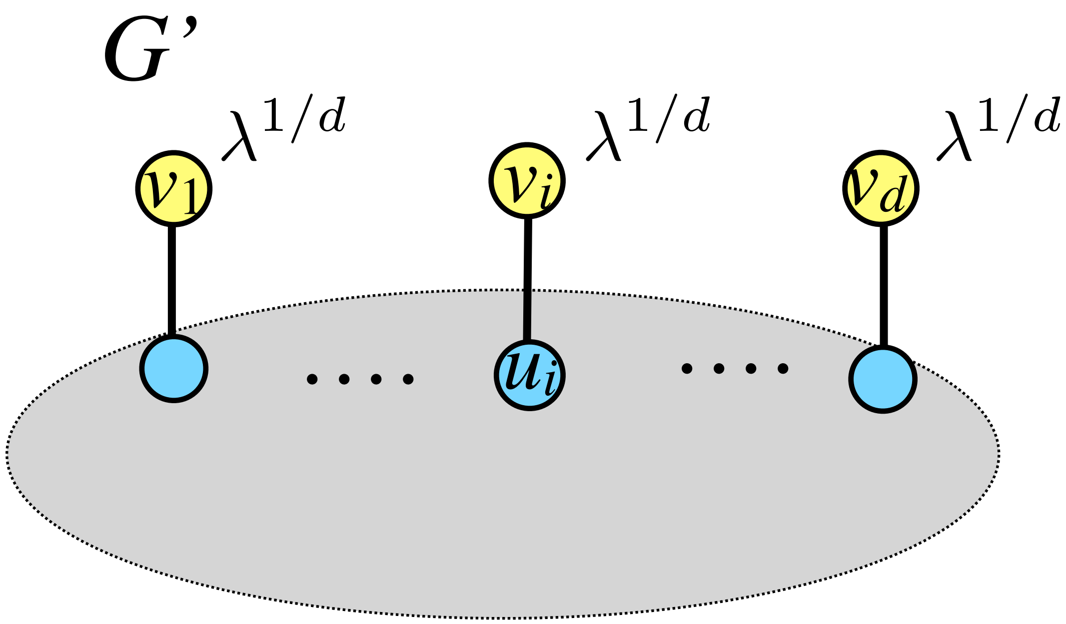
$$R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{R_{T_i}^{\sigma_i} + 1}$$

$$R_v = \frac{\Pr_{I\sim\mu_G}[v \text{ is occupied by } I]}{\Pr_{I\sim\mu_G}[v \text{ is unoccupied by } I]} = \frac{Z_G(v \text{ is occupied})}{Z_G(v \text{ is unoccupied})}$$

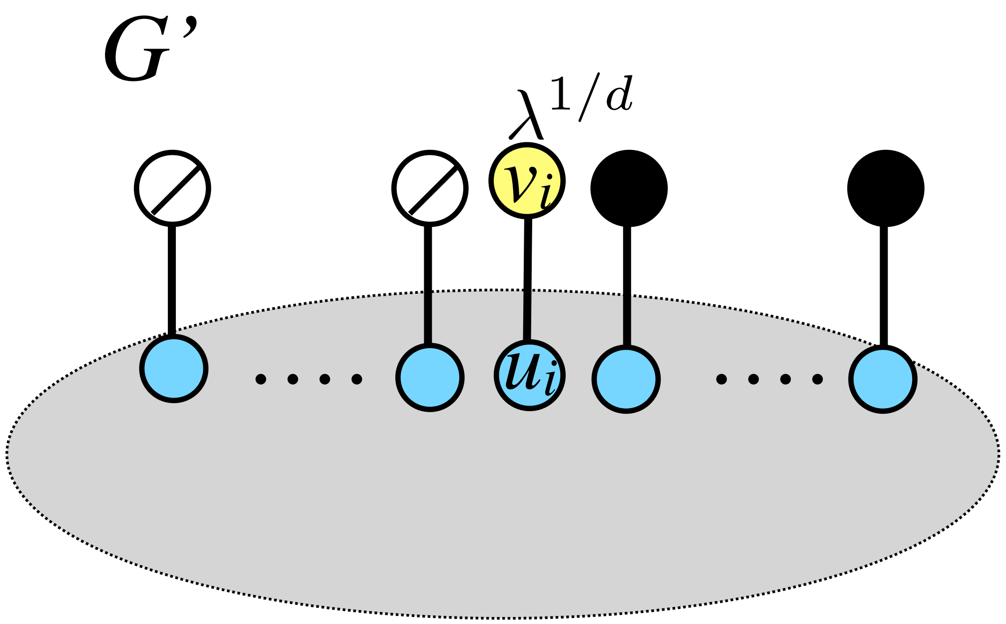


$$R_v = \frac{\Pr_{I\sim\mu_G}[v \text{ is occupied by } I]}{\Pr_{I\sim\mu_G}[v \text{ is unoccupied by } I]} = \frac{Z_G(v \text{ is occupied})}{Z_G(v \text{ is unoccupied})}$$

$$= \frac{Z_{G'}(v_1,\ldots,v_d = \bullet\cdots\bullet)}{Z_{G'}(v_1,\ldots,v_d = \circ\cdots\circ)}$$



$$R_v = \frac{\Pr_{I\sim\mu_G}[v \text{ is occupied by } I]}{\Pr_{I\sim\mu_G}[v \text{ is unoccupied by } I]} = \frac{Z_G(v \text{ is occupied})}{Z_G(v \text{ is unoccupied})}$$



$$= \frac{Z_{G'}(v_1,\ldots,v_d = \bullet\cdots\bullet)}{Z_{G'}(v_1,\ldots,v_d = \circ\cdots\circ)}$$

$$= \prod_{i=1}^{d} \frac{Z_{G'}(v_1,\ldots,v_d = \overbrace{\circ\cdots\circ}^{i-1}\,\bullet\,\overbrace{\bullet\cdots\bullet}^{d-i})}{Z_{G'}(v_1,\ldots,v_d = \underbrace{\circ\cdots\circ}_{i-1}\,\circ\,\underbrace{\bullet\cdots\bullet}_{d-i})}$$

$$= \prod_{i=1}^{d} R^{\tau_i}_{G',v_i}$$

$\tau_i:\quad v_1,\ldots,v_{i-1} = \circ\cdots\circ$

$v_{i+1},\ldots,v_d = \bullet\cdots\bullet$

$\oslash$: unoccupied

$\bullet$: occupied

$$R_v = \frac{\Pr_{I\sim\mu_G}[v \text{ is occupied by } I]}{\Pr_{I\sim\mu_G}[v \text{ is unoccupied by } I]} = \frac{Z_G(v \text{ is occupied})}{Z_G(v \text{ is unoccupied})}$$

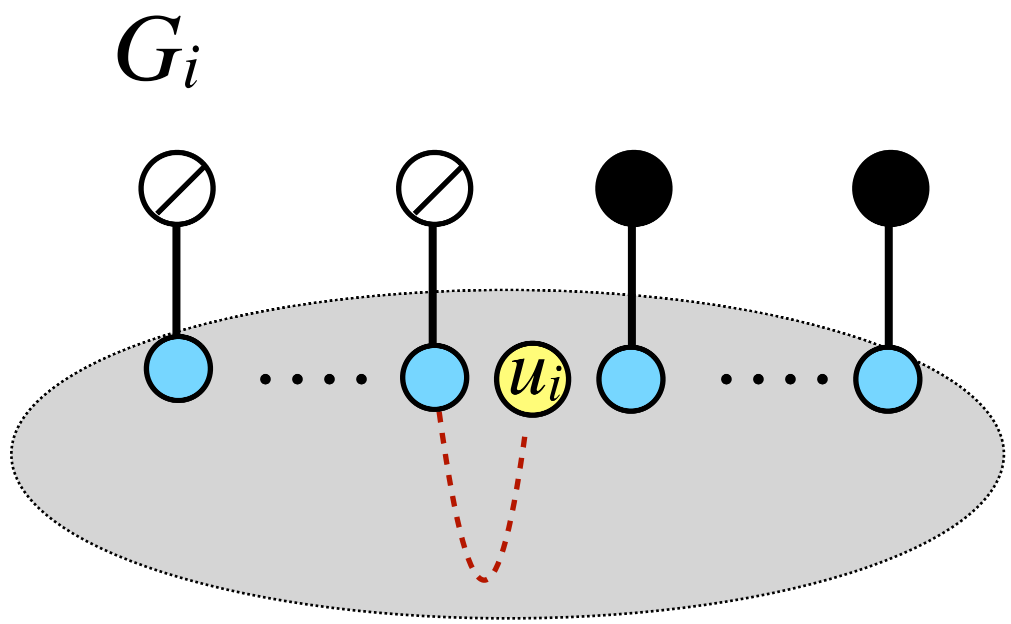
$$= \frac{Z_{G'}(v_1,\ldots,v_d = \bullet\cdots\bullet)}{Z_{G'}(v_1,\ldots,v_d = \circ\cdots\circ)}$$

$$= \prod_{i=1}^{d} \frac{Z_{G'}(v_1,\ldots,v_d = \overbrace{\circ\cdots\circ}^{i-1}\,\bullet\,\overbrace{\bullet\cdots\bullet}^{d-i})}{Z_{G'}(v_1,\ldots,v_d = \underbrace{\circ\cdots\circ}_{i-1}\,\circ\,\underbrace{\bullet\cdots\bullet}_{d-i})}$$

$$= \prod_{i=1}^{d} R^{\tau_i}_{G',v_i}$$

τ_i : $v_1,\ldots,v_{i-1} = \circ\cdots\circ$
$v_{i+1},\ldots,v_d = \bullet\cdots\bullet$

$$= \prod_{i=1}^{d} \frac{\lambda^{1/d}}{R^{\tau_i}_{G_i,u_i}+1} = \lambda \prod_{i=1}^{d} \frac{1}{R^{\tau_i}_{G_i,u_i}+1}$$



⊘ : unoccupied

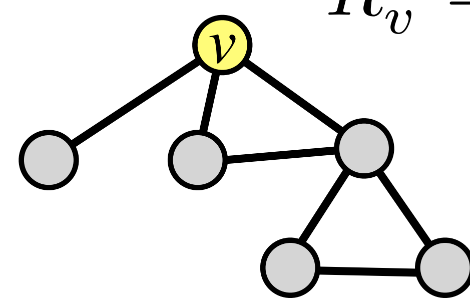
● : occupied

Tree Recurrence

hardcore model:

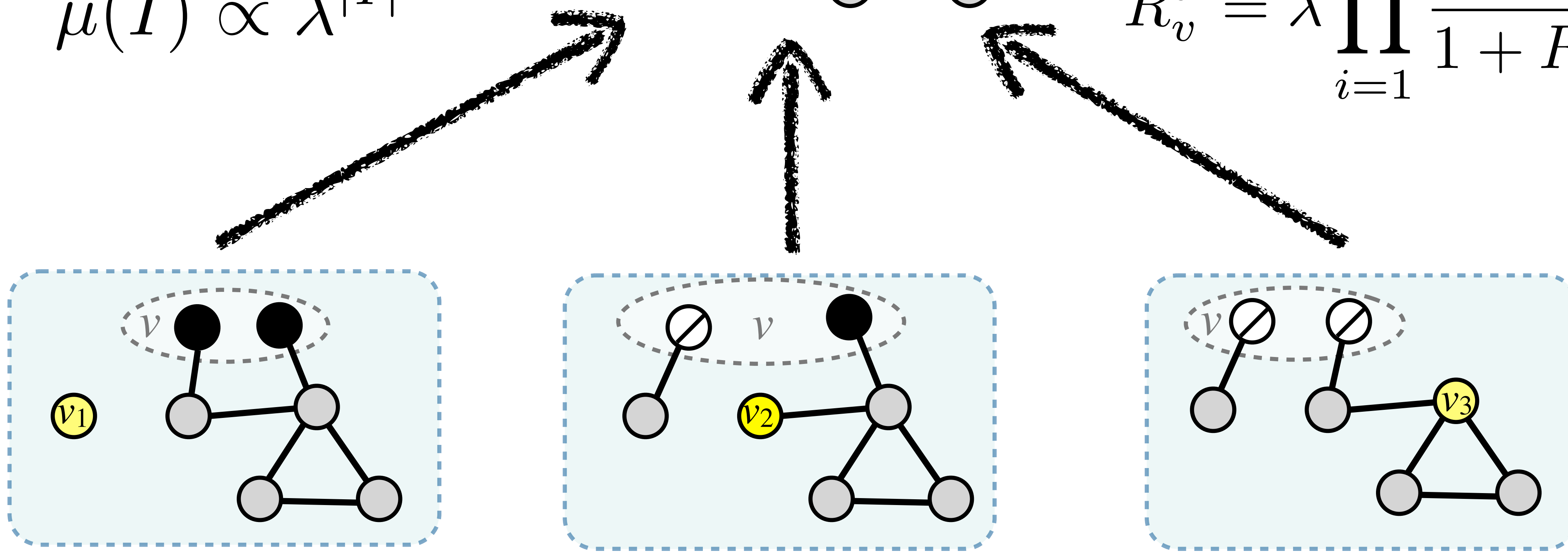
independent set I in G

$$\mu(I) \propto \lambda^{|I|}$$



$$R_v^{\sigma} = \frac{\Pr[v \text{ is occupied } \mid \sigma]}{\Pr[v \text{ is unoccupied } \mid \sigma]}$$

$$R_v^{\sigma} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_i^{\sigma_i}}$$



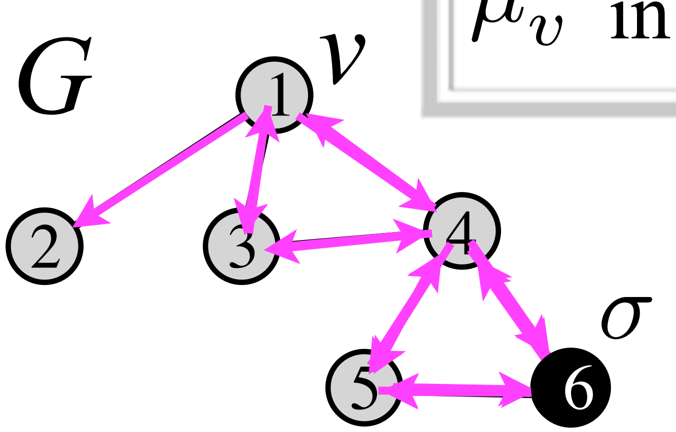
$$R_1^{\sigma_1} = \frac{\Pr[v_1 \text{ is occupied } \mid \sigma_1]}{\Pr[v_1 \text{ is unoccupied } \mid \sigma_1]} \quad R_2^{\sigma_2} = \frac{\Pr[v_2 \text{ is occupied } \mid \sigma_2]}{\Pr[v_2 \text{ is unoccupied } \mid \sigma_2]} \quad R_3^{\sigma_3} = \frac{\Pr[v_3 \text{ is occupied } \mid \sigma_3]}{\Pr[v_3 \text{ is unoccupied } \mid \sigma_3]}$$

● : occupied ⊘ : unoccupied

Self-Avoiding Walk Tree

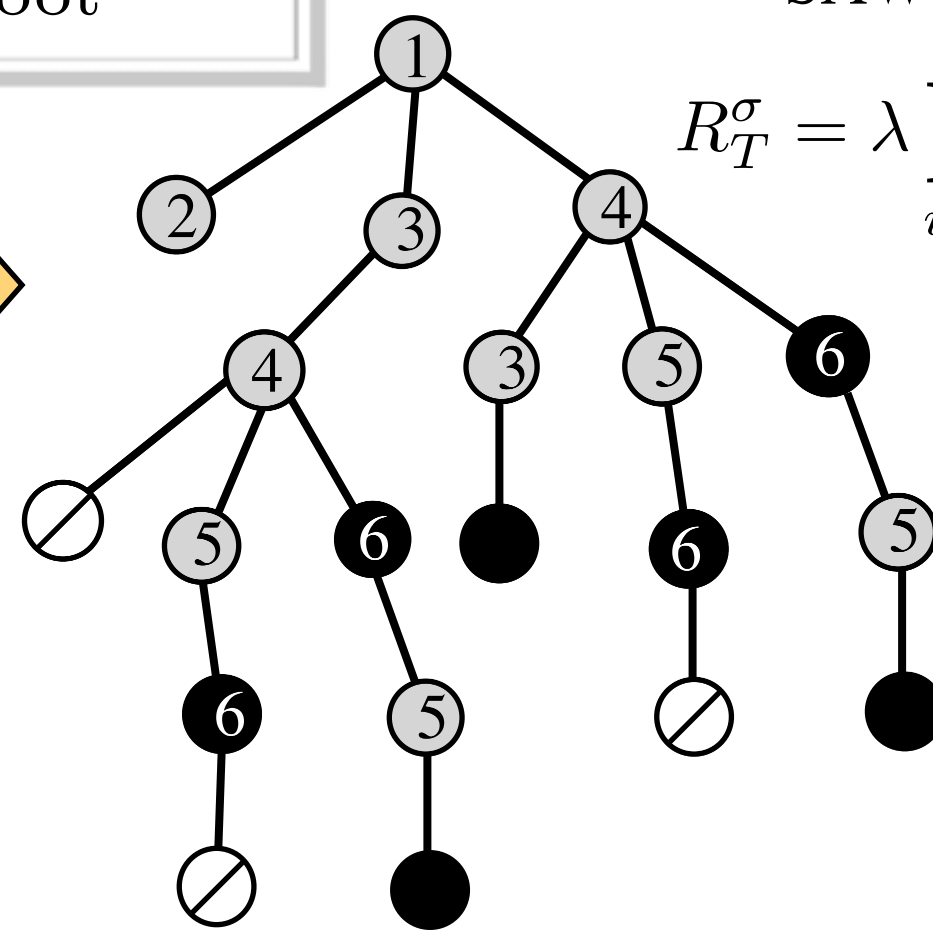
(Godsil 1981; Weitz 2006)

μ_v^σ in G = μ_{root}^σ in T



$T = T_{\text{SAW}}(G, v)$

$$R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{T_i}^{\sigma_i}}$$



SSM in trees of max-deg ≤Δ:

SSM and approx. inference in graphs with max-deg ≤Δ

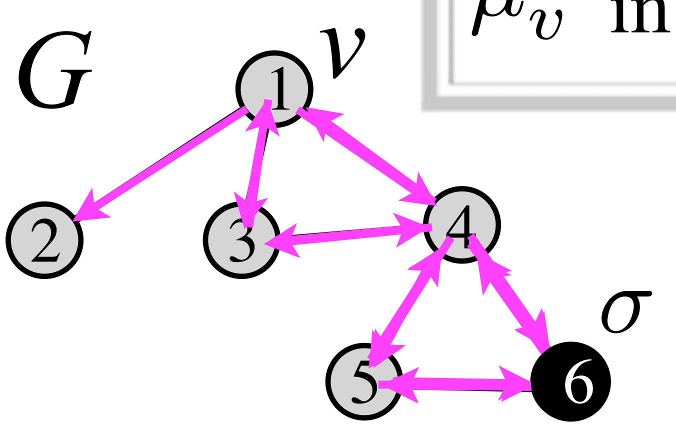
⊘ if cycle closing edge > cycle starting edge

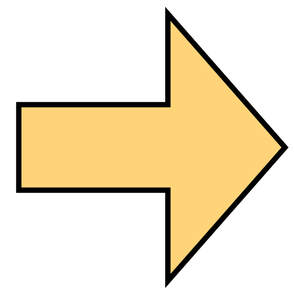
● if cycle closing edge < cycle starting edge

Self-Avoiding Walk Tree

(Godsil 1981; Weitz 2006)

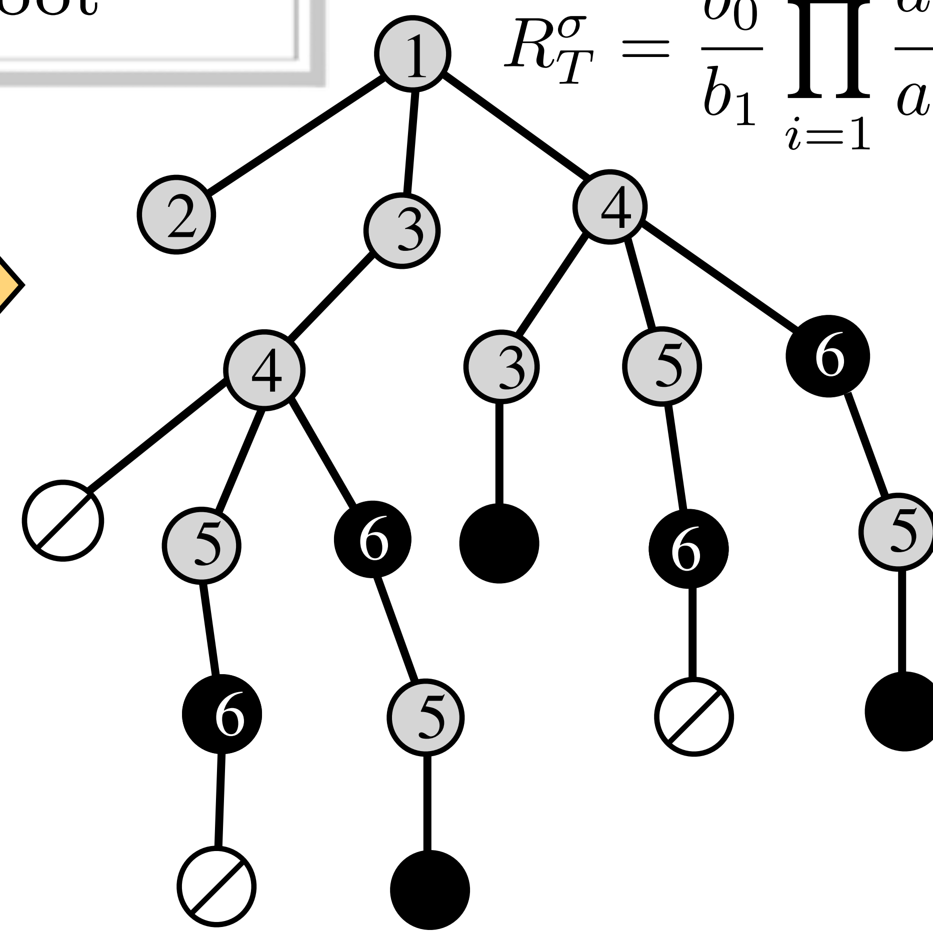
$$\mu_v^\sigma \text{ in } G = \mu_{\text{root}}^\sigma \text{ in } T$$



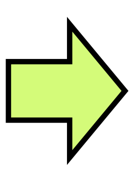


$$T = T_{\text{SAW}}(G, v)$$

$$R_T^\sigma = \frac{b_0}{b_1} \prod_{i=1}^{d} \frac{a_{00} R_{T_i}^{\sigma_i} + a_{01}}{a_{10} R_{T_i}^{\sigma_i} + a_{11}}$$



SSM in trees of max-deg ≤Δ:



SSM and approx. inference in graphs with max-deg ≤Δ

hold for 2-spin systems

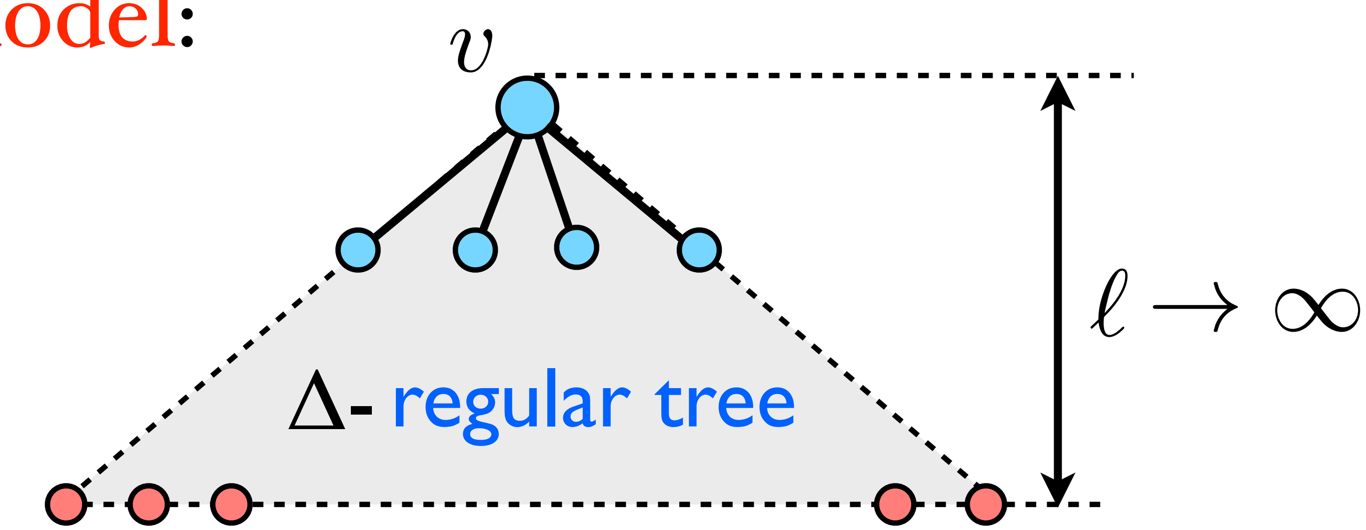
$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} \quad b = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

⊘ if cycle closing edge > cycle starting edge

● if cycle closing edge < cycle starting edge

Correlation Decay

hardcore model:



σ: all leaves are *occupied*

τ: all leaves are *unoccupied*

uniqueness threshold: $$\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$$

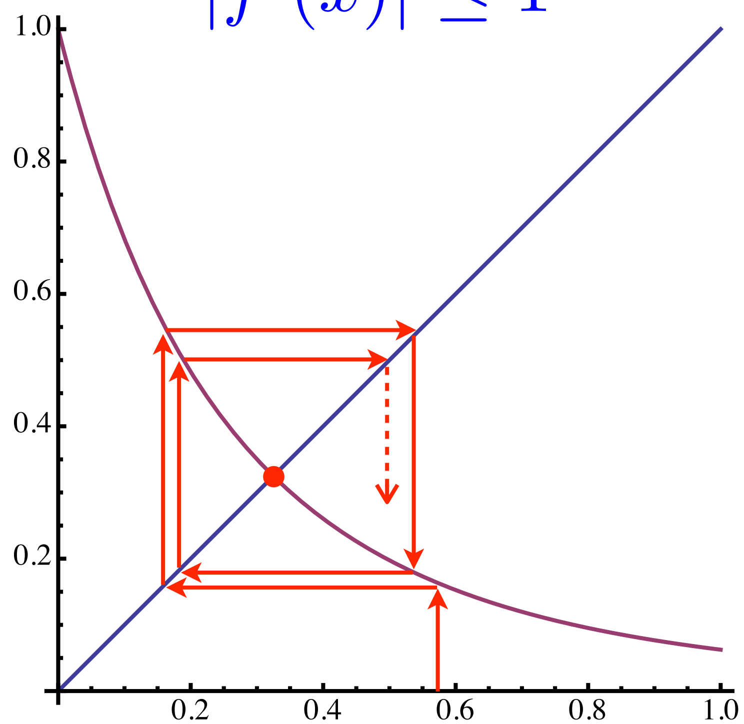
hardcore recurrence: $$R_T^{\sigma} = \lambda \prod_{i=1}^{\Delta-1} \frac{1}{1+R_{T_i}^{\sigma_i}}$$

single-variate dynamical system: $$f(x) = \frac{\lambda}{(1+x)^{\Delta-1}}$$

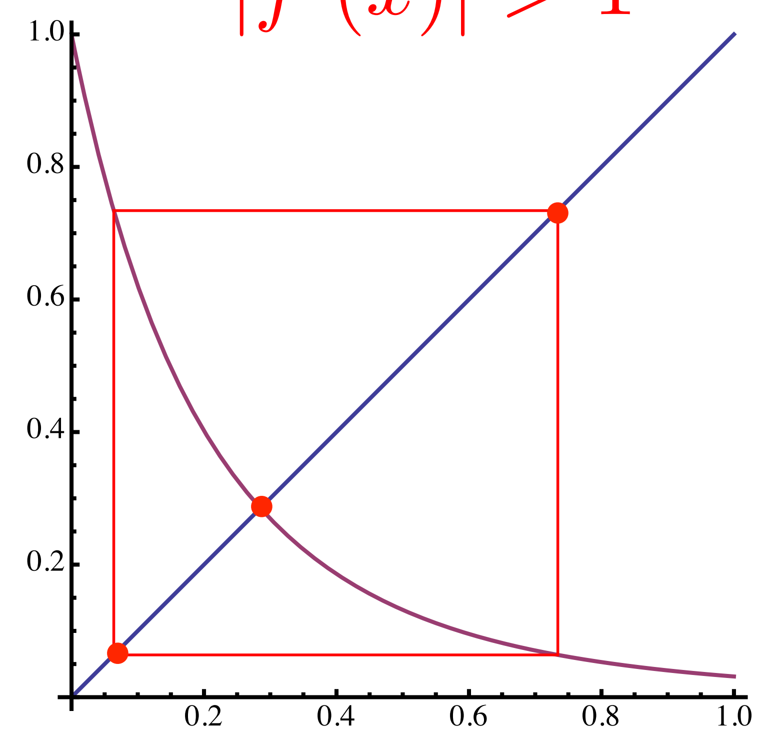
single-variate dynamical system: $f(x) = \dfrac{\lambda}{(1+x)^{\Delta-1}}$

unique fixed point: $\hat{x} = f(\hat{x})$

$|f'(\hat{x})| \le 1$



$|f'(\hat{x})| > 1$



$\lambda \le \lambda_c(\Delta)$

⇔ $|f'(\hat{x})| \le 1$

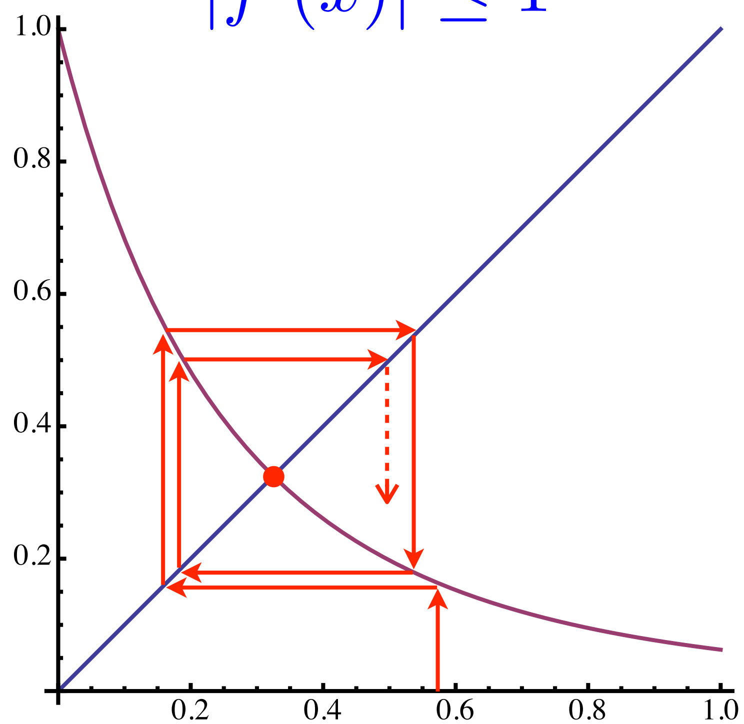
$\exists x^- < \hat{x} < x^+$

$$\begin{cases} x^+ = f(x^-) \\ x^- = f(x^+) \end{cases}$$

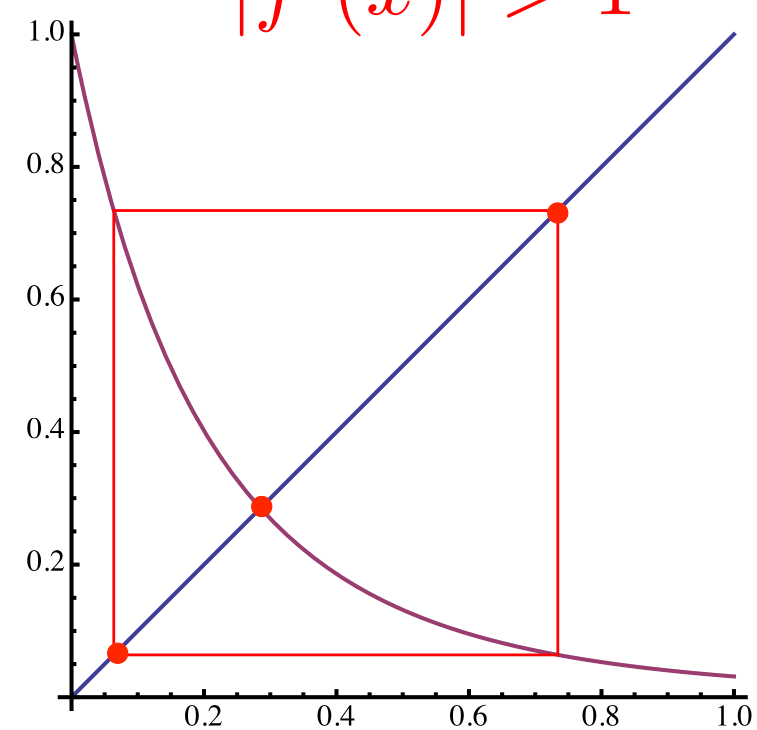
single-variate dynamical system: $f(x) = \frac{\lambda}{(1+x)^{\Delta-1}}$

unique fixed point: $\hat{x} = f(\hat{x})$

$|f'(\hat{x})| \leq 1$



$|f'(\hat{x})| > 1$



$$\lambda \leq (1-\delta)\lambda_c(\Delta)$$

$$\Rightarrow \quad \boxed{|f'(\hat{x})| \leq 1 - \Theta(\delta)}$$

$$\exists x^- < \hat{x} < x^+$$

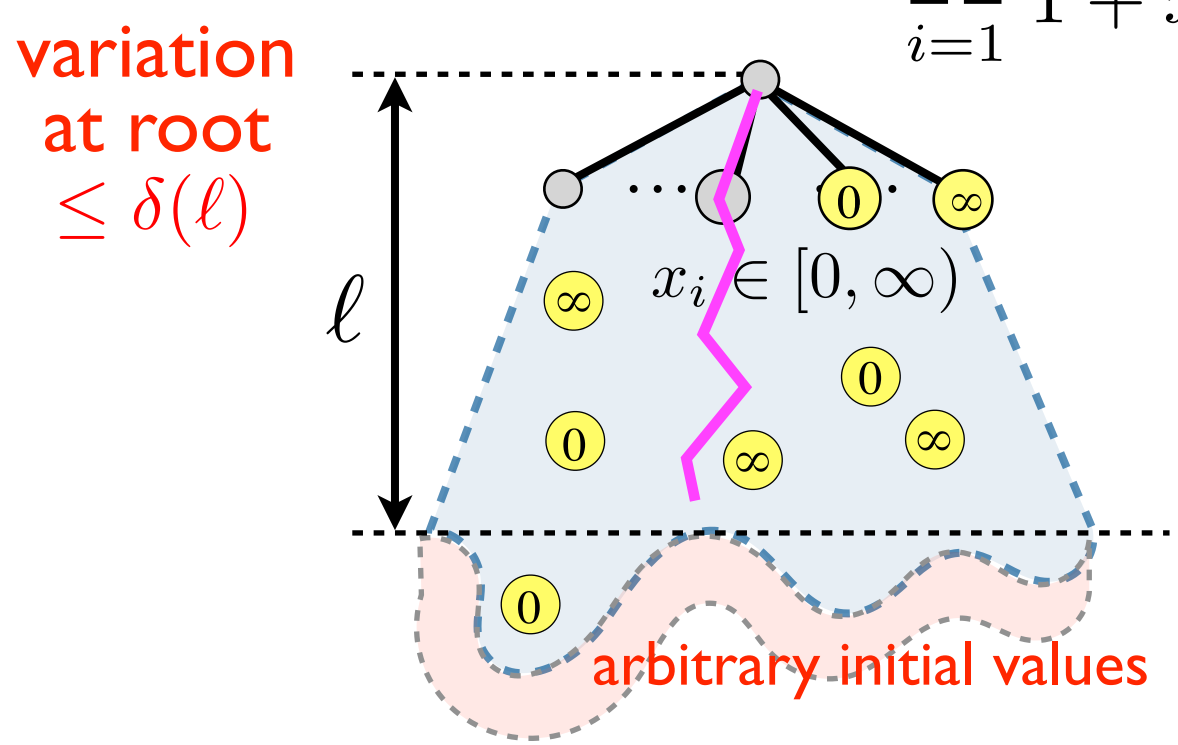
$$\begin{cases} x^+ = f(x^-) \\ x^- = f(x^+) \end{cases}$$

Correlation Decay

hardcore recurrence: $R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1+R_{T_i}^{\sigma_i}}$ $\quad R_{T_i}^{\sigma_i} \in [0,\infty)$

(marginal ratio)

dynamical system: $f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$ where $d \le \Delta - 1$



$$\alpha \triangleq \sup_{\vec{x}\in[0,\infty)^d} \sum_{i=1}^{d} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right|$$

$$= \sup_{\vec{x}\in[0,\infty)^d} \lambda \prod_{i=1}^{d} \frac{1}{1+x_i} \sum_{i=1}^{d} \frac{1}{1+x_i}$$

$$\le \lambda \cdot (\Delta - 1)$$

Mean-Value Thm: if $\lambda < \frac{1}{\Delta-1}$

$$|R_T^\sigma - R_T^\tau| = |f(\vec{x}) - f(\vec{x}')| = |\langle \nabla f(\vec{\xi}), (\vec{x} - \vec{x}') \rangle|$$

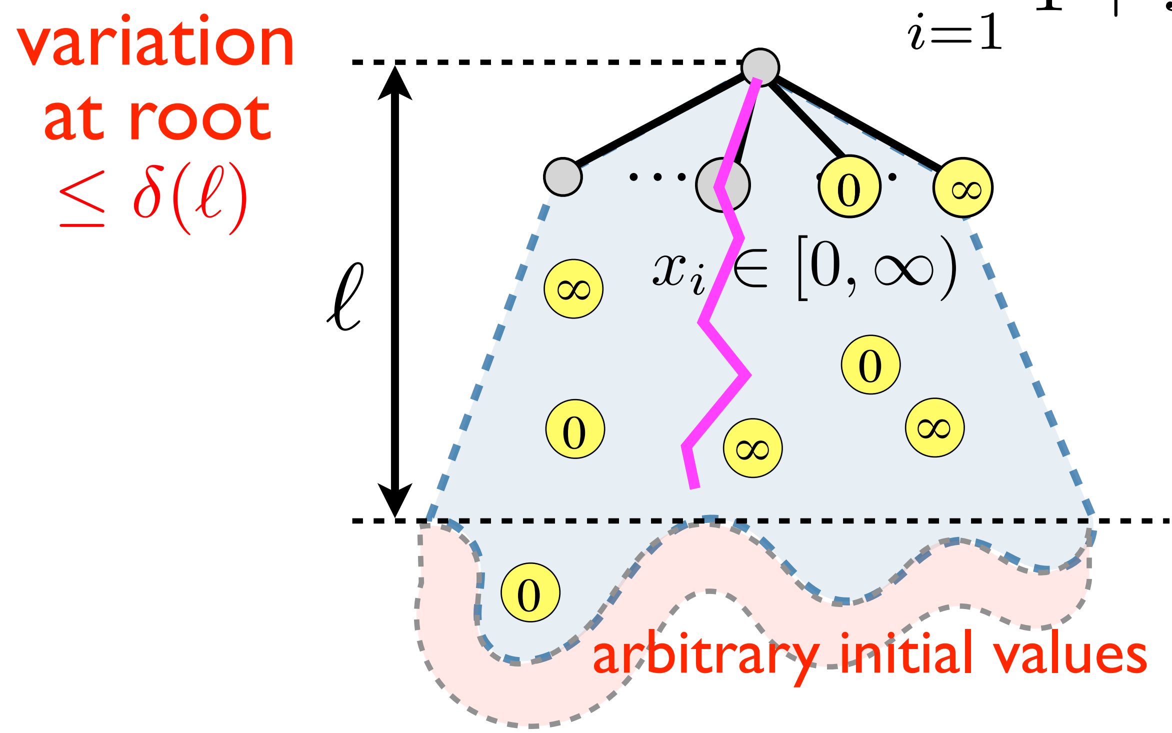
$$\le \|\nabla f(\vec{\xi})\|_1 \|\vec{x} - \vec{x}'\|_\infty \le \alpha \cdot \max_i |x_i - x_i'| = \exp(-\Omega(\ell))$$

Correlation Decay

hardcore recurrence: $R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1+R_{T_i}^{\sigma_i}}$ $\qquad R_{T_i}^{\sigma_i} \in [0,\infty)$

(marginal ratio)

dynamical system: $f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$ where $d \le \Delta - 1$



$$\alpha \triangleq \sup_{\vec{x}\in[0,\infty)^d} \sum_{i=1}^{d} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right|$$
$$= \sup_{\vec{x}\in[0,\infty)^d} \lambda \prod_{i=1}^{d} \frac{1}{1+x_i} \sum_{i=1}^{d} \frac{1}{1+x_i}$$
$$\le \lambda \cdot (\Delta - 1)$$

if $\lambda < \frac{1}{\Delta-1}$: $\left|p_T^\sigma - p_T^\tau\right| \le \left|R_T^\sigma - R_T^\tau\right| = \exp(-\Omega(\ell))$

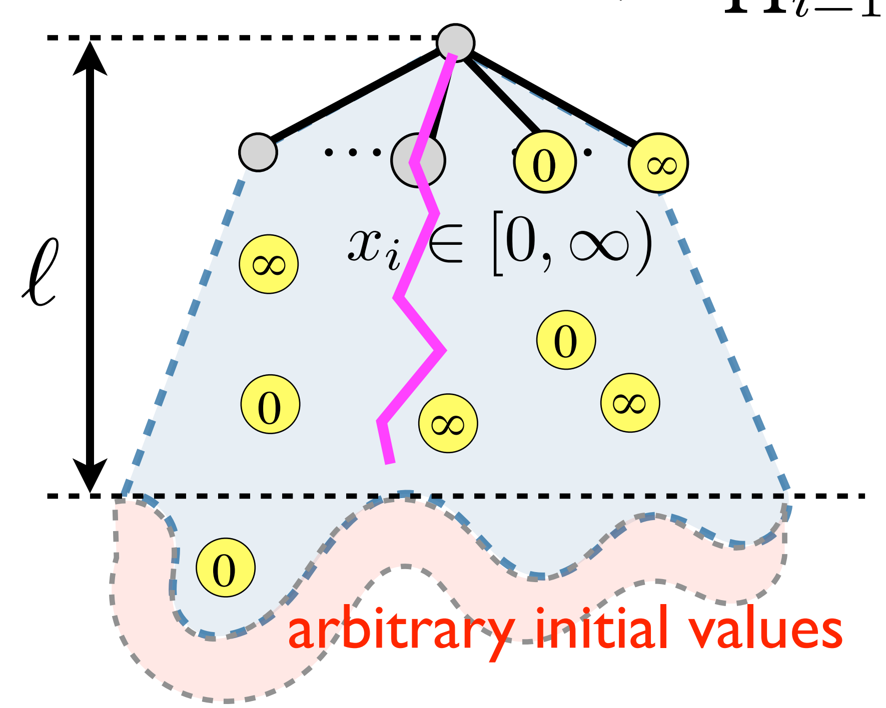
$R = \frac{1-p}{p}$ SSM with exponential decay!

Correlation Decay

hardcore recurrence: $p_T^\sigma = \dfrac{1}{1+\lambda\prod_{i=1}^d p_{T_i}^{\sigma_i}}$ $\qquad p_{T_i}^{\sigma_i} \in [0,1]$

(marginal probability)

dynamical system: $f(\vec{x}) = \dfrac{1}{1+\lambda\prod_{i=1}^d x_i}$ where $d \le \Delta - 1$



$$\alpha \triangleq \sup_{\vec{x}\in[0,1]^d} \sum_{i=1}^d \left|\frac{\partial f(\vec{x})}{\partial x_i}\right|$$

$$= \sup_{\vec{x}\in[0,1]^d} f(\vec{x})(1-f(\vec{x}))\sum_{i=1}^d \frac{1}{x_i}$$

unbounded!

Mean-Value Thm: $|f(\vec{x}) - f(\vec{x}')| = |\langle \nabla f(\vec{\xi}), (\vec{x}-\vec{x}')\rangle|$

$$\le \|\nabla f(\vec{\xi})\|_1 \|\vec{x}-\vec{x}'\|_\infty \le \alpha \cdot \max_i |x_i - x_i'|$$

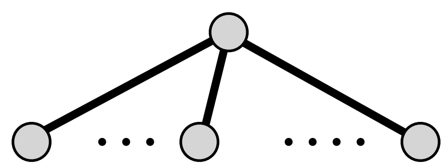
when $p \mapsto R = \frac{1-p}{p}$: the decay rate α is reduced

The Potential Method

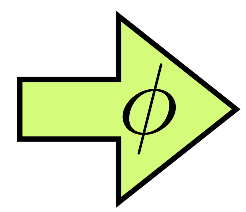
[Restrepo-Shin-Tetali-Vigoda-Yang 11]

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$



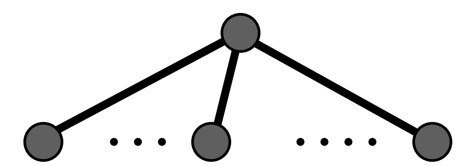
x_i



$y_i = \phi(x_i)$

potential (**message**):

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \dots, \phi^{-1}(y_d)))$$



y_i

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| = \left|\langle \nabla f^{\phi}(\vec{\xi}), (\vec{y} - \vec{y}') \rangle\right| \leq \|\nabla f^{\phi}(\vec{\xi})\|_1 \|\vec{y} - \vec{y}'\|_\infty$$

$$\|\nabla f^{\phi}(\vec{\xi})\|_1 = \sum_{i=1}^{d} \left| \frac{\partial f(\vec{x})}{\partial x_i} \right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)}$$

(where $\xi_i = \phi(x_i)$, and denote $\Phi(x) = \phi'(x)$)

Choose:

$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| = \left|\langle \nabla f^{\phi}(\vec{\xi}), (\vec{y} - \vec{y}')\rangle\right| \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty}$$

$$\alpha = \sum_{i=1}^{d} \left|\frac{\partial f(\vec{x})}{\partial x_i}\right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)}$$

(where $\xi_i = \phi(x_i)$, and denote $\Phi(x) = \phi'(x)$)

$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

(Jensen)

$$= \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \leq \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}}$$

(where $f(x) = \frac{\lambda}{(1+x)^d}$)

let $z_i = -\ln(x_1 + 1)$

then $f(\vec{x}) = \lambda \exp\left(\sum_{i=1}^{d} z_i\right)$

and $\sqrt{\frac{x_i}{x_i+1}} = \sqrt{e^{z_i}(e^{-z_i} - 1)}$

is concave in z_i

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| = \left|\langle \nabla f^{\phi}(\vec{\xi}), (\vec{y} - \vec{y}')\rangle\right| \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty}$$

$$\alpha = \sum_{i=1}^{d} \left|\frac{\partial f(\vec{x})}{\partial x_i}\right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)}$$

(where $\xi_i = \phi(x_i)$, and denote $\Phi(x) = \phi'(x)$)

$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

(Jensen)

$$= \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \leq \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}} \leq \sqrt{|f'(\hat{x})|} = \sqrt{\frac{d\hat{x}}{1+\hat{x}}}$$

(where $f(x) = \frac{\lambda}{(1+x)^d}$) (where $\hat{x} = f(\hat{x})$ is fixpoint) original

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{}{1}$$

Mean Value Theor

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}$$

$$\alpha = \sum_{i=1}^{d} \left| \frac{\partial f(\vec{x}}{\partial x_i} \right.$$

(where $\xi_i = \phi(x_i)$,

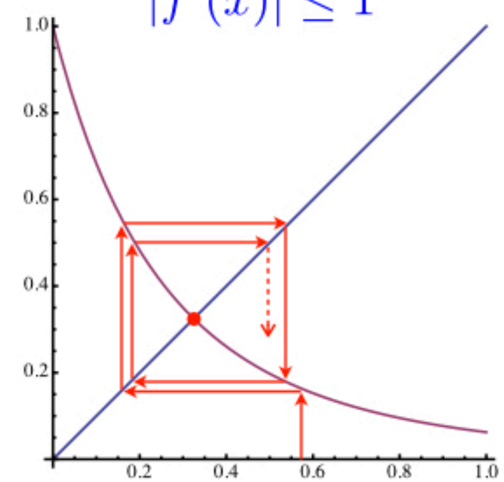
$\phi(x) =$

so t

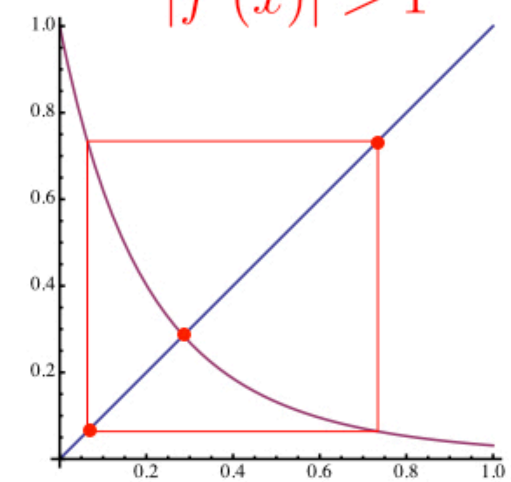
single-variate dynamical system: $f(x) = \frac{\lambda}{(1+x)^{\Delta-1}}$

unique fixed point: $\hat{x} = f(\hat{x})$

$|f'(\hat{x})| \le 1$



$|f'(\hat{x})| > 1$



$\lambda \le (1-\delta)\lambda_c(\Delta)$

$\Rightarrow$ $|f'(\hat{x})| \le 1 - \Theta(\delta)$

$\exists x^- < \hat{x} < x^+$

$$\begin{cases} x^+ = f(x^-) \\ x^- = f(x^+) \end{cases}$$

(Jensen)

$$= \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \le \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}} \le \sqrt{|f'(\hat{x})|} = \sqrt{\frac{d\hat{x}}{1+\hat{x}}}$$

(where $f(x) = \frac{\lambda}{(1+x)^d}$) (where $\hat{x} = f(\hat{x})$ is fixpoint)

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| = \left|\langle \nabla f^{\phi}(\vec{\xi}), (\vec{y} - \vec{y}')\rangle\right| \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty}$$

$$\alpha = \sum_{i=1}^{d} \left|\frac{\partial f(\vec{x})}{\partial x_i}\right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)}$$

(where $\xi_i = \phi(x_i)$, and denote $\Phi(x) = \phi'(x)$)

$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

(Jensen) (when $\lambda < \lambda_c$)

$$= \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \leq \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}} \leq \sqrt{|f'(\hat{x})|} < 1$$

(where $f(x) = \frac{\lambda}{(1+x)^d}$) (where $\hat{x} = f(\hat{x})$ is fixpoint)

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

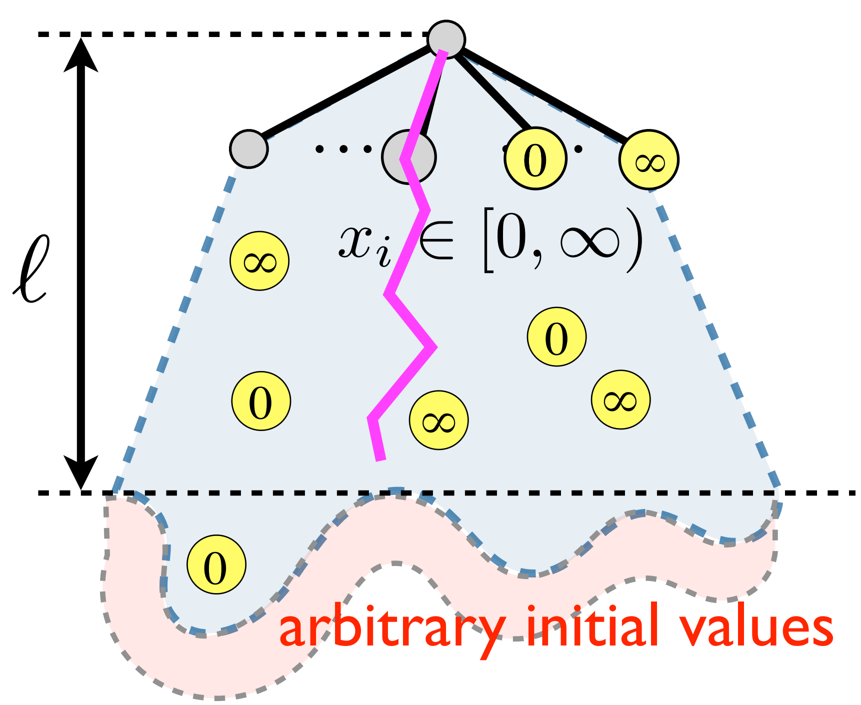
$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| \quad \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty} \quad \leq \alpha^{\ell-1}|\phi(\lambda) - \phi(0)|$$

when $\lambda < \lambda_c$: $\leq \exp(-\Omega(\ell)) \cdot |\phi(\lambda) - \phi(0)|$

$$|p_T^{\sigma} - p_T^{\tau}| \leq |R_T^{\sigma} - R_T^{\tau}| \quad \leq |\phi(R_T^{\sigma}) - \phi(R_T^{\tau})| \cdot \sup_{x \in [0,\lambda]} \frac{1}{\Phi(x)}$$

$$\leq \exp(-\Omega(\ell)) \cdot |\phi(\lambda) - \phi(0)| \cdot \sup_{x \in [0,\lambda]} \frac{1}{\Phi(x)}$$

$$= \exp(-\Omega(\ell))$$



SSM with exponential decay!

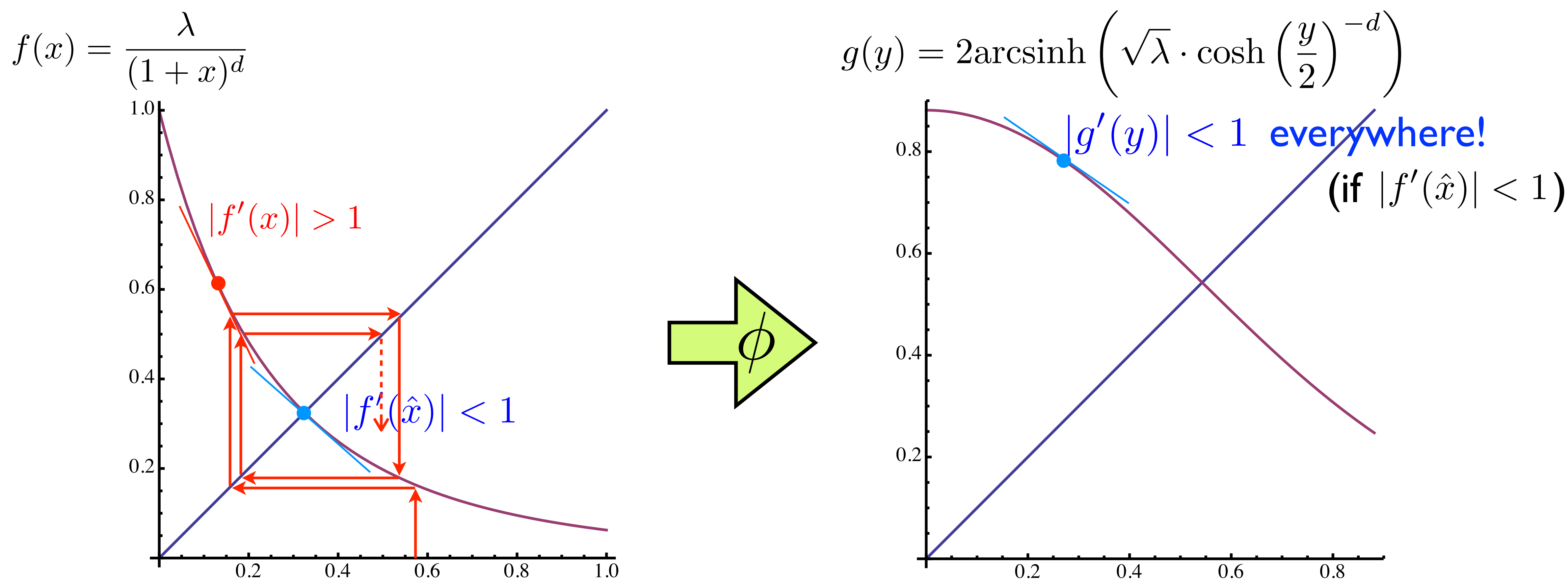
$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

$$f(x) = \frac{\lambda}{(1+x)^d} \xrightarrow{\ \phi\ } g(y) = \phi(f(\phi^{-1}(y)))$$

$$x \xrightarrow{\ \phi(x) = 2\sinh^{-1}(\sqrt{x})\ } y$$

Always contract!



$$|g'(y)| = |f'(x)| \frac{\phi'(f(x))}{\phi'(x)} = |f'(\hat{x})| = 1 \quad \text{(when } \lambda = \lambda_c = \tfrac{d^d}{(d-1)^{d+1}}\text{)}$$

(at the fixpoint $\hat{x} = f(\hat{x})$)

Good Potential Functions

monotone $\phi(x)$ denote $\Phi(x) = \phi'(x)$

①: consider $f(x) = \dfrac{\lambda}{(1+x)^d}$, when $\lambda = \lambda_c = \frac{d^d}{(d-1)^{d+1}}$:

$|f'(x)|\dfrac{\phi'(f(x))}{\phi'(x)} = \dfrac{df(x)\Phi(f(x))}{(x+1)\Phi(x)}$ is maximized at fixpoint $\hat{x} = f(\hat{x})$

Go ... s

①: cons

$|f'(x$ ... $= f(\hat{x})$

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| = \left|\langle \nabla f^{\phi}(\vec{\xi}), (\vec{y} - \vec{y}')\rangle\right| \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty}$$

$$\alpha = \sum_{i=1}^{d} \left|\frac{\partial f(\vec{x})}{\partial x_i}\right| \frac{\Phi(f(\vec{x}))}{\Phi(x_i)} \leq f(\vec{x})\Phi(f(\vec{x})) \sum_{i=1}^{d} \frac{1}{(1+x_i)\Phi(x_i)}$$

(where $\xi_i = \phi(x_i)$, and denote $\Phi(x) = \phi'(x)$)

$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

(Jensen)

$$= \sqrt{\frac{f(\vec{x})}{1+f(\vec{x})}} \sum_{i=1}^{d} \sqrt{\frac{x_i}{1+x_i}} \leq \sqrt{\frac{df(x)}{1+f(x)}} \sqrt{\frac{dx}{1+x}}$$

(where $f(x) = \frac{\lambda}{(1+x)^d}$)

let $z_i = -\ln(x_1 + 1)$

then $f(\vec{x}) = \lambda \exp\left(\sum_{i=1}^{d} z_i\right)$

and $\sqrt{\frac{x_i}{x_i+1}} = \sqrt{e^{z_i}(e^{-z_i} - 1)}$ is concave in z_i

Good Potential Functions

monotone $\phi(x)$ denote $\Phi(x) = \phi'(x)$

①: consider $f(x) = \dfrac{\lambda}{(1+x)^d}$, when $\lambda = \lambda_c = \frac{d^d}{(d-1)^{d+1}}$:

$|f'(x)|\dfrac{\phi'(f(x))}{\phi'(x)} = \dfrac{df(x)\Phi(f(x))}{(x+1)\Phi(x)}$ is maximized at fixpoint $\hat{x} = f(\hat{x})$

②: $h(z) = \dfrac{\mathrm{e}^z}{\Phi(\mathrm{e}^{-z} - 1)}$ is concave

Go… s

①: cons…

$|f'(x$ … $= f(\hat{x})$

②: $h(z) =$ …

original:

$$f(\vec{x}) = \lambda \prod_{i=1}^{d} \frac{1}{1+x_i}$$

potential:

$$f^{\phi}(\vec{y}) = \phi(f(\phi^{-1}(y_1), \ldots, \phi^{-1}(y_d)))$$

Mean Value Theorem:

$$|f^{\phi}(\vec{y}) - f^{\phi}(\vec{y}')| \quad \leq \alpha \cdot \|\vec{y} - \vec{y}'\|_{\infty} \quad \leq \alpha^{\ell-1}|\phi(\lambda) - \phi(0)|$$

when $\lambda < \lambda_c$:

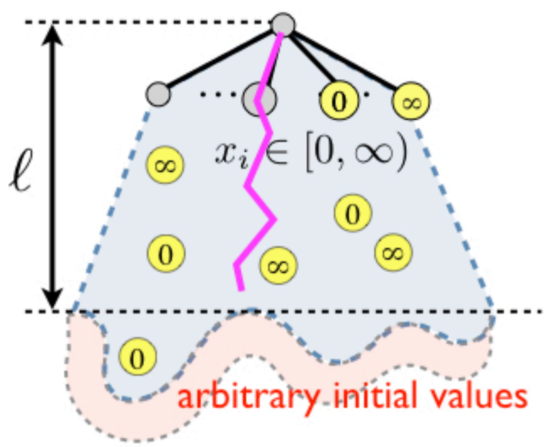
$$\leq \exp(-\Omega(\ell)) \cdot |\phi(\lambda) - \phi(0)|$$

$$|p_T^{\sigma} - p_T^{\tau}| \leq |R_T^{\sigma} - R_T^{\tau}| \quad \leq |\phi(R_T^{\sigma}) - \phi(R_T^{\tau})| \cdot \sup_{x \in [0,\lambda]} \frac{1}{\Phi(x)}$$

$$\leq \exp(-\Omega(\ell)) \cdot |\phi(\lambda) - \phi(0)| \cdot \sup_{x \in [0,\lambda]} \frac{1}{\Phi(x)}$$

$$= \exp(-\Omega(\ell))$$

SSM with exponential decay!



$$\phi(x) = 2\sinh^{-1}(\sqrt{x}) = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right)$$

so that $\Phi(x) = \phi'(x) = \frac{1}{\sqrt{x(x+1)}}$

Good Potential Functions

monotone $\phi(x)$ denote $\Phi(x) = \phi'(x)$

①: consider $f(x) = \dfrac{\lambda}{(1+x)^d}$, when $\lambda = \lambda_c = \frac{d^d}{(d-1)^{d+1}}$:

$$|f'(x)|\frac{\phi'(f(x))}{\phi'(x)} = \frac{df(x)\Phi(f(x))}{(x+1)\Phi(x)}$$ is maximized at fixpoint $\hat{x} = f(\hat{x})$

②: $h(z) = \dfrac{\mathrm{e}^z}{\Phi(\mathrm{e}^{-z}-1)}$ is concave

③: $|\phi(\lambda) - \phi(0)| \cdot \sup_{x\in[0,\lambda]} \dfrac{1}{\Phi(x)} < \infty$

[Li-Lu-Y. 13]:

$$\Phi(x) = \frac{1}{\sqrt{x(x+1)}}$$

[Restrepo-Shin-Tetali-Vigoda-Yang 11]:

$$\Phi(x) = \frac{1}{\Delta x + 1}$$

[Li-Lu-Y. 12]: $\Phi(x) = x^{-\frac{\Delta_c(\lambda)}{2(\Delta_c(\lambda)-1)}}$

[Peters-Regts 17]:

$$\Phi(x) = \frac{1}{(x+1)\left(1+\frac{\ln(x+1)}{2\hat{x}_c - \ln(\hat{x}_c+1)}\right)}$$

An Abstract View

BP operator $F : \mathbb{R}^N_{\geq 0} \rightarrow \mathbb{R}^N_{\geq 0}$

Jacobian $J = J(\vec{x})$: $$J_{ij} = \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right|$$

converges to the unique fixpoint when $J(\vec{x})\mathbf{1} < \mathbf{1}$

$$\forall i : \quad \sum_j \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right| < 1$$

An Abstract View

BP operator $F : \mathbb{R}^N_{\geq 0} \to \mathbb{R}^N_{\geq 0}$

monotone $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote $\Phi(x) = \phi'(x)$

Jacobian $J = J(\vec{x}):$ $$J_{ij} = \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right|$$

$J^\phi = J^\phi(\vec{x}):$ $$J^\phi_{ij} = \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right| \frac{\Phi(F_i(\vec{x}))}{\Phi(x_j)}$$

converges to the unique fixpoint when $J^\phi(\vec{x})\mathbf{1} < \mathbf{1}$

$$\forall i : \quad \sum_j \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right| \frac{\Phi(F_i(\vec{x}))}{\Phi(x_j)} < 1$$

$$\forall i : \quad \sum_j \left| \frac{\partial F_i(\vec{x})}{\partial x_j} \right| \frac{1}{\Phi(x_j)} < \frac{1}{\Phi(F_i(\vec{x}))}$$

An Abstract View

BP operator $F : \mathbb{R}^N_{\geq 0} \to \mathbb{R}^N_{\geq 0}$

Jacobian $J = J(\vec{x})$: $J_{ij} = \left| \dfrac{\partial F_i(\vec{x})}{\partial x_j} \right| = \begin{cases} \frac{F_i(\vec{x})}{x_j + 1} & j \in N_i \\ 0 & j \notin N_i \end{cases}$

converges to the unique fixpoint when

$$J(\vec{x})\boldsymbol{v}(\vec{x}) < \boldsymbol{v}(F(\vec{x}))$$

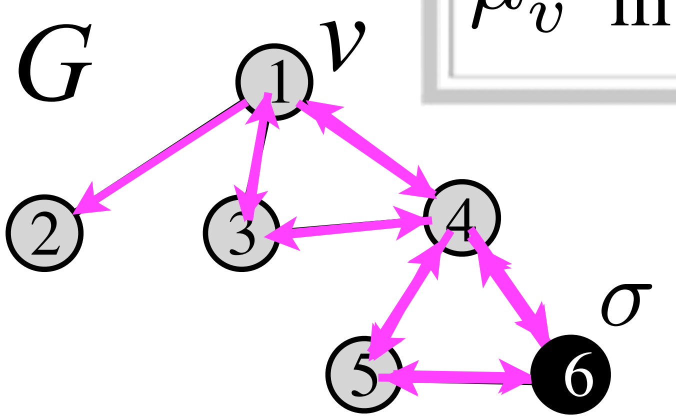
for some $\boldsymbol{v} : \mathbb{R}^N_{\geq 0} \to \mathbb{R}^N_{\geq 0}$

where $\boldsymbol{v}(\vec{x}) = (v_i(x_i))_i$ for $v_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$\boldsymbol{v} = v_1 \oplus v_2 \oplus \cdots \oplus v_N$ $\qquad v_i(x) = \sqrt{x(x+1)}$

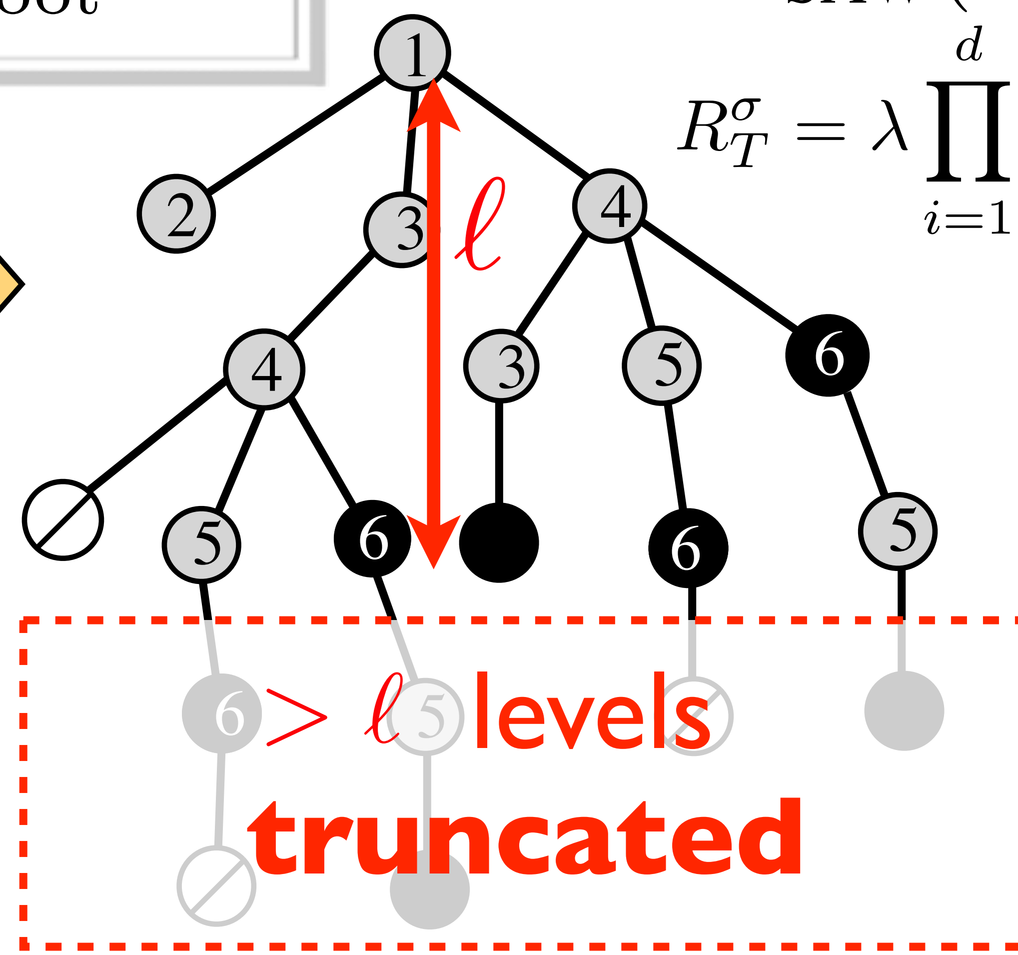
Approximate Counting

$$\mu_v^\sigma \text{ in } G = \mu_{\text{root}}^\sigma \text{ in } T$$



$$T = T_{\text{SAW}}(G, v)$$

$$R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{T_i}^{\sigma_i}}$$



μ_v^σ is approximated within

additive error $\alpha^{\Omega(\ell)} \leq O(\frac{\epsilon}{n})$

ε-approx. of $Z_G(\lambda)$

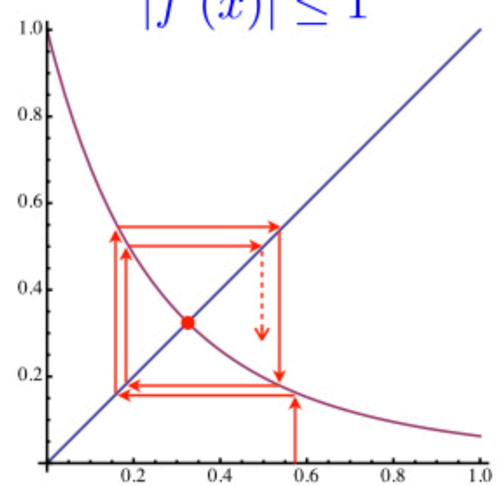
∅ if cycle closing edge > cycle starting edge

● if cycle closing edge < cycle starting edge

single-variate dynamical system: $f(x) = \frac{\lambda}{(1+x)^{\Delta-1}}$

unique fixed point: $\hat{x} = f(\hat{x})$

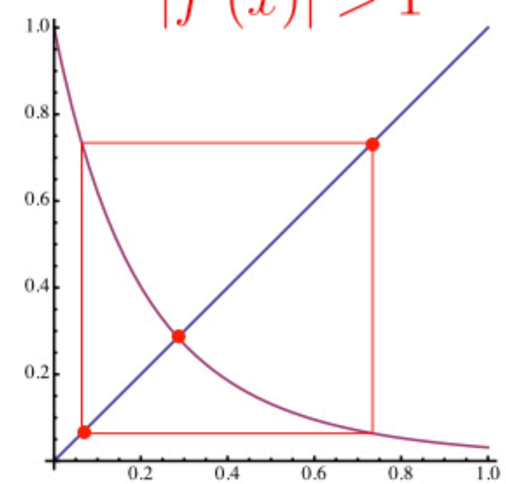
$|f'(\hat{x})| \le 1$



$\lambda \le (1-\delta)\lambda_c(\Delta)$

$|f'(\hat{x})| \le 1 - \Theta(\delta)$

$|f'(\hat{x})| > 1$



$\exists x^- < \hat{x} < x^+$

$$\begin{cases} x^+ = f(x^-) \\ x^- = f(x^+) \end{cases}$$

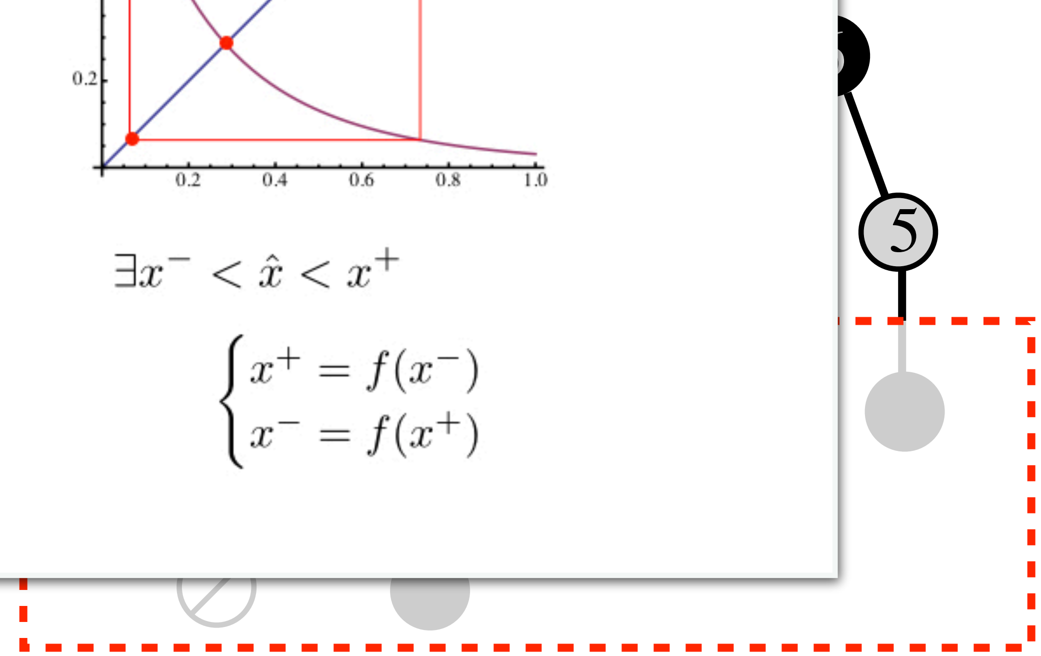
μ_v^σ is a

additive e

ε-approx. of $Z_G(\lambda)$

AW(G, v)

$= \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{T_i}^{\sigma_i}}$

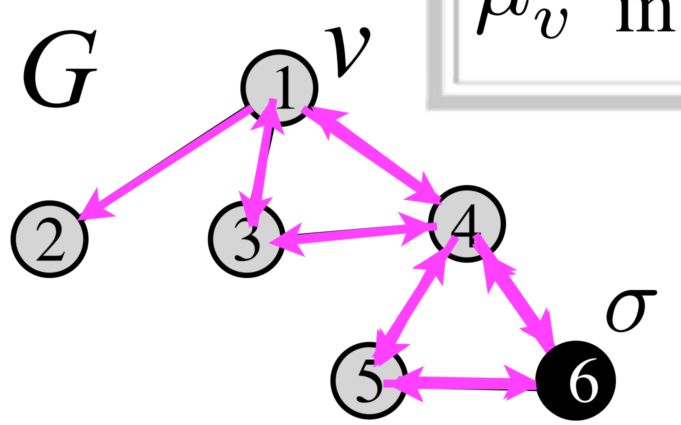


∅ if cycle closing edge > cycle starting edge

● if cycle closing edge < cycle starting edge

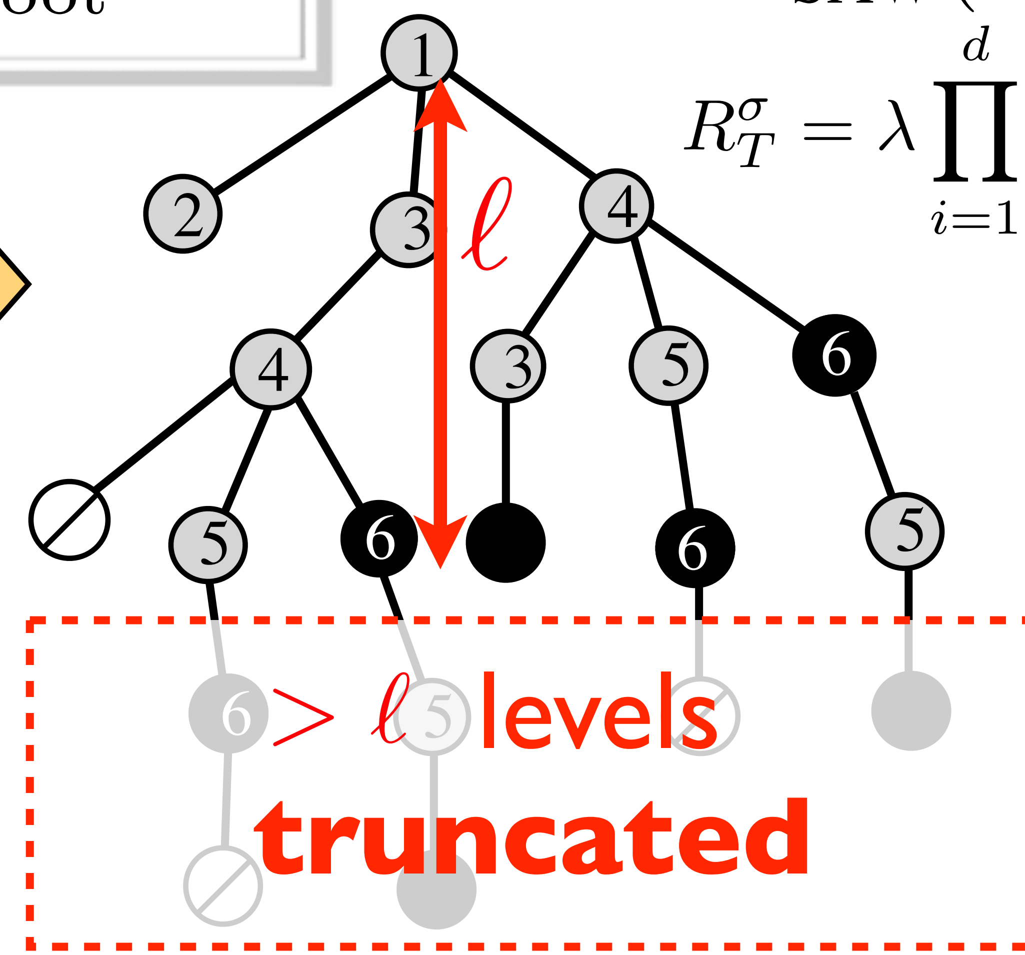
Approximate Counting

$$\mu_v^\sigma \text{ in } G = \mu_{\text{root}}^\sigma \text{ in } T$$



$$T = T_{\text{SAW}}(G, v)$$

$$R_T^\sigma = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{T_i}^{\sigma_i}}$$



○ if cycle closing edge > cycle starting edge

● if cycle closing edge < cycle starting edge

μ_v^σ is approximated within

additive error $\alpha^{\Omega(\ell)} \le O(\frac{\epsilon}{n})$

ε-approx. of $Z_G(\lambda)$

total time cost:

$$\le n\Delta^\ell = \left(\frac{n}{\epsilon}\right)^{O(\log \Delta)}$$

when $\lambda \le (1-\delta)\lambda_c(\Delta)$

for constant $\delta > 0$

Counting Independent Set

hardcore model: undirected graph $G(V,E)$ with max-degree Δ

uniqueness regime: $\lambda \leq (1-\delta)\lambda_c(\Delta)$ for constant $\delta > 0$

ε-approx.: $(1-\epsilon)Z_G(\lambda) \leq \hat{Z} \leq (1+\epsilon)Z_G(\lambda)$

deterministic approximate counting:

time cost $n^{O(\log \Delta)}$ FPTAS when $\Delta = O(1)$

randomized approximate counting:

$\Delta \geq \Delta_0(\delta)$, girth ≥ 7 ⇒ $\tilde{O}\left(n^2\right)$ time by Glauber dynamics

[Efthymiou-Hayes-Štefankovič-Vigoda-Y. 16]

Counting Matchings

monomer-dimer model: $G(V,E)$ with max-degree Δ

$$Z_G(\lambda) = \sum_{M\text{: matching in } G} \lambda^{|M|}$$

TSAW: $f(\vec{x}) = \dfrac{1}{1+\lambda\sum_{i=1}^{d} x_i}$ [Godsil 1981]

SSM with rate: $1-\Theta\left(\frac{1}{\sqrt{\lambda\Delta}}\right)$

ε-approx.: $(1-\epsilon)Z_G(\lambda) \leq \hat{Z} \leq (1+\epsilon)Z_G(\lambda)$

deterministic approximate counting:

time cost $n^{O\left(\sqrt{\lambda\Delta}\log\Delta\right)}$ FPTAS when $\Delta = O(1)$

[Bayati-Gamarnik-Katz-Nair-Tetali 2007]

randomized approximate counting:

Poly(n,1/ε) time by Jerrum-Sinclair chain

Correlation Decay Method

- Deterministic approximate counting by recurrences for marginal probabilities:
  - SAW-tree; [Weitz ’06] [Bayati Gamarnik Katz Nair Tetali 2007]…
  - computation tree. [Gamarnik Katz ’07] [Nair Tetali’07]…
- Deterministic approximate counting by non-vanishing polynomials.
  [Peters Regts ’17] [Bezakova Galanis Goldberg Štefankovič 18]…
- Rapid mixing of dynamics by coupling.
  [Mossel Sly ’13] [Efthymiou Hayes Štefankovič Vigoda Y. ’16]…

Thank you!