Rectangle Inequalities for Data Structure Lower Bounds

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Nanjing University

Nexus of Information and Computation Theories
Fundamental Inequalities and Lower Bounds Theme
@ Institut Henri Poincaré

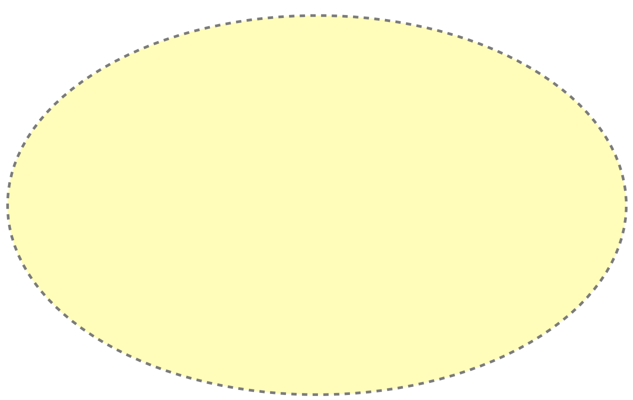
Online Note

"Yitong Yin: Simple average-case lower bounds for approximate near-neighbor from isoperimetric inequalities."

http://arxiv.org/abs/1602.05391

Nearest Neighbor Search
(*NNS*)

metric space (X, dist)



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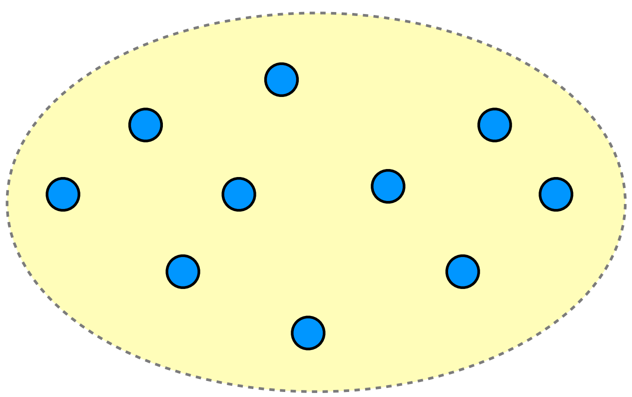
$\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in X^n$



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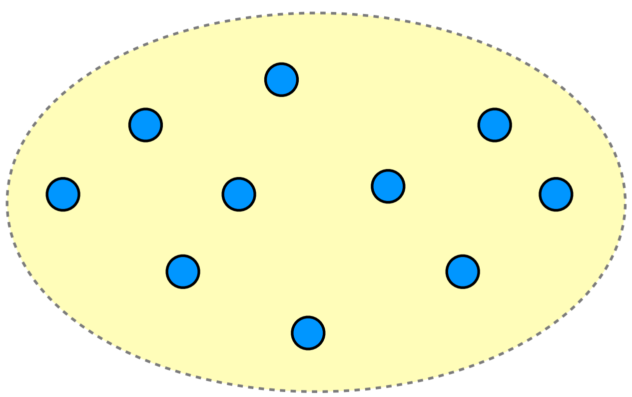

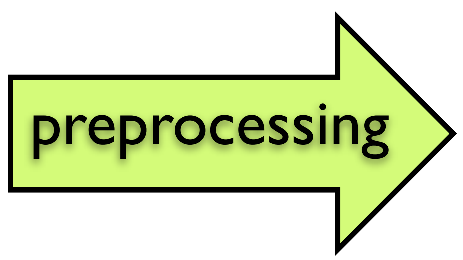

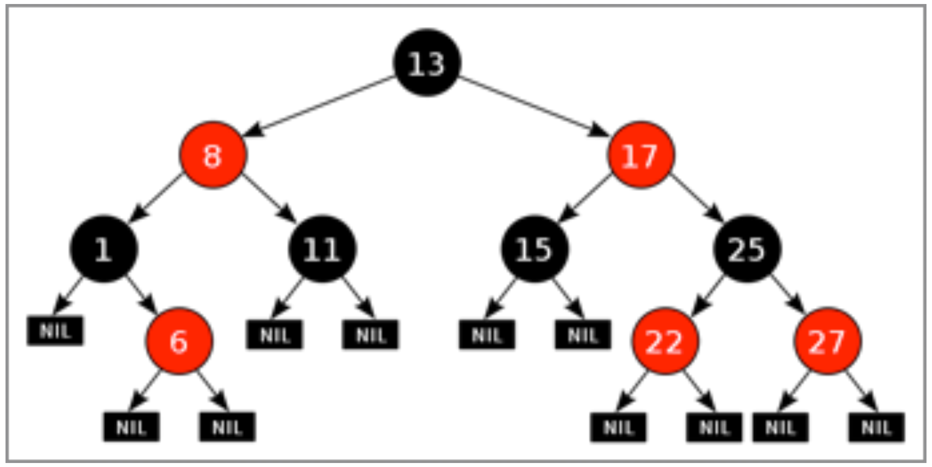

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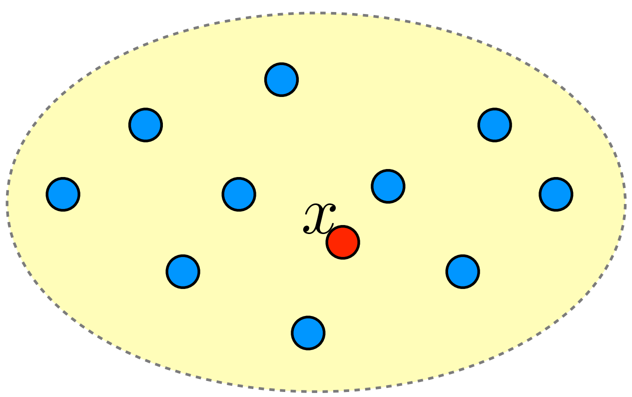

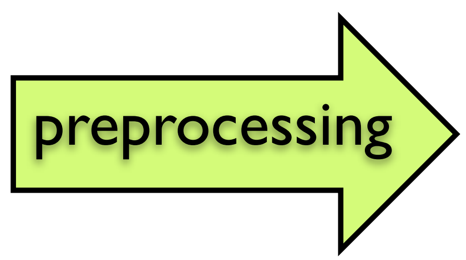

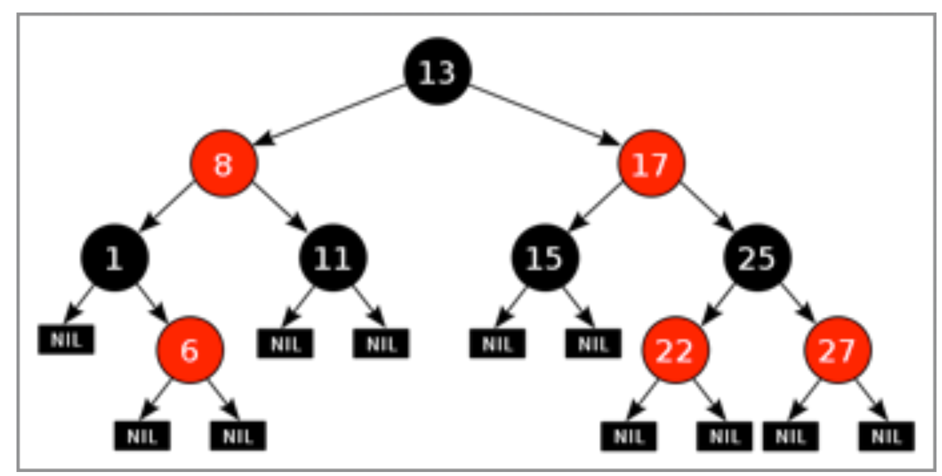

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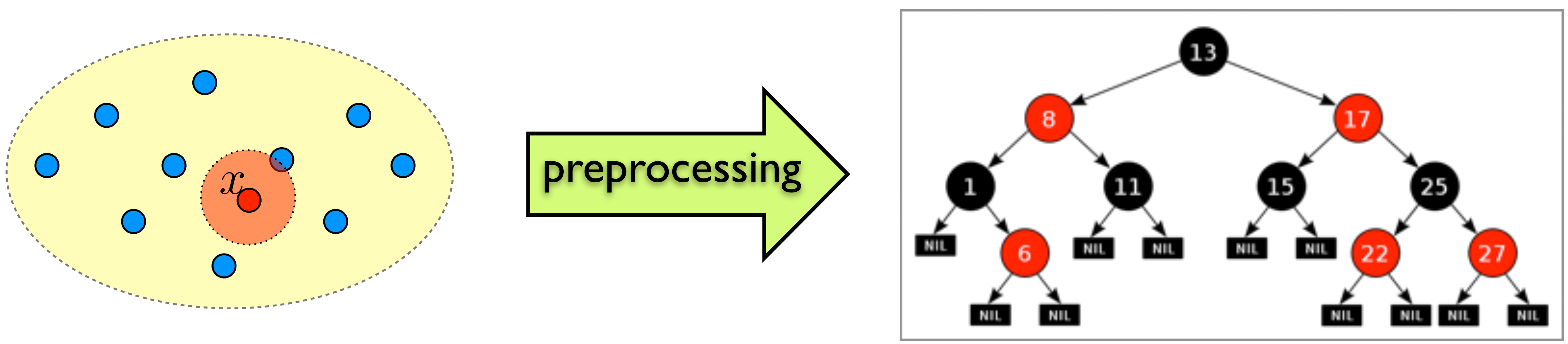
access

data structure



output: database point y_i closest to the query point x

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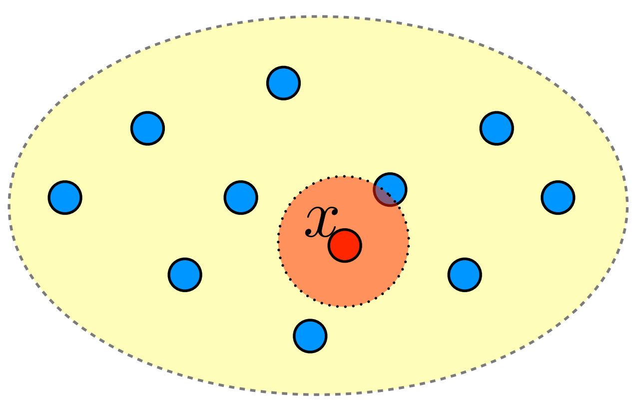

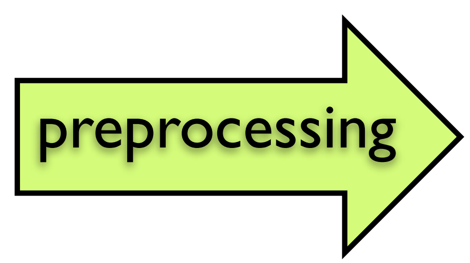

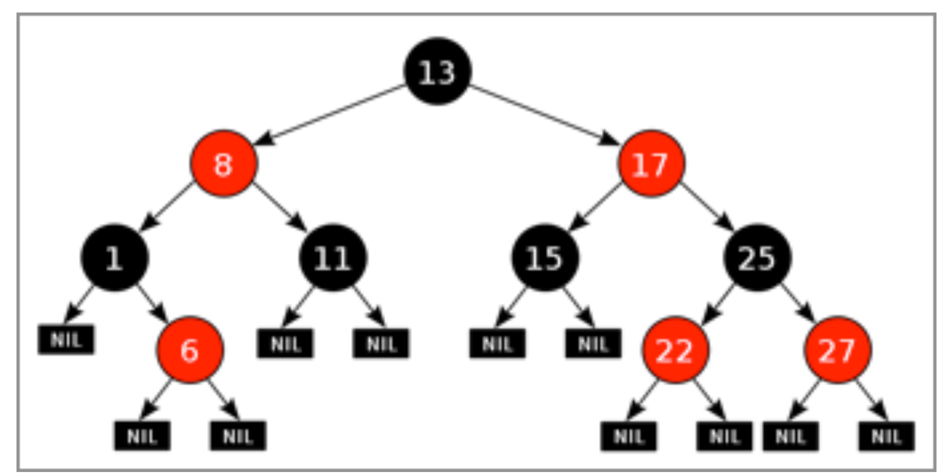

output: database point y_i closest to the query point x

applications: *database, pattern matching, machine learning, ...*

Near Neighbor Problem
(λ-*NN*)

metric space (X, dist)

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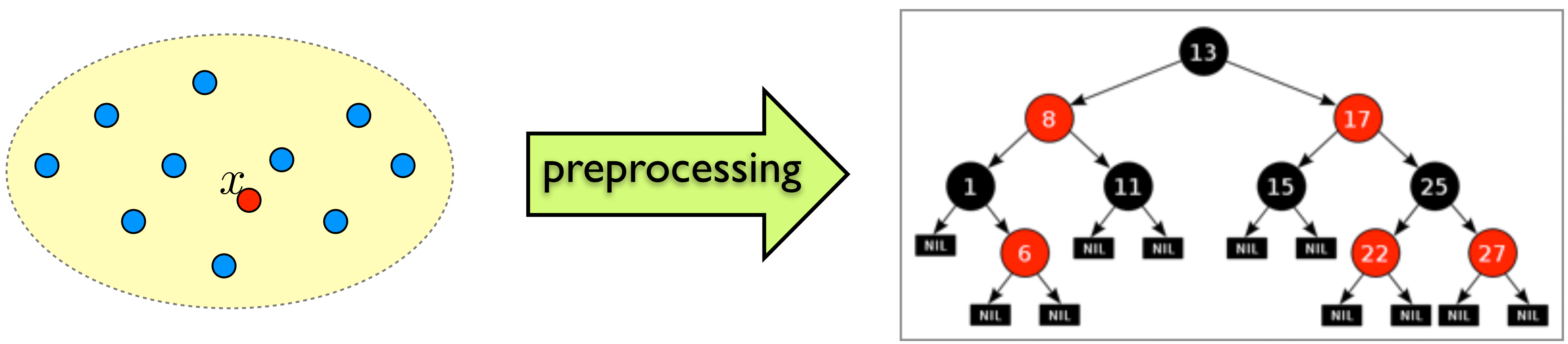

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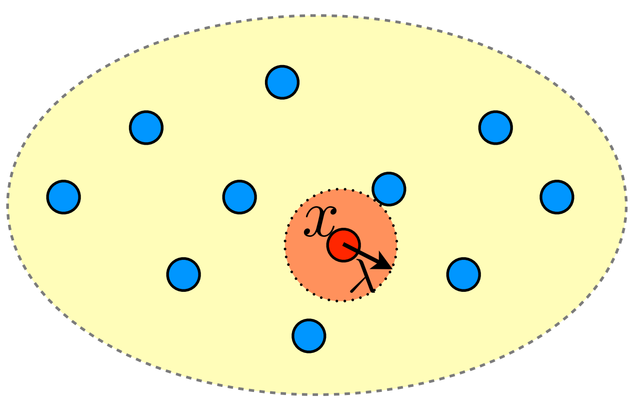
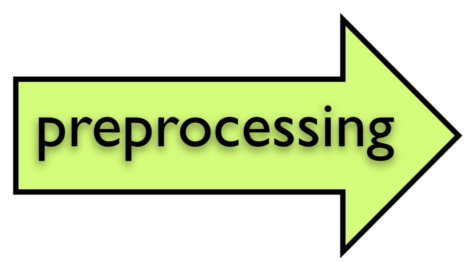
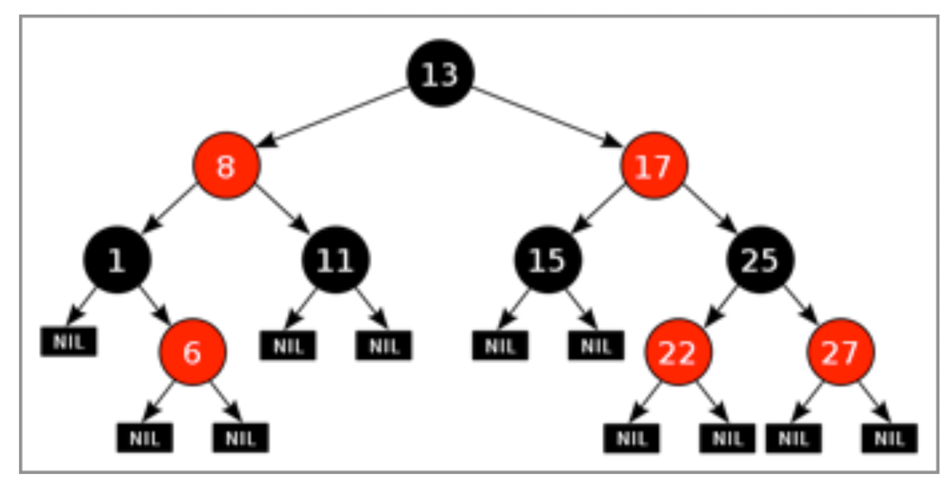
radius λ







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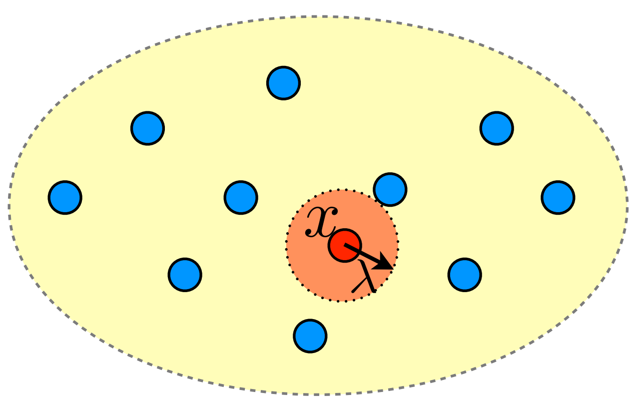

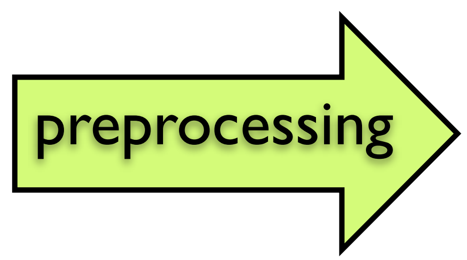

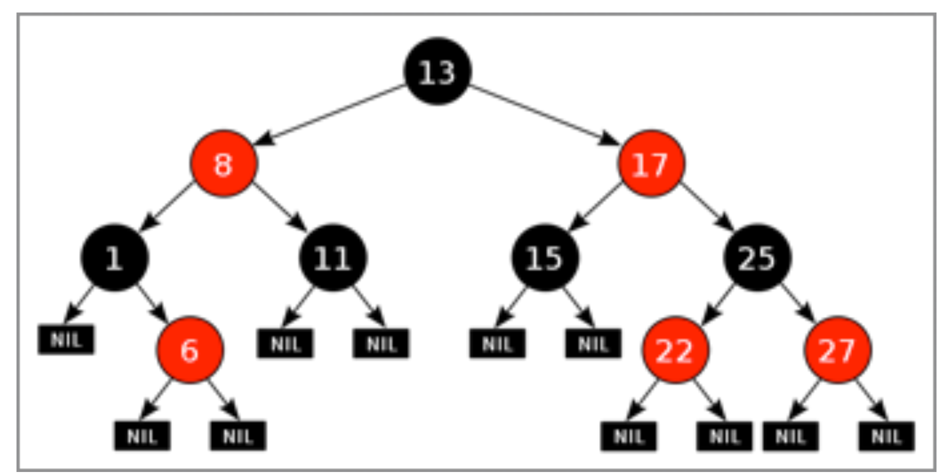

λ-NN: answer "yes" if $\exists y_i$ that is λ-close to x

"no" if all y_i are λ-faraway from x

Approximate Near Neighbor
(ANN)

metric space (X, dist)

query $x \in X$

access

database

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data structure

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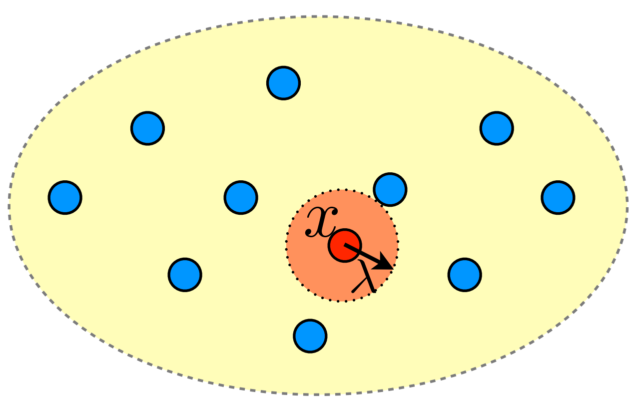

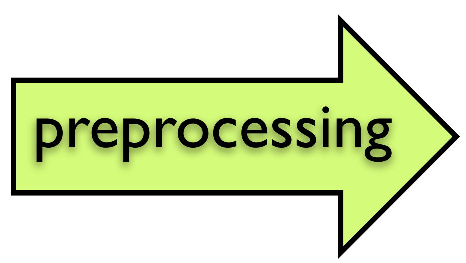

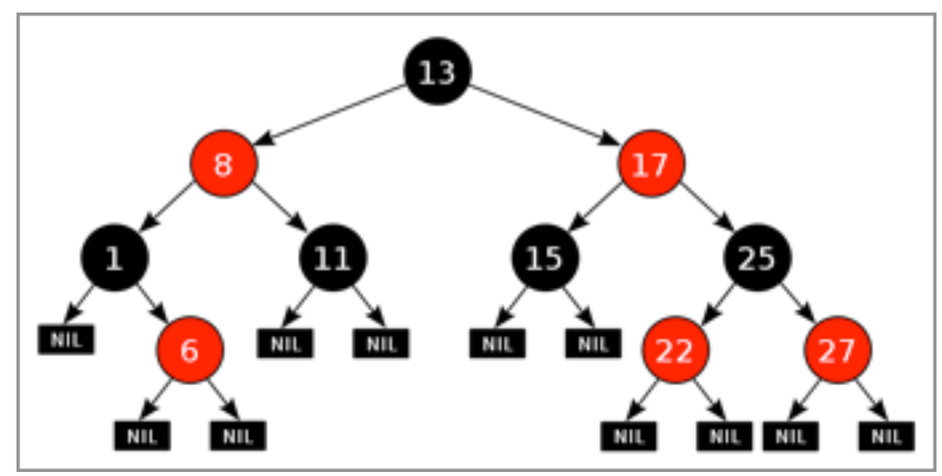

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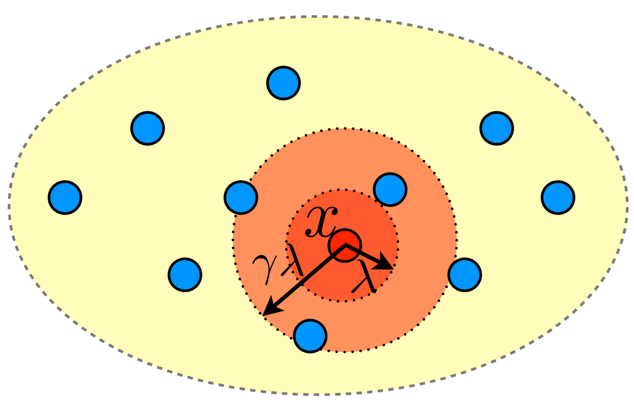
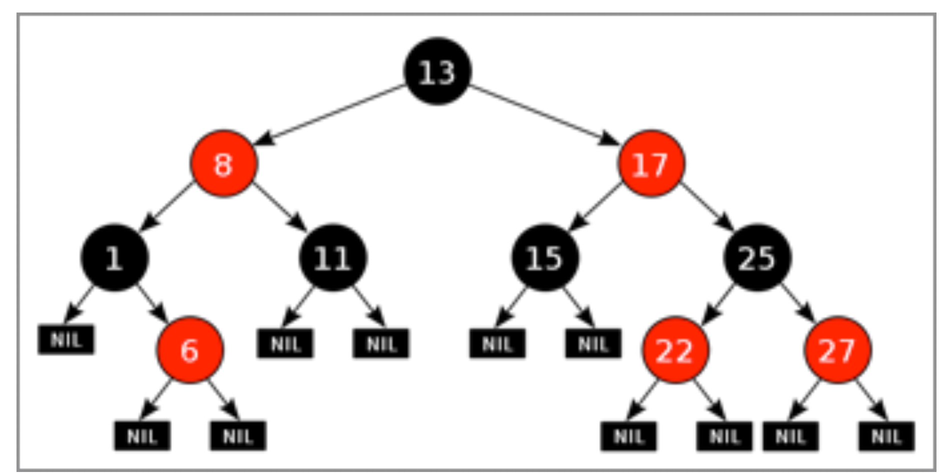


preprocessing

approx ratio $\gamma \geq 1$



(γ, λ)-*ANN*: answer "yes" if $\exists y_i$ that is λ-close to x

"no" if all y_i are $\gamma\lambda$-faraway from x

arbitrary if otherwise

Approximate Near Neighbor
(ANN)

metric space (X,dist)

database

$\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in X^n$

query $x \in X$

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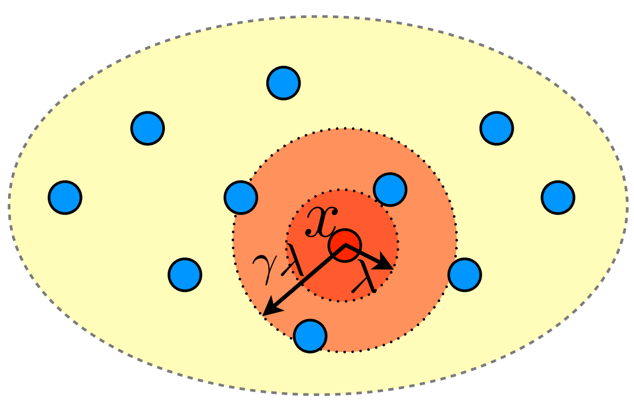

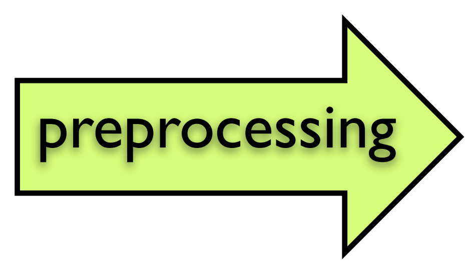

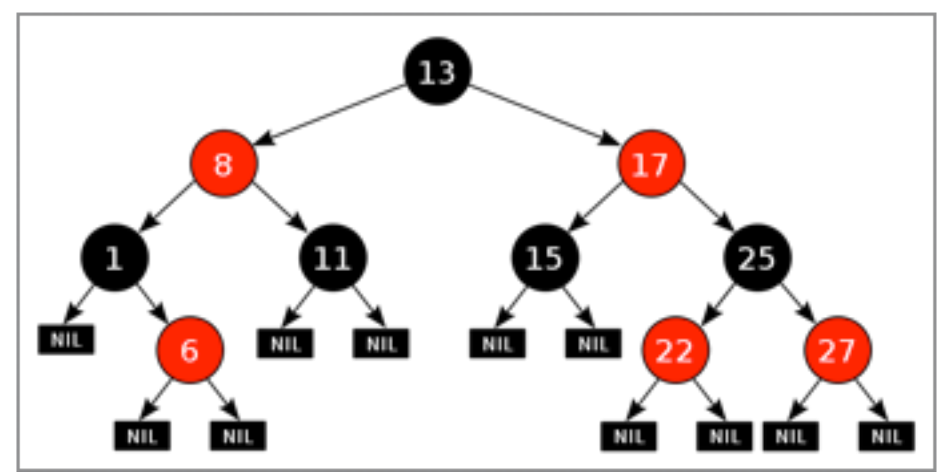

Hamming space $X = \{0,1\}^d$

$\text{dist}(x,z) = \|x - z\|_1$

Hamming distance

Approximate Near Neighbor
(ANN)

metric space (X, dist)

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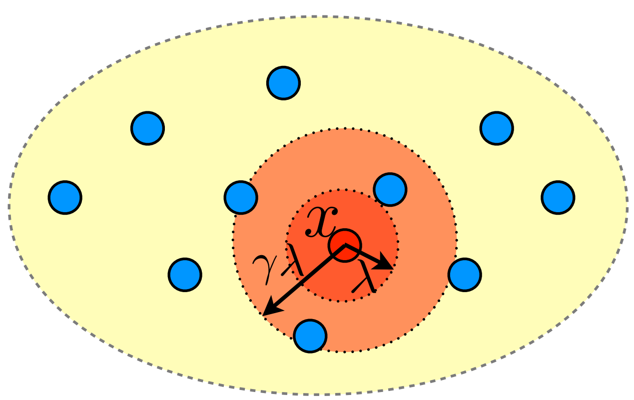

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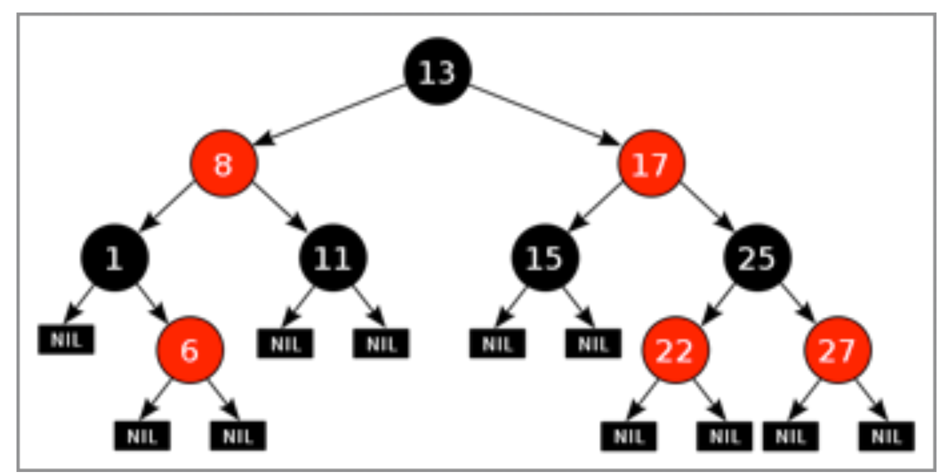

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Hamming distance

Curse of dimensionality!

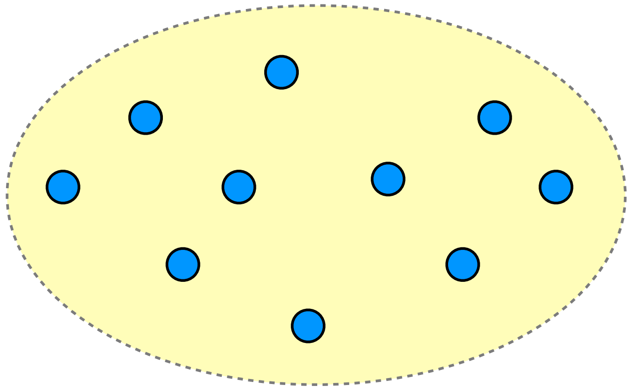
Cell-Probe Model

data structure problem:

$$f : X \times Y \to Z$$

database

$$y \in Y$$



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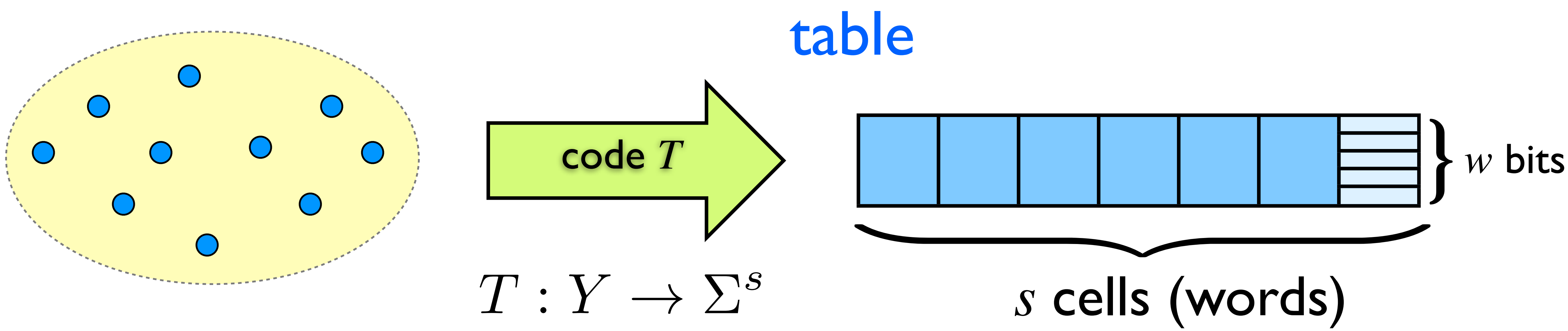

$$T : Y \to \Sigma^s$$

where $\Sigma = \{0,1\}^w$

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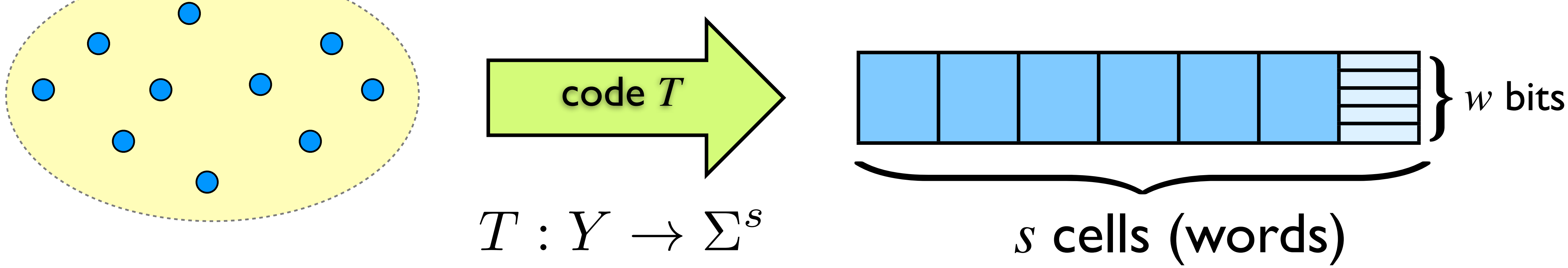

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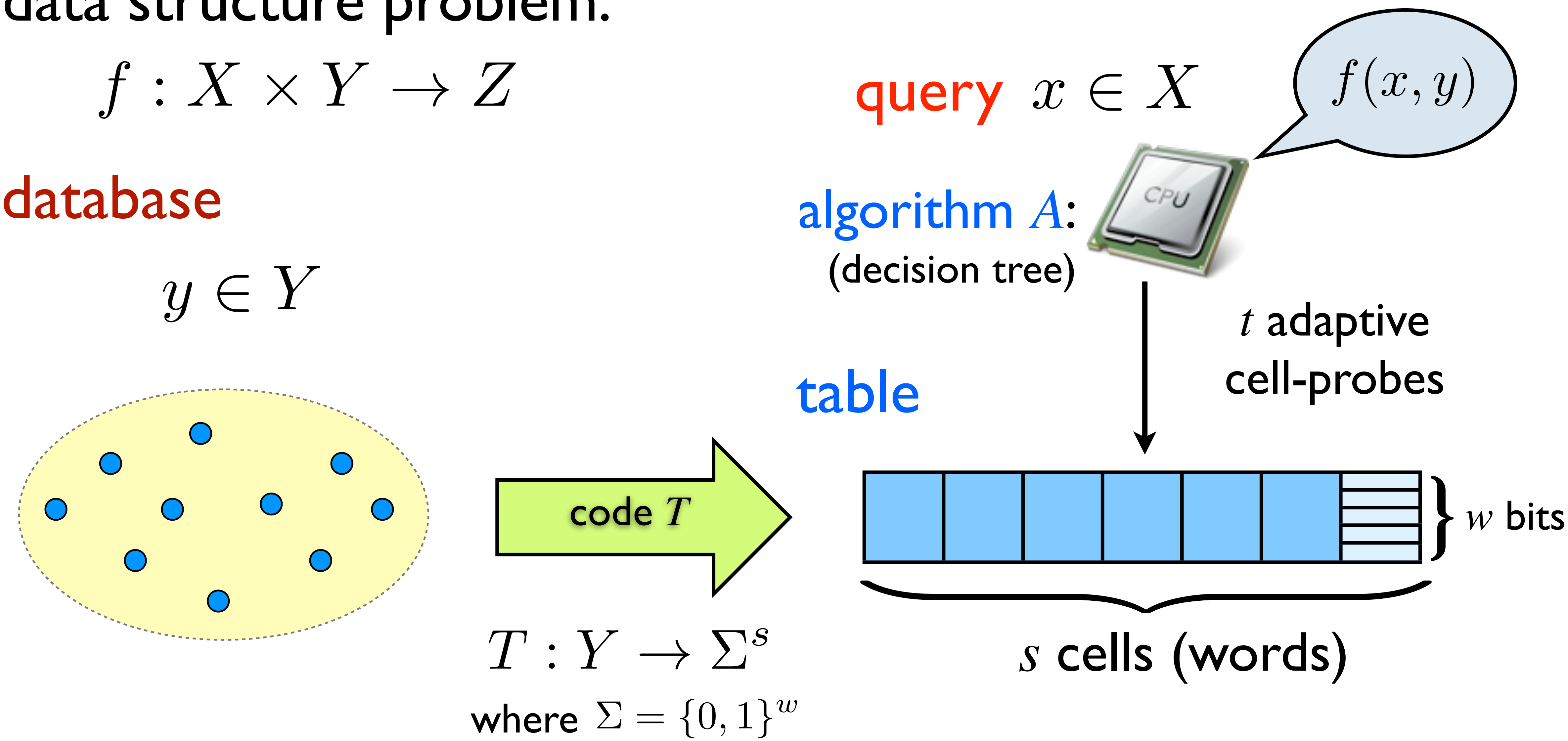

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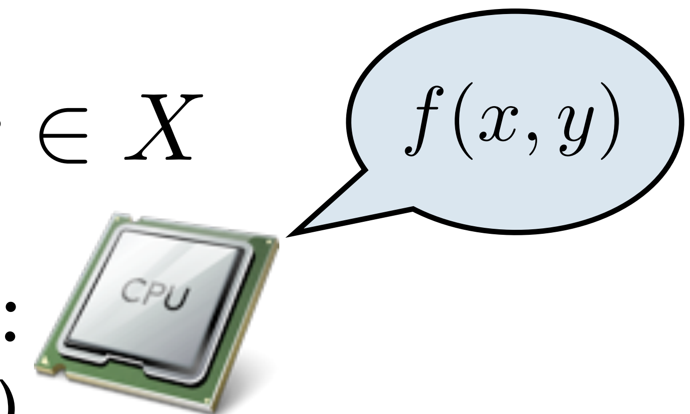


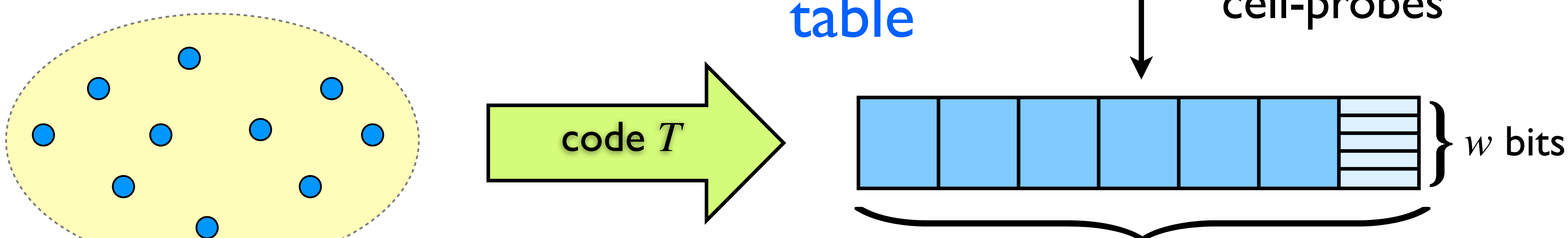
algorithm A:
(decision tree)

database

$y \in Y$

t adaptive
cell-probes

table



$T : Y \to \Sigma^s$
where $\Sigma = \{0,1\}^w$

s cells (words)

protocol: the pair (A, T)

Cell-Probe Model

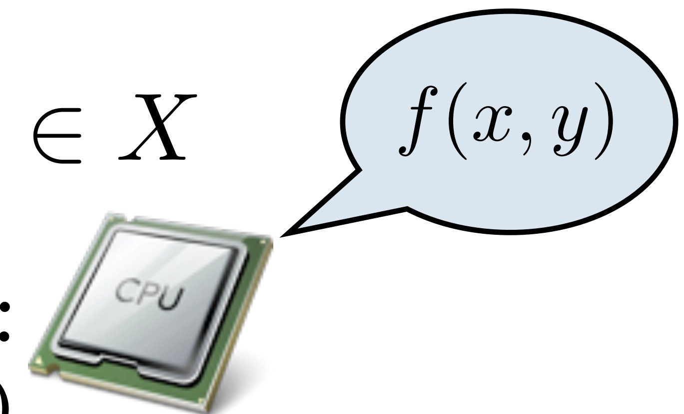
data structure problem:

$f : X \times Y \to Z$

database

$y \in Y$

query $x \in X$



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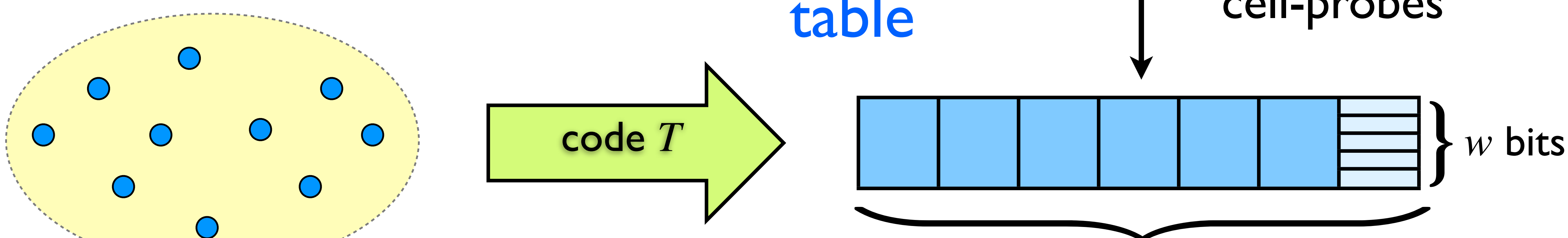

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(s, w, t)-cell-probing scheme

Lower Bounds for Hamming NNS

Hamming space $X = \{0,1\}^d$ database $\boldsymbol{y} \in X^n$

time: t cell-probes; space: s cells, each of w bits

	deterministic	randomized
exact		
approx.		

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data structure problem: $f : X \times Y \to Z$

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- In data-dependent LSH [Andoni Razenshteyn 2015]:
  a key step is to solve the problem on random input.

Lower Bounds for Hamming NNS

Hamming space $X = \{0,1\}^d$ database $\boldsymbol{y} \in X^n$

time: t cell-probes; space: s cells, each of w bits

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worst-case

Approximate Near Neighbor

deterministic or *LV randomized* algorithm for

(γ, λ)-***ANN*** in Hamming space $\{0,1\}^d$

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Metric Expansion

metric space (X,dist)

λ-neighborhood: $\forall x \in X$, denote $N_\lambda(x) = \{z \in X \mid \text{dist}(x,z) \le \lambda\}$

$\forall A \subseteq X$, denote $N_\lambda(A) = \{z \in X \mid \exists x \in A \text{ s.t. } \text{dist}(x,z) \le \lambda\}$

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vertex expansion, “blow-up” effect

for metric space (X, dist), distribution μ over X:

deterministic or *LV randomized* algorithm for

(γ, λ)-***ANN*** in metric space (X, dist)

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query $x \sim \mu$, database $y=(y_1, y_2,\ldots,y_n) \sim \mu^n$

$$\Rightarrow \quad t = \Omega\left(\frac{\log \Phi}{\log \frac{sw}{n \log \Psi}}\right) \quad \text{or} \quad t = \Omega\left(\frac{n \log \Psi}{w + \log s}\right)$$

Asymmetric Communications

$$f : X \times Y \to \{0, 1\}$$

$x \in X$   $y \in Y$

Asymmetric Communications

$f(x, y)$

$$f : X \times Y \to \{0, 1\}$$

$x \in X$

a bits in total →

← b bits in total

$y \in Y$

[a,b]-protocol: Alice sends a total of ≤a bits
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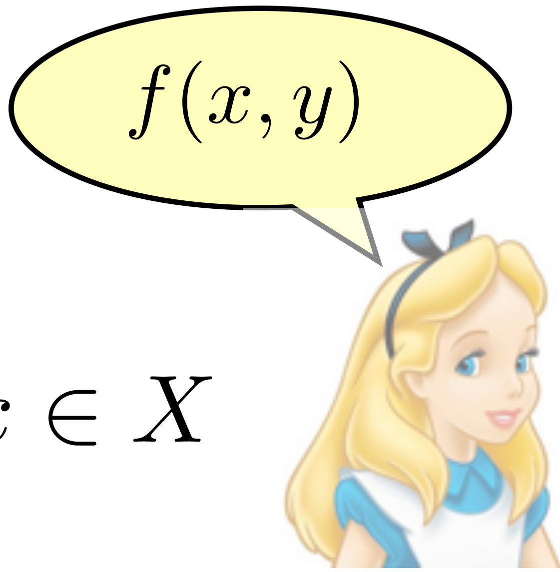
Asymmetric Communications



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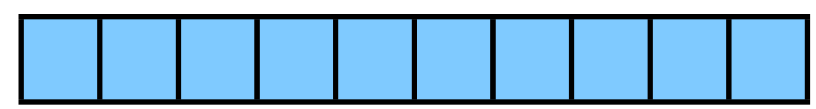

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(s,w,t)-cell-probing scheme ⇒ [$t \log s$, tw]-protocol

The *Richness* Lemma

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$f : X \times Y \to \{0,1\}$ distributions μ over X, ν over Y

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monochromatic 1-rectangle: $A \times B$ with $A \subseteq X$, $B \subseteq Y$
$\forall (x,y) \in A \times B$, $f(x,y)=1$

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Richness lemma (Miltersen, Nisan, Safra, Wigderson, 1995)

f is 0.01-dense under $\mu\times\nu$

f has $[a,b]$-protocol

$\Rightarrow$ f has 1-rectangle $A\times B$ with

$$\begin{cases} \mu(A) \geq 2^{-O(a)} \\ \nu(B) \geq 2^{-O(a+b)} \end{cases}$$

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New Richness lemma

f is 0.01-dense under $\mu\times\nu$
f has average-case (s,w,t)-cell-probing scheme under $\mu\times\nu$

⇒ $\forall \Delta \in [320000t, s]$,
f has 1-rectangle $A\times B$ with

$$\begin{cases} \mu(A) \geq 2^{-O(t \log (s/\Delta))} \\ \nu(B) \geq 2^{-O(\Delta \log (s/\Delta) + \Delta w)} \end{cases}$$

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when $\Delta = O(t)$, it becomes the richness lemma (with slightly better bounds)

New Richness lemma

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$\exists$ constant fraction (under ν) of "good" databases y:

New Richness lemma

f is 0.01-dense under $\mu\times\nu$

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⟹

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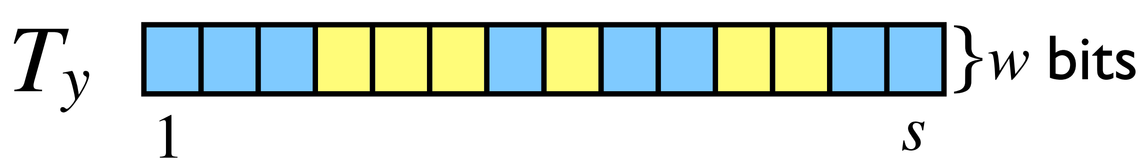

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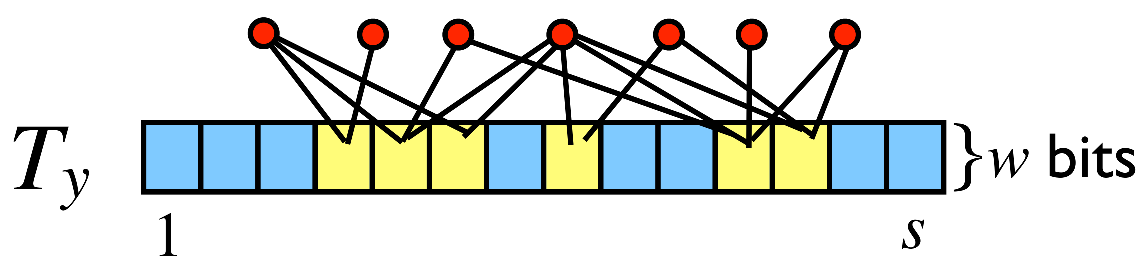

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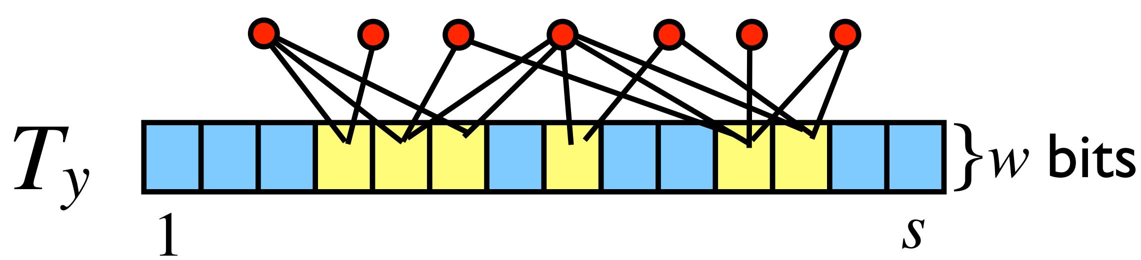

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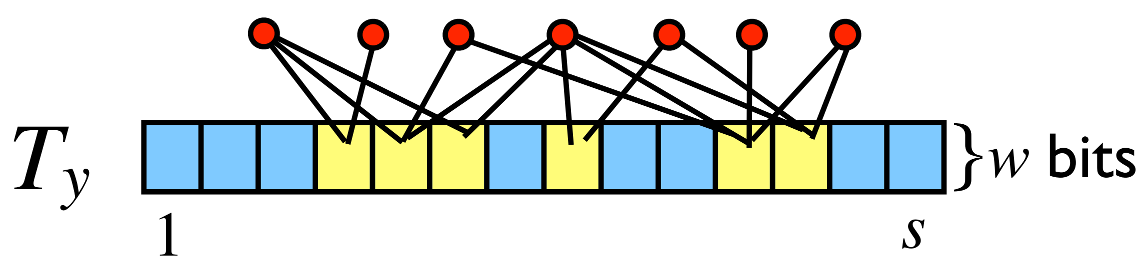

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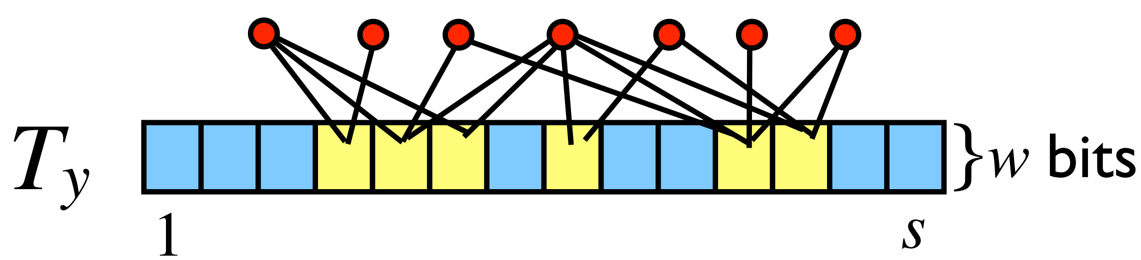

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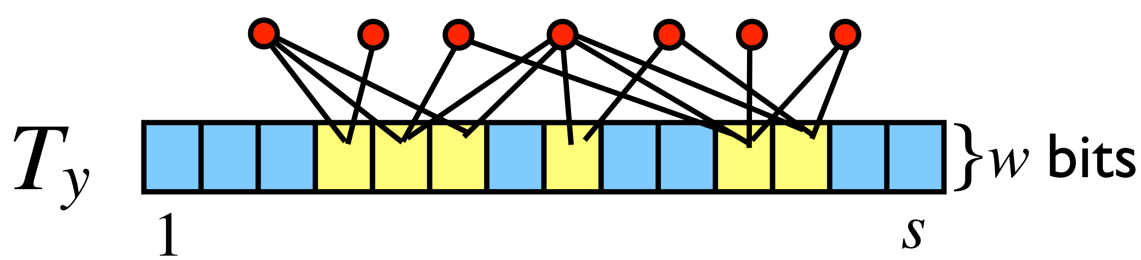

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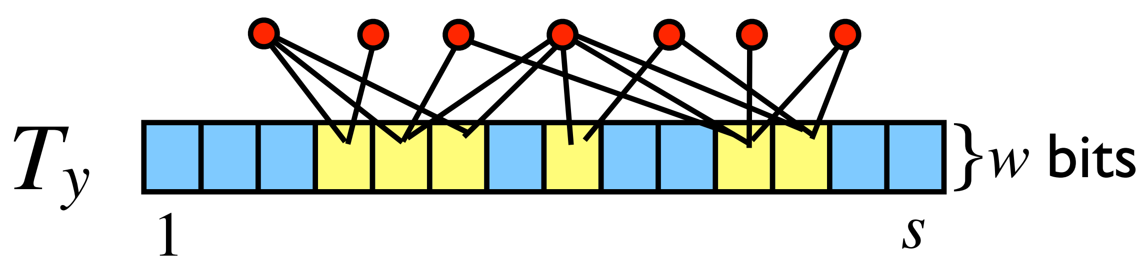

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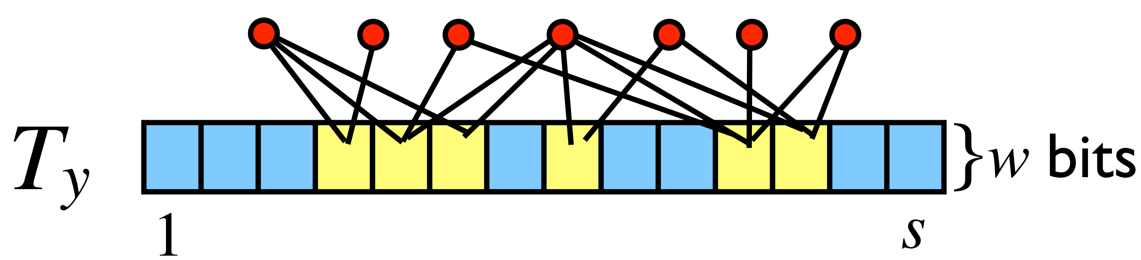

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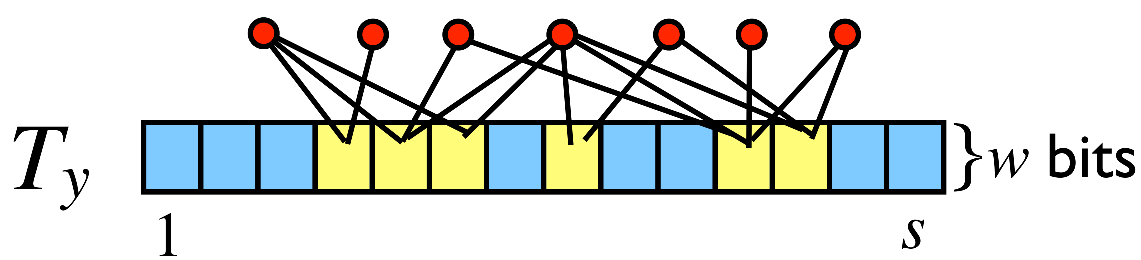

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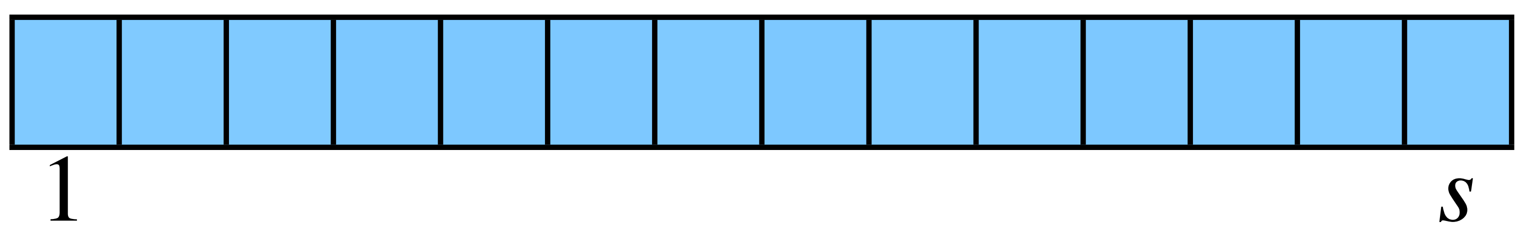

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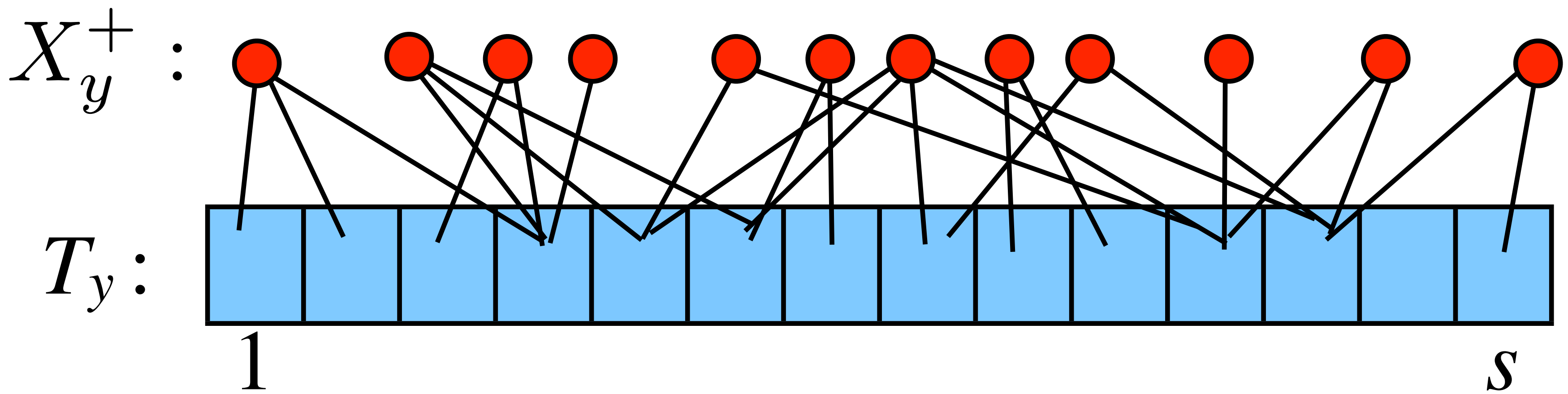

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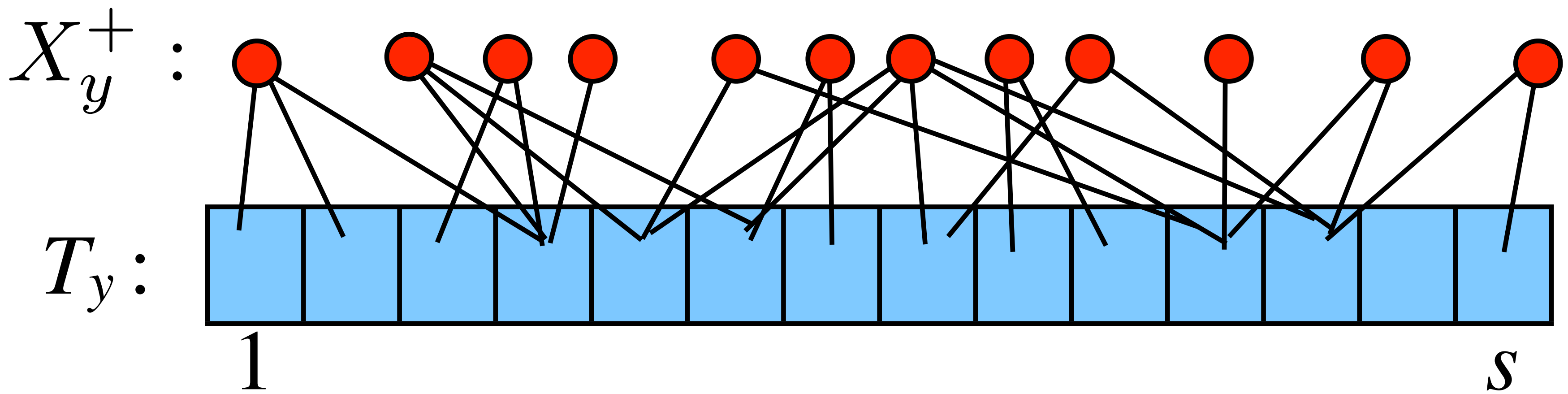

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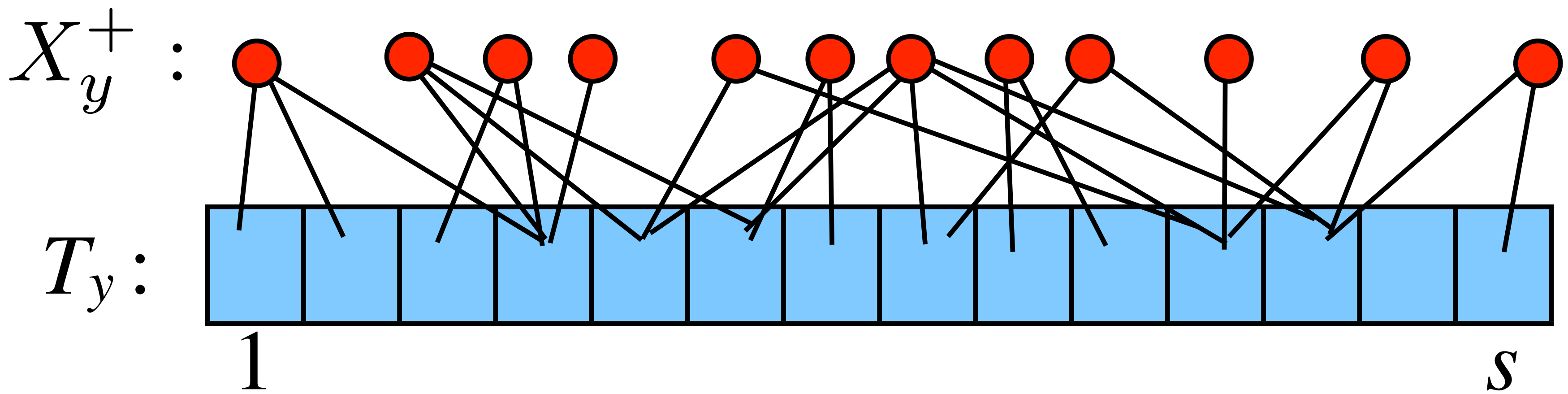

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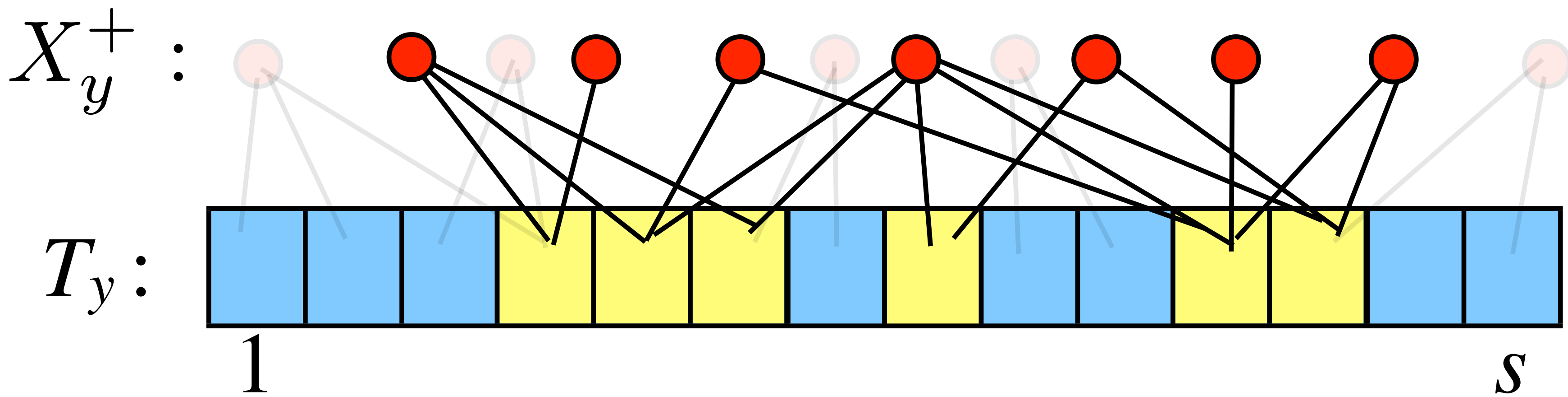

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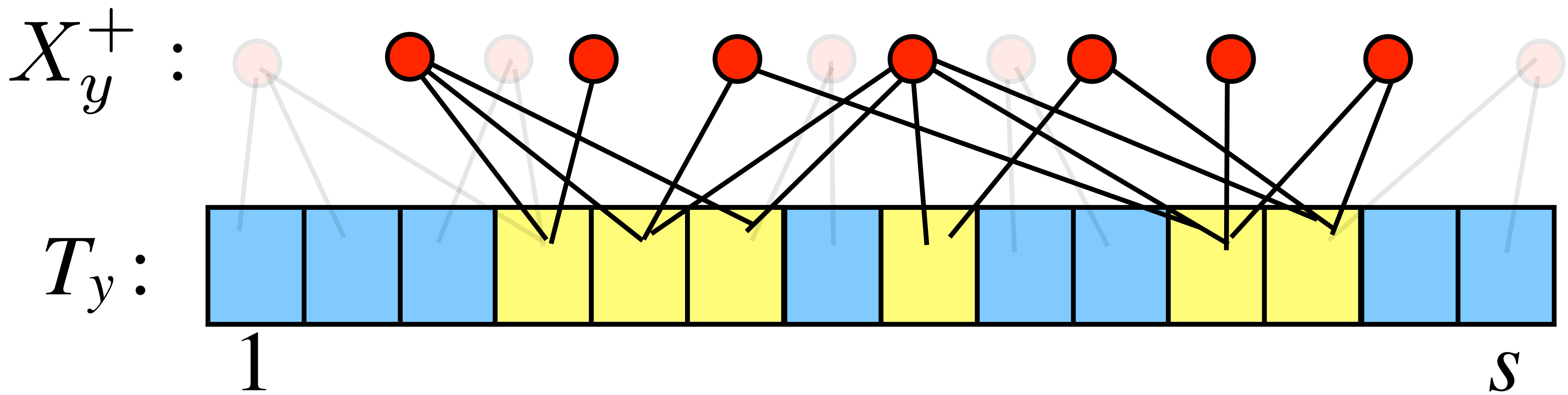

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probabilistic method ⇒ $\forall\ \Delta \geq 320000t$, $\exists$ *sub*-hypergraph *induced* by Δ vertices of measure $\frac{1}{2}\left(\frac{\Delta}{2s}\right)^{80000t} \geq 2^{-O(t\log\frac{s}{\Delta})}$ (under μ_y^+)

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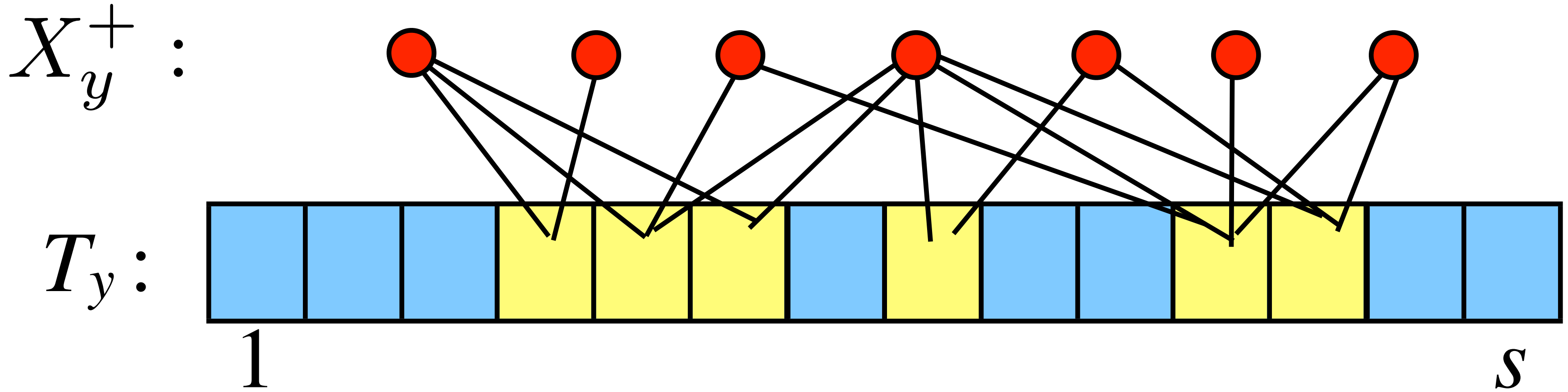

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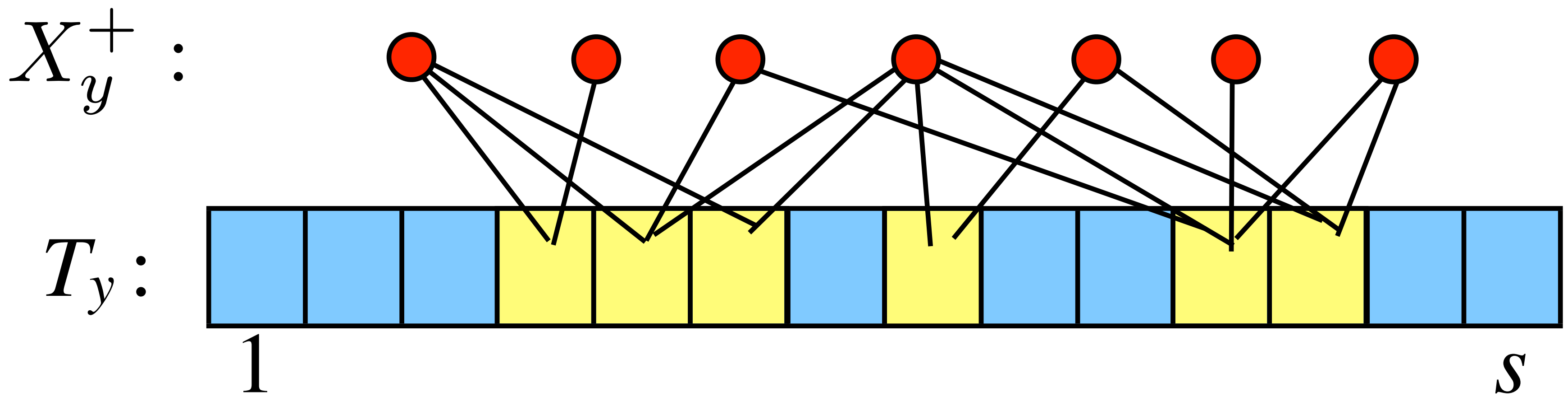

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fraction of positive queries (under μ)

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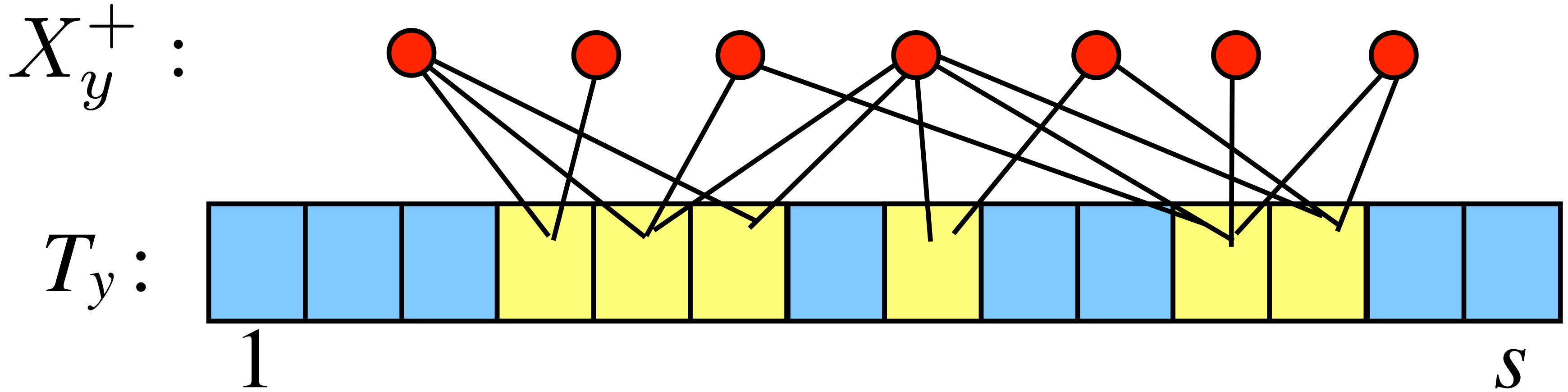

$\forall \ \Delta \geq 320000t$:

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good $y \longmapsto \omega$ resolving $2^{-O(t \log (s/\Delta))}$ positive queries

A New Richness Lemma

$f : X \times Y \to \{0,1\}$ distributions μ over X, ν over Y

New Richness lemma

f is 0.01-dense under $\mu\times\nu$

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⇒

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Examples: partial match, membership, range query, ...

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Theorem

Assume:

- density of 0s in g is $\leq 0.99/n$ under $\mu\times\nu$;
- g has no 1-rectangle $A\times B$ with $\mu(A)\geq 1/\Phi$ and $\nu(B)\geq 1/\Psi$.

If f has an average-case (s,w,t)-cell-probing scheme under $\mu\times\nu^n$

$$\Rightarrow \quad t = \Omega\left(\frac{\log \Phi}{\log \frac{sw}{n \log \Psi}}\right) \quad \text{or} \quad t = \Omega\left(\frac{n \log \Psi}{w + \log s}\right)$$

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$$f(x,y) = \bigwedge_{i=1}^{n} g(x, y_i)$$

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- density of 0s in g is $\leq 0.99/n$ under $\mu\times\nu$; ⇒ (union bound) **f is 0.01-dense under $\mu\times\nu^n$**
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New Richness lemma

f is 0.01-dense under $\mu\times\nu^n$

f has average-case (s,w,t)-cell-probing scheme under $\mu\times\nu^n$

⇒ $\forall\Delta\in[320000t,s]$,

f has 1-rectangle $A\times B'$ with

$$\begin{cases} \mu(A) \geq 2^{-O(t\log(s/\Delta))} \\ \nu^n(B') \geq 2^{-O(\Delta\log(s/\Delta)+\Delta w)} \end{cases}$$

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case 1: $\Delta < 320000t$

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case 1: $\Delta < 320000t$ ⇨ $t = \Omega\left(\frac{n\log\Psi}{w+\log s}\right)$

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union bound $\Rightarrow$ f is 0.01-dense under $\mu\times\nu^n$

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case 1: $\Delta < 320000t$ $\Rightarrow$ $t = \Omega\left(\frac{n\log\Psi}{w+\log s}\right)$

case 2: otherwise

$$f(x,y) = \bigwedge_{i=1}^{n} g(x, y_i)$$

Assume:

- density of 0s in g is $\leq 0.99/n$ under $\mu\times\nu$; → (union bound) f is 0.01-dense under $\mu\times\nu^n$
- g has no 1-rectangle $A\times B$ with $\mu(A)\geq 1/\Phi$ and $\nu(B)\geq 1/\Psi$.

New Richness lemma

f is 0.01-dense under $\mu\times\nu^n$

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case 1: $\Delta < 320000t$ ⟹ $t = \Omega\left(\frac{n\log\Psi}{w+\log s}\right)$

case 2: otherwise ⟹ $1/\Phi > \mu(A) \geq 2^{-O(t\log(s/\Delta))}$

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data structure problem $f : X \times Z^n \to \{0,1\}$

$$f(x,y) = \bigwedge_{i=1}^{n} g(x, y_i)$$

with point-wise function $g : X \times Z \to \{0,1\}$

distributions μ over X, ν over Z, ν^n over $Y = Z^n$

Theorem

Assume:

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(γ, λ)-ANN: $$f(x,y) = \bigwedge_{i=1}^{n} g(x, y_i)$$

$$g(x, y_i) = \begin{cases} 1 & \text{dist}(x, y_i) > \gamma\lambda \\ 0 & \text{dist}(x, y_i) \le \lambda \\ * & \text{otherwise} \end{cases}$$

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If f has an average-case (s,w,t)-cell-probing scheme under $\mu\times\mu^n$

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  $\forall A\subseteq X$, $\mu(A)\ge 1/\Phi \Rightarrow \mu(N_\lambda(A))\ge 1\text{-}1/\Psi$

metric space (X,dist), distribution μ over X:

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Theorem

Assume:

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If (γ,λ)-*ANN* has avg.-case (s,w,t)-cell-probing scheme under $\mu\times\nu^n$

$$t = \Omega\left(\frac{\log\Phi}{\log\frac{sw}{n\log\Psi}}\right) \quad \text{or} \quad t = \Omega\left(\frac{n\log\Psi}{w+\log s}\right)$$

Hamming space $X=\{0,1\}^d$, uniform distribution μ over X:

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Harper's Isoperimetric inequality:

$$\forall A \subseteq X,\ \mu(A) \geq \mu(N_r(\mathbf{\underline{0}})) \Rightarrow \mu(N_\lambda(A)) \geq \mu(N_{r+\lambda}(\mathbf{\underline{0}}))$$

"Hamming balls have the smallest vertex-expansion."

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- λ-neighborhoods are $(2^{\Theta(d)}, 2^{\Theta(d)})$-expanding under μ:

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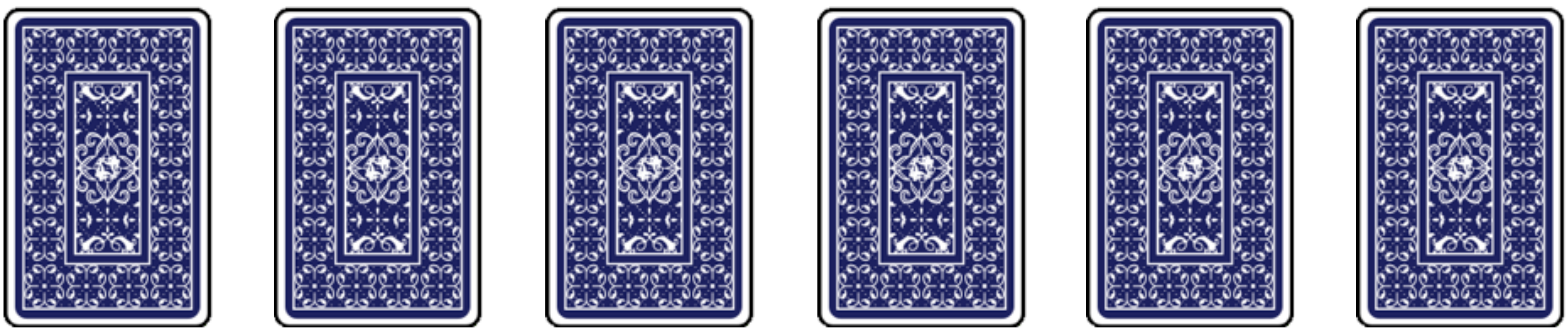
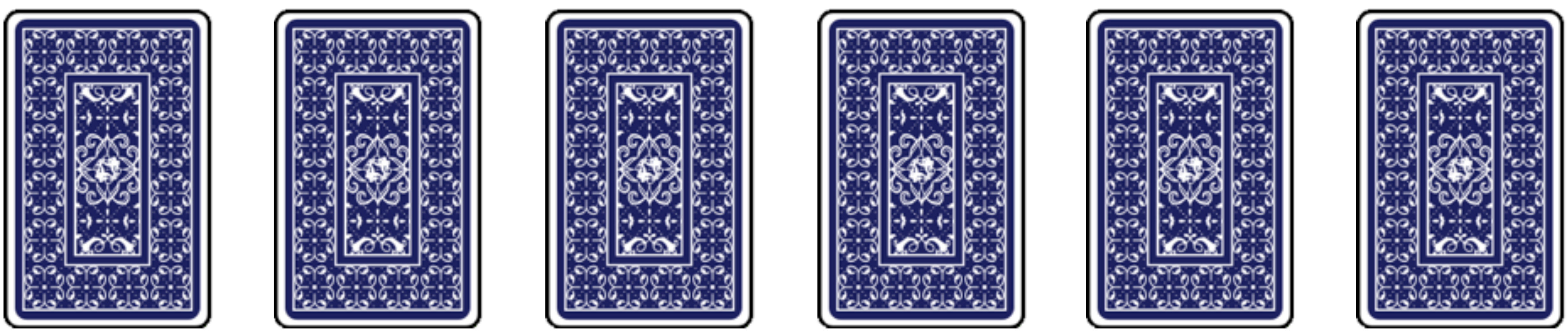
Certificates in Data Structures



certificate: the cells whose contents *uniquely identify* the answer

Certificates in Data Structures

for one-dimensional nearest neighbor search
(predecessor search)



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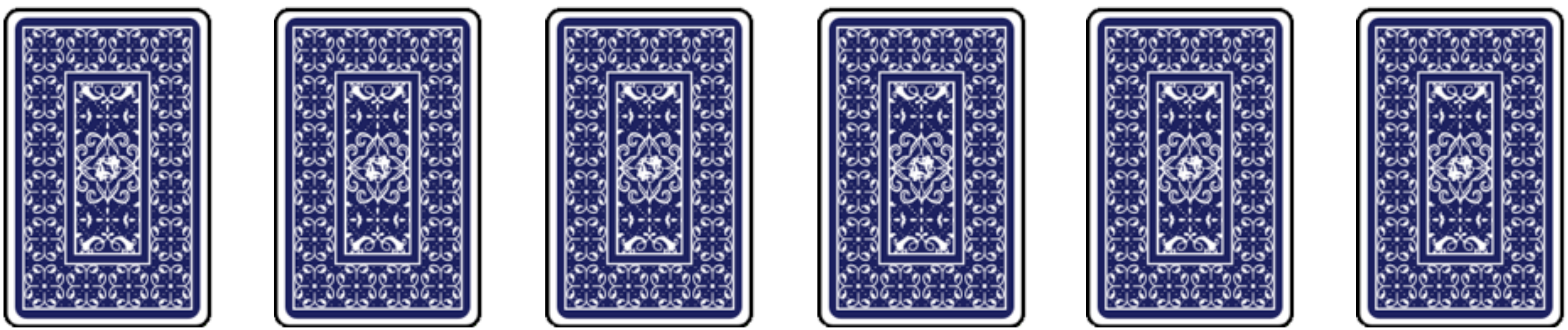
assuming the data structure is the sorted table:



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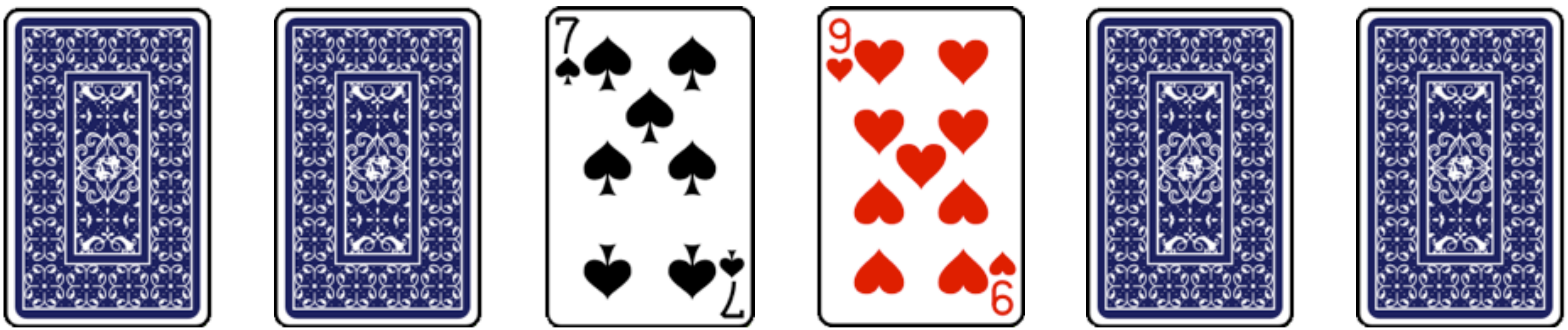

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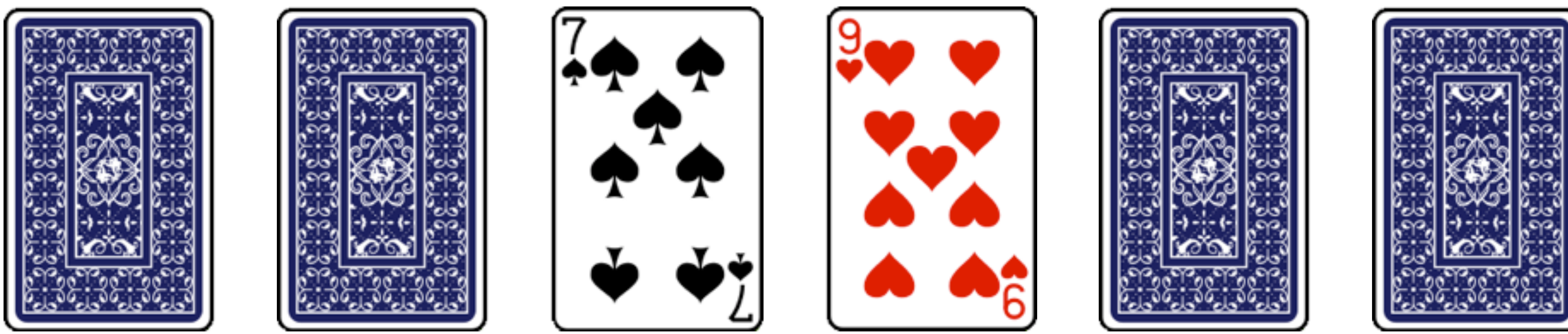

certifies the nearest neighbor of "8.5"

certificate: the cells whose contents *uniquely identify* the answer

Lower Bounds for Hamming NNS

Hamming space $X = \{0,1\}^d$ database $\boldsymbol{y} \in X^n$

time: t cell-probes; space: s cells, each of w bits

	deterministic	randomized
exact	average-case: $t = \Omega\left(\frac{d}{\log \frac{sw}{nd}}\right)$	average-case: $t = \Omega\left(\frac{d}{\log s}\right)$ [Barkol Rabani 2000] worst-case: $t = \Omega\left(\frac{d}{\log \frac{sw}{n}}\right)$ [Pătraşcu Thorup 2006]
approx.	average-case: $t = \Omega\left(\frac{d}{\log \frac{sw}{nd}}\right)$	worst-case, search problem: $t = \Theta\left(\frac{\log\log d}{\log\log\log d}\right)$ for $s = \mathrm{poly}(n)$ [Chakrabarti Regev 2004] average-case: $t = \Omega\left(\frac{\log n}{\log \frac{sw}{n}}\right)$ [Panigrahy Talwar Wieder 2008, 2010]

Thank you!